GENERALIZED HELICES, TWISTINGS AND FLATTENINGS OF CURVES IN n-SPACE

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Abstract

We define the concept of twisting of a n-space curve as flattening point of its tangent indicatrix and show that this is equivalent to having higher order of contact with some generalized helix. We prove that generic closed curves in $\mathbb{R}^3$ have at least two twistings, or at least four under some appropiate geometric conditions. We also provide lower bounds for the number of twistings on some classes of closed curves embedded in odd dimensional spaces.

1. Introduction

By a flattening of a n-space curve is meant a point at which the osculating hyperplane has contact of order at least $n+1$ with the given curve. In the case of 3-space, these are the well known torsion-zero points. We can say that at these points the curve is closer to be a plane curve. The minimum number of such points on a closed space curve has been a classical object of study. Several examples of curves with nowhere vanishing torsion may be found among the (q,p) curves on the standard torus whose equations in polar coordinates are $r = a + cos(n\theta)$, $z = sin(n\theta)$, where $n = p/q$ and a is the proportion between the radii of the torus. Such a curve winds q times in the horizontal and p times in the vertical sense around the torus. It was shown in [3] that the (q,p) curve has nonvanishing torsion if and only if we have $n^2 > 1$ and $(2n^2+1)/(n^2-1) < a < n^2+1$. Moreover, curves with never vanishing torsion and curvature must enter twice inside its convex envelope, as proven in [4].

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On the other hand, curves with never vanishing curvature lying on the boundary of their convex envelopes always have at least four torsion zero points (see [11] for a proof in the generic case, or [13] in the general case). Analogous results for closed curves in higher dimensional spaces under some convexity conditions (which appear to be more restrictive than the above one in the three-dimensional case) are due to M. Barner [2] and V. I. Arnol'd [1].

We introduce in this paper the concept of twisting of a n-space curve as flattenings of its tangent indicatrix. We show that this points correspond to points where the curve has higher contact with some (generalized) helix, so the curve appears to "twist in a somehow regular manner" at them. We get the following results for $n = 3$:

1) *Any closed curve in* $\mathbb{R}^3$ *has at least two twisting points.*

2) *Any closed curve in* $\mathbb{R}^3$ *with nonvanishing curvature and no parallel tangents with the same orientation has at least four twistings.*

3) *Any closed curve in* $\mathbb{R}^3$ *with nonvanishing torsion and no parallel osculating planes with the same orientation has at least four twistings.*

In the attempt to extend these results to curves in higher dimensional spaces we give a definition generalizing the concept of helix in 3-space (section 1). We then see that at each point of a given curve in $\mathbb{R}^n$ there are some generalized helices having contact of order at least n with the curve and that the point is a twisting when this contact can be taken to be of order $n + 1$.

We conjecture that:

Generic closed curves in odd dimensional spaces must have at least two twistings.

Finally, we discuss the possibility of applying the existing results on flattenings of curves in higher dimensional spaces to obtain higher lower bounds for the number of twistings of closed curves in n-space.

2. Generalized helices in $\mathbb{R}^n$

We define a **generalized helix** as a curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that its tangent vector forms a constant angle with a given direction v at $\mathbb{R}^n$.

It is not difficult to see that this is equivalent to asking that the tangent indicatrix of α, $\alpha_T : \mathbb{R} \longrightarrow S^{n-1} \hookrightarrow \mathbb{R}^n$ is contained in a $(n-2)$-sphere in S^{n-1}. In particular, we have that this $(n-2)$-sphere is of maximum radius (or an "equator") if and only if α is a $(n-1)$-flat curve, in the sense that it lies in a hyperplane of $\mathbb{R}^n$ (orthogonal to the direction v). So $(n-1)$-flat curves can be regarded as a particular case of generalized helix in $\mathbb{R}^n$.

Proposition. *A curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a generalized helix if and only if the function $det(\alpha''(t), \alpha'''(t), ..., \alpha^{(n+1)}(t))$ is identically zero, where $\alpha^{(i)}$ represents the ith derivative of α with respect to its arc-length.*

Proof: It follows from the above definition that α is a generalized helix if and only if α_T is a $(n-1)$-flat curve. Consider the Frenet frame of α_T as a curve in $\mathbb{R}^n$ and the corresponding curvature functions $k_1^T(t), ..., k_{n-1}^T(t)$ (see [5]). We have that

$$\alpha_T \text{ is } (n-1)\text{-flat} \Leftrightarrow k_{n-1}^T(t) = 0 \ \forall t.$$

And now an easy exercise in vector calculus shows that

$$k^T{}_{n-1}(t) = 0 \Leftrightarrow det({\alpha_T}'(t), {\alpha_T}''(t), ..., {\alpha_T}^{(n)}(t)) = 0.$$

The result then follows from observing that ${\alpha_T}'(t) = \alpha''(t)$.

