COMPLEX STRUCTURES ON $\mathbb{R}H^4$ AND $\mathbb{C}H^2$

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Abstract

Left invariant complex structures in $\mathbb{R}H^4$ and $\mathbb{C}H^2$ are classified. Using the model of these symmetric spaces given by Heintze, we take the correponding Lie algebras and then we classify left invariant complex structures. On the real hyperbolic space there exists only one complex structure (which correspond to the canonical one) and on the complex case there are three, one of them is a Kählerian structure with respect to the symmetric metric. Then we study the associated hermitian geometry after attaching a left invariant metric to the Lie algebra.

1. Introduction

The invariant complex structures on a compact semisimple Lie group or, more generally on compact homogeneous manifolds with finite fundamental groups was treated by Wang [W]. In the non-compact case, Morimoto [M] showed that there always exists an invariant complex structure on any even dimensional reductive Lie group. D. Snow [Sn1] gave a classification of regular invariant complex structures on reductive Lie groups.

On the solvable case, J. E. Snow [Sn2] classified invariant complex structures on four dimensional solvable real Lie groups with conmutator of dimension less than three and L. Barberis [B] classified left invariant hypercomplex stuctures on four dimensional Lie groups.

The symmetric spaces $\mathbb{R}H^4$ and $\mathbb{C}H^2$ modelized as real Lie groups with a left invariant metric are examples on dimension four of real solvable Lie groups whose conmutator has dimension three. The purpose of this note is to classify complex structures on the corresponding Lie algebras. The existence problem is

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reduced to find certain subalgebras on the complexification of these Lie algebras and then after introducing an equivalence relation, to get representatives of each class.

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2. Preliminares

Let G be an even-dimensional connected real Lie group and $\mathfrak{g}$ its Lie algebra of left invariant vector fields on G which is identified with the tangent space of G at the unit element e. A complex structure on the underlying manifold consists of an atlas of holomorphic coordinate systems. A complex structure on G is said to be left invariant when the left multiplication by each element in G is holomorphic. Because of the Newlander-Niremberg theorem and the left invariant condition we have the following equivalent definition:

Definition 1. *A left invariant complex structure on G is an endomorphism $J \in End(\mathfrak{g})$ such that:*

(1) $J^2 = -I$

(2) $0 = N_J(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \quad \forall\, X,Y \in \mathfrak{g}$

Condition (2) is called the **integrability condition**.

Let $\mathfrak{g}^{\mathbb{C}} = \{X + iY, X, Y \in \mathfrak{g}\}$ be the complexification of $\mathfrak{g}$ and σ be the conjugation in $\mathfrak{g}^{\mathbb{C}}$, that is $\sigma : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{g}^{\mathbb{C}}$, $\sigma(X + iY) = X - iY$. There exists a natural extension of J to $\mathfrak{g}^{\mathbb{C}}$ denoted also by J. Now in $\mathfrak{g}^{\mathbb{C}}$ we have the subspaces corresponding to the eigenvalues i and $-i$. In terms of these subspaces there is an equivalent definition of a left invariant complex structure.

Proposition 2. *A real Lie group G has a left invariant complex structure if* *and only if* $\mathfrak{g}^{\mathbb{C}}$ *admits a decomposition:*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{m} \oplus \sigma, \tag{3}$$

where $\mathfrak{m}$ *is a subalgebra of* $\mathfrak{g}^{\mathbb{C}}$.

Proof. Condition (1) in definition 1 is equivalent to have a decomposition of $\mathfrak{g}^{\mathbb{C}}$ into a direct sum of subspaces (one is the conjugated of the other). Condition (2) is equivalent to the fact that these subspaces are subalgebras.

Thus, there exists a one to one correspondence between left invariant complex structures J and subalgebras that satisfy (3). $\square$

A subalgebra $\mathfrak{m}$ as in (3) is said an (invariant) **complex subalgebra**.

