hypersurface in the sphere $S^4(c)$ with constant principal curvature 0 and $\pm\sqrt{3c}$, then M^3 is *proper pseudo-symmetric* (not semi-symmetric) with $\phi(p) = c$, for every p in M (see [DVY], Examples 1,2). It follows from Proposition 2.2 that the Cartan hypersurface $f : M^3 \to S^4(c)$ is not pseudo-parallel. Moreover, if $c = 0$, it is well-known that the cone over a Clifford torus of dimension two, $g : \mathbb{R}_+ \times S^1(1) \times S^1(1) \to \mathbb{R}^4$, is a minimal hypersurface which has three different principal curvatures $0, \pm\lambda \neq 0$. It follows that g is a semi-symmetric hypersurface, but by the Proposition 2.2 cannot be a pseudo-parallel hypersurface. This shows that pseudo-symmetry and semi-symmetry do not imply pseudo-parallelism, in general. However:

Proposition 2.4. *If $n \geq 3$ and $f : M^n \to \mathbb{R}^{n+1}$ is an isometric immersion, then f is a proper pseudo-parallel immersion if and only if M^n is a proper pseudo-symmetric manifold.*

Proof. It follows directly from Proposition 2.2 and Theorem 1 given in [DDV].

□

We observe explicitly that, in the example above, the cone over the Clifford torus is not proper pseudo-symmetric.

3. Proof of 1.1 and 1.2

Let $f : M^n \to Q^N(c)$ be an isometric immersion. We choose a local orthonormal frame field $\{e_1, ..., e_n, e_{n+1}, ..., e_N\}$ adapted to f. Then the components of the second fundamental form α are given by $h^\sigma_{ij} = \langle \alpha(e_i, e_j), e_\sigma \rangle$, $i, j \in \{1, \dots n\}$, $\sigma \in \{n+1, \dots N\}$. The components of the covariant derivative of α are given by $h^\sigma_{ijk} = \left\langle \left(\overline{\nabla}_{e_k}\alpha\right)(e_i, e_j), e_\sigma \right\rangle = \overline{\nabla}_{e_k} h^\sigma_{ij}$, and the components of the second covariant derivative of α are given by $h^\sigma_{ijkl} = \left\langle \left(\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\alpha\right)(e_i, e_j), e_\sigma \right\rangle = \overline{\nabla}_{e_l} h^\sigma_{ijk} = \overline{\nabla}_{e_l}\overline{\nabla}_{e_k} h^\sigma_{ij}$. We have that f is pseudo-parallel if and only if

$$h^\sigma_{ijkl} = h^\sigma_{ijlk} - \phi\left\{\delta_{ki}h^\sigma_{lj} - \delta_{li}h^\sigma_{kj} + \delta_{kj}h^\sigma_{il} - \delta_{lj}h^\sigma_{ik}\right\}, \tag{10}$$

where $i, j, k, l = 1, ..., n$ and $\sigma = n+1, ..., N$.