$\square$

Remark: In the particular case of $n = 3$, we have that the above conditions are equivalent to the familiar definition of helix : *a curve for which the rational function $\frac{\tau}{k}$ is constant*, where k and τ denote respectively the curvature and torsion of the considered curve (see [15]).

For instance, curves with constant curvature and torsion form a particular class of helices. It is a straightforward (but tedious!) exercise to see that for n odd the curves having all their curvature functions constant satisfy our definition. It can also be seen that this is not the case for even dimensional spaces.

3. Special contacts of a curve: flattenings and twistings

Suppose that X_1 and X_2 are submanifolds with a common point P in $\mathbb{R}^n$. It is classically known that X_2 has a **contact of order k** with X_1 at P if given some point Q in X_1 we have that

$$\lim_{Q\to P} \frac{QB}{PQ^r} = \begin{cases} 0 \text{ if } r = 1, 2, ..., k \\ \text{finite and } \neq 0 \text{ for } r = k+1. \end{cases}$$

where QB represents the distance of Q to the manifold X_2 and PQ the distance between the points P and Q in X_1.

Given two pairs of submanifolds (X_1, X_2) and (X_1', X_2') in $\mathbb{R}^n$ **the contact of X_1 with X_2 at a common point P is said to be the same than the contact of X_1' with X_2' at P'** if and only if there exist a diffeomorphism $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ taking X_1, X_2 and P respectively to X_1', X_2' and P'. Clearly this is an equivalence relation between pairs of submanifolds of $\mathbb{R}^n$.

For a pair of submanifolds X_1 and X_2 with a common point P, it is always possible to find local coordinates for $\mathbb{R}^n$ in a neighbourhood of P such that X_1 is locally given as the image of some embedding $g : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, P)$ and $X_2 = f^{-1}(0)$, for some submersion $f : (\mathbb{R}^n, P) \longrightarrow (\mathbb{R}^p, 0)$. The composite map $f \circ g : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^p, 0)$ is known as the **contact map** for X_1 and X_2 at P.

J. Montaldi [9] proved that the singularity type ($\mathcal{K}$-class) of the map $f \circ g$ at 0 completely characterizes the contact of X_1 and X_2 at P and that this actually independs of the choice of the maps g and f.

If the first submanifold is a curve, given by $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$, and X is cut out by $f : (\mathbb{R}^n, P) \longrightarrow (\mathbb{R}^p, 0)$ at P (i. e., $X = f^{-1}(0)$ in a neighbourhood of P) then it is not difficult to show that both of them have contact of order k at P

if and only if

$$dh(0) = ... = d^k h(0) = 0 \text{ and } d^{k+1} h(0) \neq 0,$$

where $h = f \circ \alpha : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^p, 0)$ is the contact map for α and X at P.

From the classification of singularities of maps from to $\mathbb{R}$ to $\mathbb{R}^p$ it follows that whenever X_1 and X_2 have contact of order k at P it is possible to find coordinates for $\mathbb{R}$ and $\mathbb{R}^p$ such that the corresponding contact map is written as $h(t) = (t^{k+1}, 0, ..., 0)$, see [10] for instance. In particular, if X_2 is a hypersurface we shall have that the contact map $h : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ can be put as $h(t) = t^{k+1}$ by means of such changes of variables.

Suppose now that $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a curve and H a hyperplane defined by the equation $< v, x > +\rho = 0$, with $v = (v_1, ..., v_n) \in S^{n-1}$ as orthogonal direction and ρ a positive real number, i.e., the distance of H to the origin. So H is cut out by the function

$$f(x_1, ..., x_n) = v_1.x_1 + ... + v_n.x_n + \rho.$$

If H is tangent to α at a point t_0, we have that the vector v must be normal to the curve α at t_0 and that $\rho = \alpha(t_0).v$. Moreover, it is straightforward to verify that H is the osculating hyperplane of α at t_0 if and only if α has contact of order at least $n-1$ with H at $\alpha(t_0)$.

We shall say that t_0 is a **flattening** of α if the contact of α with the osculating hyperplane at t_0 is of order at least n.