Definition 3. *Two left invariant complex structures* J_1 *and* J_2 *are equivalent if there exists an automorphism* x *of* $\mathfrak{g}$ *such that* $xJ_1 = J_2x$.

In the terminology of complex subalgebras this equivalence relation is stated in the next proposition.

Proposition 4. *Let* $\mathfrak{m}_i$ *be complex subalgebras corresponding to complex structures* J_i, $i = 1, 2$. *Then* J_1 *is equivalent to* J_2 *if and only if there exists* $x \in Aut(\mathfrak{g}^{\mathbb{C}})$ *such that* $x\,\sigma = \sigma x$ *and* $x\mathfrak{m}_1 = \mathfrak{m}_2$.

Troughout this work all complex structures and complex subalgebras considered will be left invariant.

Non-compact rank-one symmetric spaces admits a solvable group of isometries acting simply and transitively on them [H]. So these groups endowed with a left invariant metric became isometric to the symmetric spaces in which they act. In this way we can see $\mathbb{C}H^2$ and $\mathbb{R}H^4$, the only two non-compact rank-one four dimensional symmetric spaces, as a real Lie algebra with a left invariant metric.

Thus, $\mathbb{R}H^4$ corresponds to $\mathfrak{g} = \mathbb{R}A \;\; \rtimes \;\; \mathfrak{n}$, where $\mathfrak{n} =< B, C, D, >$ is an abelian ideal and $adA_{|\mathfrak{n}} = Id$ and $\mathbb{C}H^2$ corresponds to $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{n}$, $\mathfrak{n} =< B, C, D >$ the three dimensional real Heisenberg algebra and the bracket relations:

$$[A, B] = B, \quad [A, C] = \frac{1}{2}C, \quad [A, D] = \frac{1}{2}D \tag{4}$$

In both cases, the metric that makes the basis $\{A, B, C, D\}$ an orthonormal basis is the symmetric metric and realize the isometry between $(G, <, >)$ and the corresponding symmetric spaces.

Lemma 5. *Let $\mathfrak{g}$ be the real Lie algebra, $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{n}$, then*

a) When $\mathfrak{n}$ is abelian, the automorphisms of $\mathfrak{g}$ in the basis $\{A, B, C, D\}$ as above have a matrix $A = (a_{i,j})$ with $0 = a_{12} = a_{13} = a_{14}$.

b) When $\mathfrak{n}$ is the Heisenberg ideal, the automorphisms of $\mathfrak{g}$ in the basis $\{A, B, C, D\}$ as above have the following matricial representation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & e & f & g \\ b & 0 & a_{11} & a_{12} \\ c & 0 & a_{21} & a_{22} \end{pmatrix}$$

where $e = a_{11}a_{22} - a_{12}a_{21}$, $f = 2(ca_{11} - ba_{21})$, $g = 2(ca_{12} - ba_{22})$.

THE CLASSIFICATION

The technique used to determine the existence of a complex subalgebra is basically the same in both cases.

First we take the basis $\{A, B, C, D\}$ as a basis of $\mathfrak{g}^{\mathbb{C}}$. We suppose that a subalgebra $\mathfrak{m}$ as in (3), exists. So, we may take linearly independant elements X, Y of $\mathfrak{m}$ as follows:

$$X = A + b_1B + c_1C + d_1D \qquad\qquad Y = b_2B + c_2C + d_2D$$

As we need $[X, Y] \in \mathfrak{m}$, we have

$$[X, Y] = \beta Y, \qquad \text{for some} \quad \beta \in \mathbb{C}. \tag{5}$$

(5) give us a system of equations. The existence of solutions is equivalent to the existence of a sum $\mathfrak{m}+\sigma\mathfrak{m}$, $\mathfrak{m}$ subalgebra. Then we put some extra conditions on the coefficients of X and Y to force the sum to be a direct sum.

Proposition 6. *Let $\mathfrak{g}$ be the real Lie algebra $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{n}$, $\mathfrak{n}$ a three dimensional abelian ideal and $adA_{|\mathfrak{n}} = Id$. Then $\mathfrak{g}$ admits (up to equivalence) only one complex structure, given by:*

$$JA = B \qquad JC = D$$

Moreover every almost complex structure is integrable.