Lemma. *A point t_0 is a flattening of α if and only if*

$$det(\alpha'(t_0), \alpha''(t_0), ..., \alpha^{(n)}(t_0)) = 0,$$

where $\alpha^{(i)}$ represents the ith derivative of α with respect to its arc-length.

Proof: Write $f(t) = \alpha(t).b$, where b is the binormal vector of α at t_0 (that is, $b = N_{n-1}(t_0)$ in the standard notation of the Frenet frame [5] of α).

We have that t_0 is a flattening of α if and only if

$$f'(t_0) = f''(t_0) = ... = f^{(n)}(t_0) = 0.$$

That is, $\alpha'(t_0).b = ... = \alpha^{(n)}(t_0).b = 0$, which is equivalent to asking that all the vectors $\alpha'(t_0), ..., \alpha^{(n)}(t_0)$ belong to the osculating hyperplane, and hence $det(\alpha'(t_0), ..., \alpha^{(n)}(t_0)) = 0$.

□

Remark: As in the case of curves in 3-space, it can be seen that if α is a curve for which the first $n-2$ derivatives are linearly independent at each point, t_0 is a flattening of α if and only if $k_{n-1}(t_0) = 0$, where k_{n-1} is the $(n-1)$th curvature function of α.

All the curves that we consider in what follows will have this property. It is possible to show that all these curves form an open and dense subset of the set of embeddings of $\mathbb{R}$ in $\mathbb{R}^n$ with the Whitney C^∞-topology. In this sense, we say that they are generic.

A hypersphere of $\mathbb{R}^n$ whose contact with α at t_0 is of order at least n is called **osculating hypersphere** of α at t_0.

A **conformal flattening** or **vertex** of α is a point at which α has contact of order at least $n+1$ with its osculating hypersphere.

Let $\{T(t), N_1(t), ..., N_{n-1}(t)\}$ be the Frenet frame of the curve α. The $(i+1)$-subspace generated by $\{T(t), N_1(t), ..., N_i(t)\}$ shall be called **osculating** $(i+1)$**-subspace** of α at t and its intersection with the osculating hypersphere, **osculating** i**-sphere** of α at t. It is not difficult to see that the osculating $(i+1)$-subspace of α at t is also generated by the vectors $\{\alpha'(t), ..., \alpha^{(i+1)}(t)\}$.

It can also be shown that the curve α has contact of orden at least k with its osculating $(k-1)$-spheres at each point.

In the particular case that X_1 and X_2 are a couple of curves, having contact of orden k at the point t_0, the definition of contact is equivalent (see [8]) to the existence of parametrizations $\alpha : I \longrightarrow \mathbb{R}^n$ and $\beta : J \longrightarrow \mathbb{R}^n$ such that

$\alpha(t_0) = \beta(t_0) = P$, for some $t_0 \in I \cap J$, and

$$\begin{array}{rcl} \alpha^{(i)}(t_0) & = & \beta^{(i)}(t_0), \quad i = 1, \ldots, k, \\ \alpha^{(r+1)}(t_0) & \neq & \beta^{(k+1)}(t_0). \end{array}$$

It is easy to deduce that if α and β have contact of order k at t_0 then their osculating i-spaces at t_0 coincide for $i = 1, ..., k$, as well as the value of their curvature functions k_i, $i = 1, ..., k-1$ at the point t_0. The reciprocal is not necessarily true as we can guess from a quick glance to the expression of the derivatives $\alpha^{(i)}$ and $\beta^{(i)}$ in terms of the Frenet frames and the curvature functions and their derivatives.

It follows from the definition of order of contact that given a curve α and a submanifold X in $\mathbb{R}^n$, if there exists some curve β in X whose contact with α is of order k at P, then the contact of α with X at P is of order at least k. Moreover, we have the following:

Lemma. *If α has contact of order k with some m-dimensional submanifold X at P then exists some curve β passing through P in X such that α and β have contact of orden k.*

Proof: If α has contact of order k with X at P, we can locally write the contact map as $h = f \circ \alpha : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^{n-m}, 0)$ for some $f : (\mathbb{R}^n, P) \longrightarrow (\mathbb{R}^{n-m}, 0)$. Then as we have previously said, convenient changes of variables allow us to put $h(t) = (t^{k+1}, 0, ..., 0)$ in a neighbourhood of 0. But then it us possible to restrict ourselves to some curve β lying on X (determined up to $(k+1)$-jet) in such a way that the new contact function (of α with β) is given by

$$\begin{array}{rcl} \hat{h} : (\mathbb{R}, 0) & \longrightarrow & (\mathbb{R}^{n-1}, 0) \\ t & \longmapsto & (t^{k+1}, 0, ..., 0) \end{array}$$

and hence α and β must have contact of order k.