Proof. In this situation from (5) we have:

$$\begin{aligned} b_2 &= \beta b_2 \\ c_2 &= \beta c_2 \\ d_2 &= \beta d_2 \end{aligned}$$

As at least one of the coefficients of Y should be different from zero, we get $\beta = 1$.

Now we will select the coefficients of X and Y to make $\{X, Y, \sigma X, \sigma Y\}$ a basis of $\mathfrak{g}^{\mathbb{C}}$.

If $c_2 = 0$, then $d_2 \neq 0$ and we can choose a basis as $X = A + b_1B + c_1C$, $Y = b_2B + D$ with $Imb_2Imc_1 \neq 0$.

If $c_2 \neq 0$, then we can take the basis as $X = A + b_1B + d_1D$, $b_2B + C + d_2D$ and $Imb_1Imd_2 + Imb_2Imd_1 \neq 0$.

Finally any complex subalgebra is equivalent to $\mathfrak{m} =< A + iB, C + iD >$. In fact, let x be the automorphism of $\mathfrak{g}^{\mathbb{C}}$ given by $x(A) = A + Reb_1B + Rec_1C + Red_1D$, $x(B) = Imb_1B + Imc_1C + Imd_1D$, $x(C) = Reb_2B + Rec_2C + Red_2D$, $x(D) = Imb_2B + Imc_2C + Imd_2D$. Then $x\sigma = \sigma x$ and $x\mathfrak{m} = \mathfrak{q}$ with $\mathfrak{q}$ any of the complex subalgebras above. □

Remarks: 6.1. Let J be the complex structure corresponding to the complex subalgebra of proposition 6. Consider the action of $GL(4,\mathbb{R})$ on the space of almost complex structures $((x, J) \longrightarrow xJx^{-1})$. It is known that the isotropy at any almost complex structure is $GL(2,\mathbb{C})$. In our case, as any almost complex structure is integrable, the space of complex structures is $GL(4,\mathbb{R})/GL(2,\mathbb{C})$. An analogous result holds for $\mathbb{R}H^n$. In fact, I. Dotti-L.Barberis [B-D] showed that any almost complex structure on $\mathbb{R}H^{2n}$ is integrable.

6.2. Using known identifications, the real Lie group G corresponding to this Lie algebra is the semi-direct product $(\mathbb{R}^3 \ltimes \mathbb{R}^+, *)$ where the multiplication is $(X,a)*(Y,b) = (X + aY, ab)$.

We have seen that G admits (up to equivalence) only one complex structure. Multiplication on the left by (X,a) on $\mathbb{R}^3 \ltimes \mathbb{R}^+$ is holomorphic with respect to the complex structure defined by any endomorphism J of $\mathbb{R}^{3+1}$ such that $J^2 = -Id$. Thus, the almost complex structure on G obtained by left translation of J is integrable (it correspond to the pullback of the standard complex structure on $\mathbb{R}^3 \times \mathbb{R}^+$ as an open subset of $\mathbb{R}^4$).

Proposition 7. *Let $\mathfrak{g}$ be the real Lie algebra $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{n}$, $\mathfrak{n}$ a three dimensional Heisenberg ideal and the Lie bracket relations as in* (4) *above. Then $\mathfrak{g}$ admits (up to equivalence) three complex subalgebras given by:*

$$\mathfrak{m}_1 =< A + iB, C + iD >,\ \mathfrak{m}_2 =< A - iB, C + iD >,\ \mathfrak{m}_3 =< A + iD, C + 2iB >$$

Proof. The equation (5) has the form:

$$\begin{aligned} \beta b_2 &= b_2 + c_1 d_2 - d_1 c_2 \\ \beta c_2 &= \frac{1}{2} c_2 \\ \beta d_2 &= \frac{1}{2} d_2 \end{aligned}$$

One of the coefficients c_2, d_2 must be different from zero because if $c_2 = d_2 = 0$, then $Y = b_2 B$ so, $B = \sigma B \in \mathfrak{m} \cap \sigma\mathfrak{m} = 0$, contradiction.

Suppose that c_2 is different from zero. We can take $c_2 = 1, c_1 = 0$ and we get $\beta = 1/2$ and $b_2 = 2d_1$. Thus, we obtain the complex subalgebras $\mathfrak{m}_{b_1,d_1,d_2} =< A + b_1 B + d_1 D,\ 2d_1 B + C + d_2 D >$ with $-2Imd_1^2 + Imb_1 Imd_2 \neq 0$.

If $d_2 \neq 0$ with a similar procedure as above we get $\mathfrak{m}_{b_1,c_1,c_2} =< A + b_1B + c_1C, -2c_1B + c_2C + D >$ with $2Imc_1^2 + Imb_1Imc_2 \neq 0$.

But any $\mathfrak{m}_{b_1,c_1,c_2}$ is equivalent to some $\mathfrak{m}_{b_1,d_1,d_2}$. In fact, take the automorphism of $\mathfrak{g}^{\mathbb{C}}$ given by $x(A) = A$, $x(B) = B, x(C) = -D$, $x(D) = C$. Then $x\sigma = \sigma x$ and $x(A + b_1B + c_1C) = A + b_1B - c_1D$, $x(-2c_1B + c_2C + D) = -2c_1B + C - c_2D$.

The equivalences:

(1) $\mathfrak{m}_1 =< A + iB, C + iD >$ is equivalent to $\mathfrak{m}_{b_1,d_1,d_2}$ if and only if $Imd_2 \neq 0$ and $-2Imd_1^2 + Imb_1Imd_2 > 0$.

(2) $\mathfrak{m}_2 =< A - iB, C + iD >$ is equivalent to $\mathfrak{m}_{b_1,d_1,d_2}$ if and only if $Imd_2 \neq 0$ and $-2Imd_1^2 + Imb_1Imd_2 < 0$.

(3) $\mathfrak{m}_3 =< A + iD, C + 2iB >$ is equivalent to $\mathfrak{m}_{b_1,d_1,d_2}$ if and only if $Imd_2 = 0$ and $Imd_1 \neq 0$.

(i) Suppose that there exists $x \in Aut(\mathfrak{g}^{\mathbb{C}})$ such that $x\sigma = \sigma x$ and $x\mathfrak{m}_1 = \mathfrak{m}_{b_1,d_1,d_2}$. Then there exists α, ϕ, γ, $\delta \in \mathbb{C}$ such that $x(A + iB) = \alpha X' + \phi Y'$, $x(C + iD) = \gamma X' + \delta Y'$ where X', Y' is a basis of $\mathfrak{m}_{b_1,d_1,d_2}$. Because of Lemma 5 we have $\gamma = 0$, $\alpha = 1$, $\delta \neq 0$ and the following system:

$$\begin{array}{rcl} a + ie & = & b_1 + 2d_1\phi \\ b & = & \phi \\ c & = & d_1 + \phi d_2 \\ f + ig & = & 2d_1\delta \\ a_{11} + ia_{12} & = & \delta \\ a_{21} + a_{22} & = & \delta d_2 \end{array} \tag{7}$$

The third equation implies that $Imd_1 + bImd_2 = 0$. If $Imd_2 = 0$ then we must have $Imd_1 = 0$ but this fact is not possible because $Imb_1Imd_2 - 2Imd_1^2 \neq 0$. So, $Imd_2 \neq 0$ and $b = \dfrac{-Imd_1}{Imd_2}$. Replacing δ in the sexth equation by the value given in the fifth, we get $a_{22} = Imd_2a_{11} + Red_2a_{12}$, $a_{21} = Red_2a_{11} - Imd_2a_{12}$. From the first two equations, we have $e = Imb_1 + 2bImd_1$ and by the other hand $e = a_{11}a_{22} - a_{12}a_{21}$, then:

$$a_{11}(Imd_2a_{11}+Red_2a_{12})-a_{12}(Red_2a_{11}-Imd_2a_{12}) = Imb_1+2bImd_1$$