□

We define a **twisting** of $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ as a flattenings of its tangent indicatrix $\alpha_T : \mathbb{R} \longrightarrow S^{n-1}$. It follows that if α is parametrized by its arc-length t, then t_0 is a twisting of α if and only if $det(\alpha''(t_0), ..., \alpha^{(n+1)}(t_0)) = 0$.

We shall see now that the twistings of α can also be characterized as points at which it has higher order of contact with some generalized helix.

Proposition. *Given a curve $\alpha(t)$ parametrized by arc-length in $\mathbb{R}^n$, there exists for each point $\alpha(t_0) = P$ of this curve some generalized helix $\gamma_p(t)$ whose contact with α at P is of order at least n. Moreover, if t_0 is a flattening point of α_T then we have that γ_P has order of contact at least $n+1$ with α at P.*

Proof: Given $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ parametrized by arc-length, let $\alpha_T(t) = \alpha'(t)$ denote its tangent indicatrix.

Consider now, the osculating hyperplane H of α_T at t_0. The intersection of H with the unit sphere S^{n-1} determines a $(n-2)$-sphere S which is the osculating $(n-2)$-sphere of α_T at t_0.

Since α_T has contact of order at least $n-1$ with S at P, the above lemma tells us that there must be some curve β contained in S whose contact with α_T is of order at least $n-1$. Then β is the tangent indicatrix of some helix $\gamma_P : \mathbb{R} \longrightarrow \mathbb{R}^n$, and clearly α and γ_P have contact of order at least n at t_0. If t_0 is a flattening of α_T, then this curve has contact of order at least n with S (see [16]) and therefore α has contact at least $n+1$ with γ_P at t_0.

$\square$

Remark: The above construction tells us that the helix γ_P is not unique. In fact, the curve β itself is only determined up to its $(n-1)$-th order derivatives, moreover given β there is a whole family of curves in $\mathbb{R}^n$ having β as its tangent indicatrix.

We also observe that the existence of some helix γ with order of contact at least $n+1$ with α at t_0 implies that α_T and γ_T have contact of order at least n. But since γ_T lies in some $(n-2)$-sphere S this means that this must be the osculating $(n-2)$-sphere of α_T. Consequently α_T has contact of order at least n with its osculating hyperplane and the point is a flattening of α_T. Therefore, we can assert

Corollary. *A point t_0 is a twisting of a curve α if and only if there exists some generalized helix whose order of contact with α at t_0 is at least $n+1$.*

4. Global results for closed curves in $\mathbb{R}^3$

We observe first that torsion-zero points of both the tangent indicatrix and the binormal indicatrix of a 3-space curve α are characterized by the property

$$\tau'(t_0)k(t_0) - \tau(t_0)k'(t_0) = 0,$$

where k and τ respectively denote the curvature and torsion function of α (see [15, pgs. 71-72]). Therefore, we can say: *Given $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$, any critical point of the function $\frac{\tau}{k}$ is a twisting of α.*

From this we get immediately

Theorem. *Any closed curve with non vanishing curvature in $\mathbb{R}^3$ has at least two twisting.*

Since non vanishing of the curvature is a generic condition in the sense that it is satisfied by an open and dense set of curves in the Whitney C^∞-topology, we get that any generic closed curve has at least two twisting.

Consider now the tangent indicatrix α_T of α. This is a regularly embedded curve in S^2 provided that the curvature of α does not vanish and that α does not have pairs of parallel tangents with the same orientation. If we observe now that stereographic projection from $S^2 - \{p\} \longrightarrow \mathbb{R}^2$ sends torsion-zero points of spherical curves into vertices of its plane images, we shall obtain as, an immediate consequence of the 4-vertex theorem for plane curves, the following result

Theorem. *Any closed curve with non vanishing curvature and no pair of parallel tangents with the same orientation in $\mathbb{R}^3$ has at least four twisting.*

Remark: With the above assumptions B. Segre proved [14] that the curve must also have at least four flattenings. We observe that for a generic curve the sets of flattenings and twistings must be disjoint. For otherwise, it would have points satisfying $\tau(t) = \tau'(t) = 0$ which is non generic, in the sense that this is non stable under small perturbations of the curve in the Whitney C^∞-topology.