Thus,

$$Imd_2(a_{11}^2+a_{12}^2) = Imb_1+2bImd_1$$

that is $a_{11}^2+a_{12}^2 = (Imd_2Imb_1-2Imd_1^2)/Imd_2^2$, and from this it follows $Imb_1Imd_2-2Imd_1^2>0$.

Assume now that $\mathfrak{m}_{b_1,d_1,d_2}$ is a complex subalgebra with $Imd_2\neq 0$ and $Imb_1Imd_2-2Imd_1^2>0$. Let t be the automorphism of $\mathfrak{g}^{\mathbb{C}}$ as in lemma 5, with $a_{12}=0$, $a_{11}=\sqrt{\frac{Imb_1}{Imd_2}-2(\frac{Imd_1}{Imd_2})^2}$, $a_{21}=Red_2a_{11}$, $a_{22}=Imd_2a_{11}, b=-\dfrac{Imd_1}{Imd_2}$, $a=Reb_1+2bRed_1$, $c=Red_1+bRed_2, f$, g, e defined following the lemma. Then $t\sigma=\sigma t$ and (7) holds.

(ii) Similar to the case (i).

(iii) We start as in (i), but the system of equations we have now, has the following form:

$$\begin{array}{rcl} a+ig &=& b_1+2d_1\phi\\ b+ia_{12} &=& \phi\\ c+ia_{22} &=& d_1+\phi d_2\\ f+i2e &=& 2d_1\phi\\ a_{11} &=& \delta\\ a_{21} &=& \delta d_2 \end{array} \qquad (7')$$

The last two equations of (7′) imply $Imd_2=0$. On the other hand, let $\mathfrak{m}_{b_1,d_1,d_2}$ be a complex subalgebra with $Imd_2=0$. Choose $t\in Aut(\mathfrak{g}^{\mathbb{C}})$ as in lemma 5 with $a_{12}=0$, $a_{11}=1$, $a_{21}=d_2$, $a_{22}=Imd_1, b=-\dfrac{Imb_1}{4Imd_1}$, $a=(Reb_1+Imb_1Red_1)/(2Imd_1)$. Thus, $t\sigma=\sigma t$ and the system (7′) holds. □

Remarks: 7.1 The complex subalgebra $\mathfrak{m}_1$ corresponds to the canonical complex structure on $\mathfrak{g}$ such that $(G,<,>,J)$, where $<,>$ is the symmetric metric, is the Hermitian symmetric space $\mathbb{C}H^2$, which is also a Kähler manifold.

7.2 Let J_1,J_2,J_3 be the complex structures corresponding to the complex subalgebras $\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{m}_3$, respectively. Consider the action of $Aut(\mathfrak{g})$ in the space of complex structures. The isotropy group of $Aut(\mathfrak{g})$ at either J_1 or J_2 is represented by matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & -a_{34} & a_{33} \end{pmatrix} \qquad a_{33}^2 + a_{34}^2 = 1$$

and at J_3 by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & -2a_{31} \\ a_{31} & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad a_{22} \neq 0$$

Thus, the orbit space $\{xJ_ix^{-1}\}$ though J_i, $i = 1, 2$ has dimension six and dimension five through J_3.

ABOUT THE HERMITIAN GEOMETRY

Now we will discuss about metrics compatibles with complex structures.

Recall that a hermitian inner product on a real vector space V with an almost complex structure J is an inner product h such that $h(JX, JY) = h(X, Y)$ $\forall X, Y \in V$.

Denote also by h the extension of h to a complex symmetric bilinear form of $V^{\mathbb{C}}$. Then h satisfy:

(1) $h(\overline{Z}, \overline{W}) = \overline{h(Z, W)}$ for $Z, W \in V^{\mathbb{C}}$;

(2) $h(Z, \overline{Z}) > 0$ for all non zero $Z \in V^{\mathbb{C}}$;

(3) $h(Z, \overline{W}) = 0$ for $Z \in V_i^{\mathbb{C}}$ and $W \in V_{-i}^{\mathbb{C}}$.

where $V_k^{\mathbb{C}}$ is the eigenspace corresponding to the eigenvalue $k \in \mathbb{C}$.

A metric which is invariant by left multiplication by an element x of the group G is determined by its value at the tangent space T_eG, e the unit element on G. In fact, it is defined by $< X, Y >_x = < dL_x^{-1}X, dL_x^{-1}Y >_e$, X, Y left invariant fields on G.

Let us consider $<,>$ a left invariant metric on G, corresponding to the Lie algebra $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{g}'$, $\mathfrak{g}'$ an abelian ideal and adA as in preposition 6. Choosing an orientation, the space of complex structures wich are orthogonal with respect to the metric and preserve orientation is represented by $SO(4,\mathbb{R})/U(2)$. The proof of this fact is similar to that given in Remark 6.1.

If the conmutator of $\mathfrak{g}$ is a Heisenberg ideal, the question is not so easy to determine. We will consider two examples to demonstrate that the space of orthogonal complex structures depends on the metric on $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{n}$, adA the derivation as in proposition 7.

Take h_s the symmetric metric on $\mathfrak{g}$, with the orthonormal basis $\{A, B, C, D\}$. Denote also by h_s the extension to $\mathfrak{g}^{\mathbb{C}}$. So, to have an hermitian complex structure corresponding to a complex subalgebra $\mathfrak{m}_{b_1,d_1,d_2}$ we need:

$$1)\, 1 + b_1^2 + d_1^2 = 0$$
$$2)\, 1 + (2d_1)^2 + d_2^2 = 0$$
$$3)\, 2d_1 b_1 + d_1 d_2 = 0$$

From the equation 3) we need $a)\, d_1 = 0$ or $b)\, 2b_1 + d_2 = 0$. If $b)$ holds, replacing b_1^2 by $d_2^2/2$ we get a contradiction with 2). Then $a)$ holds and this fact implies $b_1^2 = -1 = d_2^2$. Thus , the complex structures orthogonal with respect to the symmetric metric are $J_1, -J_1, J_2, -J_2$ given by:

$$J_1 A = B \qquad J_1 C = D \qquad J_1^2 = -Id$$
$$J_2 A = B \qquad J_2 C = -D \qquad J_2^2 = -Id$$

It is not difficult to see that we obtain the same results from the equations for $\mathfrak{m}_{b_1,c_1,c_2}$.

Consider now h_a the left invariant metric induced by the inner product with orthonormal basis $\{A, B, \frac{\sqrt{2}}{2}C, \frac{\sqrt{2}}{2}D\}$. This metric is hypermitian with respect to the hypercomplex structure (see [B]):

$$J_1A = B \qquad J_1D = C \qquad J_1^2 = -Id$$
$$J_2A = \frac{\sqrt{2}}{2}C \qquad J_2B = \frac{\sqrt{2}}{2} \qquad J_2^2 = -Id$$

The metric h_a is not symmetric and has negative sectional curvature.

Let $\{X, Y\}$ the basis of $\mathfrak{m}_{b_1,d_1,d_2}$ as in preposition 7. In this case to have an Hermitian complex structure we need:

$$1)\, 1 + b_1^2 + 2d_1^2 = 0$$
$$2)\, 1 + (2d_1)^2 + 2d_2^2 = 0$$
$$3)\, 2d_1b_1 + 2d_1d_2 = 0$$

From the equation 3) we have a) $d_1 = 0$ or b) $b_1 + d_2 = 0$. If a) holds, we get the complex structures as above.