By applying now similar arguments to the binormal indicatrix of a closed curve, we obtain

Theorem. *Any closed curve with non vanishing torsion and no pair of osculating planes with the same orientation in* $\mathbb{R}^3$ *has at least four twisting.*

A particularly interesting class of closed space curves with never vanishing torsion is given by the **elasticae** [6]. These are defined as critical points of the energy functional $F(\gamma) = \int_\gamma k^2(t)dt$ among all the curves of the same length and first order boundary data.

A necessary condition, arising from the Euler equation, for a curve to be an elastica is that its curvature k and torsion τ satisfy $k^2\tau =$ constant (see [7]).

Therefore $2kk'\tau + k^2\tau' = 0$ and thus we get the following

Corollary. *In an elastic curve with never vanishing curvature in* $\mathbb{R}^3$, *the twistings are also critical points of the torsion. Moreover, if such curve has no parallel binormals with the same orientation them it has at least four of these points.*

Remark: We observe that although generically twistings and singular points of the torsion do not coincide, this situation is "stable" in the set of elastic curves in the sense that it will not be destroyed by a perturbation of the curve inside this subset.

5. Global viewpoint for closed curves in higher dimensional spaces

First of all, we observe that given any curve γ in $\mathbb{R}^{n-1}$ the inverse of the stereographic projection $\xi : \mathbb{R}^{n-1} \longrightarrow S^{n-1}$, transforms the vertices of γ into the flattenings of its spherical image $\alpha = \xi \circ \gamma$ considered as a curve in n-dimensional space [12].

Now if we start with a closed curve α in $\mathbb{R}^n$ and consider its tangent indicatrix $\alpha_T \subset S^{n-1}$, the number of twisting of α will be equal to that of vertices of $\gamma = \xi^{-1} \circ \alpha_T$. Consequently, we have that any lower bound for the number of vertices of any class of regular closed embedded curves in $\mathbb{R}^{n-1}$ will be a lower bound for the number of twistings in the corresponding class of closed curves with non vanishing first curvature and no pairs of parallel tangents in $\mathbb{R}^n$.

An interesting question now consist in investigating whether the fact that closed curves in $\mathbb{R}^3$ have at least two twistings generalizes to $\mathbb{R}^n$.

The answer is that this is not the case for even dimensional spaces, as the following counterexample, passed to us by R. Uribe, shows:

The closed spherical curve $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^4$, given by

$$\gamma(t) = \frac{1}{\sqrt{5}}(cost, -sint, cos2t, sin2t)$$

has as tangent indicatrix the spherical curve

$$\gamma'(t) = \frac{1}{\sqrt{5}}(-sint, cost, -2sin2t, 2cos2t).$$

An easy verification tells us that the curve γ' has no flattenings in $\mathbb{R}^4$ and thus γ has no twistings. In fact, R. Uribe proves [17] that it is always possible to find closed curve with no twistings in even dimensional spaces.

Now, based in the case of $\mathbb{R}^3$ and in the fact that the parity of n is fundamental for the questions of existence and no existence of flattening and vertices, we formulate here the following conjecture:

Conjecture: *Generic closed curves in odd dimensional spaces have at least two twistings.*

Finally, we see that appropriate convexity conditions can rise the lower bound of the twistings of closed curves in odd-dimensional spaces.

A closed curve α in $\mathbb{R}^n$ is said to be **convex** if any hyperplane of $\mathbb{R}^n$ meets α in at most n points with the multiplicities counted.

Theorem (V. I. Arnol'd [1]). *A curve in $\mathbb{R}^{2k+1}$ whose image through some projection $\mathbb{R}^{2k+1} \longrightarrow \mathbb{R}^{2k}$ is convex has at least $2k+2$ flattening points.*

Based on Arnol'd's result R. Uribe has proven the following

Theorem. ([16]) *Any closed convex curve in $\mathbb{R}^{2k}$ has at least $2k+2$ vertices.*

And then, from the above arguments we deduce

Theorem. *Any closed curve in $\mathbb{R}^{2k+1}$ with non vanishing first curvature and no pairs of parallel tangents, whose tangent indicatrix does not meet any hypercircle of S^{2k} in more than $2k$ points (counting their multiplicities) has at least $2k+2$ twisting.*

We end up by remarking that it would be interesting to determine sufficient conditions on α implying convexity of its tangent indicatrix in the above sense.

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