Suppose b) holds. Then $2d_1^2 = -1-b_1^2$ and the complex subalgebras $\mathfrak{m}_{b_1,d_1,-b_1}$ with $2d_1^2 = -1 - b_1^2$ and $Imb_1 \neq 0$, corresponds to the complex structures compatibles with h_a.

Now we will search almost Kähler structures on $\mathfrak{g}$.

Let $(G, <,>, J)$ be an (almost) Hermitian Lie group, where both $<,>$ and the (almost) complex structure J are left invariant. Let ϕ be the *fundamental 2-form* of G, that is:

$$\phi(X, Y) = < X, JY > \qquad \text{for all} \quad X, Y \in \mathfrak{g}$$

The (almost) Hermitian Lie group G is called *(almost) Kählerian* if $d\phi \equiv 0$, explicitely:

$$< [X, Y], JZ > + < [Y, Z], JX > + < [Z, X], JY >= 0 \qquad \forall\, X, Y, Z \in \mathfrak{g} \quad (8)$$

When the conmutator is abelian, (8) implies $J(\mathfrak{g}') \subset \mathbb{R}A$, which is impossible because J is an isomorphism. Thus, there is no Kählerian metrics in $\mathbb{R}H^4$.

When the conmutator is the Heisenberg ideal generated by $\{B, C, D\}$ form (8) we have:

$$< B, JC >= 0 =< B, JD > \qquad < JA, B >=< JC, D >\neq 0 \qquad (9)$$

In particular, if $\{A, B, C, D\}$ is orthogonal with respect to $<,>$, there are only two almost complex structures with fundamental two form closed, wich are:

$$\begin{pmatrix} 0 & \mp a & 0 & 0 \\ \pm a & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp b \\ 0 & 0 & \pm b & 0 \end{pmatrix} \qquad a = |A|/|B|, \quad b = |C|/|D|$$

Moreover J is integrable (it follows from simple calculations).

The triple $(\mathfrak{g}, <,>, J)$ is a Kählerian subalgebra which corresponds to $\mathbb{C}H^2$. This fact result from the following theorem due to Heintze [H]:

Theorem. *Let $(\mathfrak{g}, <,>)$ be a solvable Lie algebra with inner product and a orthogonal complex structure such that the associated 2-fundamantal form is closed. Assume that:*

A) $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$;

B) There exists a unit vector $A_o \in \mathfrak{g}$ orthogonal to $\mathfrak{g}'$ such that $D_o : \mathfrak{g}' \longrightarrow \mathfrak{g}'$ is positive definite, where D_o is the symmetric part of $adA_{o|_{\mathfrak{g}'}} : \mathfrak{g}' \longrightarrow \mathfrak{g}'$.

Then the pair $(\mathfrak{g}, <,>)$ corresponds to the complex hyperbolic space.

Take now $\mathfrak{g} = \mathbb{R}A \bowtie \mathfrak{n}$ $\mathfrak{n}$ the Heisenberg ideal and adA as in proposition 7. Let J be a complex structure with fundamental two form closed and $<,>$ be an inner product on $\mathfrak{g}$. We would like to exhibit conditions to apply the theorem in this situation. From (9) we have that JB is orthogonal to $\mathfrak{g}'$. So, $A_o = \dfrac{JB}{|JB|}$ is a unit vector and we need that the symmetric part of adA_o be positive definite and then we apply the theorem above.

Write $JB = aA + bB + cC + dD$, from the condition that J is orthogonal we get $a = \dfrac{|B|^2}{< JB, A >}$. The matrix on the basis $\{B, C, D\}$ of $adJB$ is:

$$adJB = \begin{pmatrix} a & -d & c \\ 0 & a/2 & 0 \\ 0 & 0 & a/2 \end{pmatrix}$$

So if $< A, JB >> 0$ and $8a^2 > c^2 + d^2$ then the condition $B)$ of theorem holds and $(\mathfrak{g}, J, <,>)$ corresponds to $\mathbb{C}H^2$. (Note that if $< A, JB >< 0$, we can take $J(-B)$). $\square$

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