CURVATURE ESTIMATES AND STABILITY PROPERTIES OF CMC-SUBMANIFOLDS IN SPACE FORMS

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Dedicated to Manfredo P. do Carmo on his 70^{th} birthday

Abstract

In this survey paper, we review several results connecting geometric properties (compactness, finiteness of the Morse index, caracterization of stable surfaces) and curvature estimates for constant mean curvature submanifolds in simply-connected space forms. The results go in both directions (curvature estimates to geometric properties and conversely).

Resumo

Neste artigo descreveremos vários resultados relacionando propriedades geométricas (compacidade, índice de Morse, caracterização de superfícies estáveis) e estimativas de curvatura para subvariedades com curvatura média constante em espaços de curvatura seccional constante simplesmente conexos. Estes resultados vão em ambas direções (de estimativas de curvatura para propriedades geométricas e reciprocadamente).

1. Introduction

This paper is a survey on some recent and less recent results relating geometric properties (compactness, stability, gap phenomena) and curvature estimates for constant mean curvature (CMC) submanifolds in space forms. We do not aim at extensively reviewing the literature but rather at illustrating some general ideas. We shall in particular limit ourselves to the case of complete non-compact submanifolds and we refer to the survey [10] for the compact case. Since the results we mention (will) appear elsewhere, we only give some ideas of the proofs.

One basic tool to prove the results we are interested in is the de Giorgi – Moser – Nash (dGMN) iteration method, applied to a well chosen elliptic equation. In an attempt to popularize the dGMN method among young geometers and to make the main lines of thought more clear, we have sketched some proofs involving this method in a very informal way.

Notations. In the sequel, we will use the following notations.

• We denote by $\overline{M}$ a simply-connected space form with constant sectional curvatures $c \in \{-1, 0, 1\}$. We shall specify the dimension of $\overline{M}$ and the value of c whenever it is necessary.

• Given an isometric immersion $i : (M^n, g) \to \overline{M}^{n+p}$, we let A denote the second fundamental form of i. For any vector fields X, Y tangent to M, the normal vector field $A(X, Y)$ is the projection of $\overline{D}_X Y$ on the normal space to M in $\overline{M}$, where $\overline{D}$ is the Levi-Civitá connection on $\overline{M}$. The tensor A is a symmetric tensor on M, with values in the normal bundle of M. The mean curvature vector of the immersion i is given by $n\,\vec{H} := \mathrm{Trace}\,(\mathrm{A})$, where the trace is taken with respect to the induced metric g on M.

We also introduce the traceless analogue of the second fundamental form, namely the tensor $A^0 := A - Hg$. The norms of the tensors A, A^0 and of the mean curvature vector $\vec{H}$ will be calculated with respect to the metrics $\overline{g}, g$.

• When the immersion i has codimension 1, we also introduce the *stability operator* L which is given by $L = \Delta_g - n(c + H^2) - |A^0|^2$, where Δ_g is the non-negative Laplacian on (M, g). Of great geometric interest is the so-called Morse index of the immersion. There are in fact two definitions which we now recall.

Given a relatively compact domain $\Omega \subset M$, we denote by $\mathrm{Ind}_{\mathrm{s}}(\Omega)$ the number of negative eigenvalues of the operator L for the Dirichlet problem on Ω; we denote by $\mathrm{Ind}_{\mathrm{w}}(\Omega)$ the number of negative eigenvalues of the operator L for the Dirichlet problem on Ω, with the additional condition that the functions under consideration have mean value zero over Ω, *i.e.* $\int_\Omega f\, dv_g = 0$ (for an explanation of this definition, see [4]). The *strong and weak indices* of the immersion are defined respectively as

$$\begin{cases} \mathrm{Ind}_{\mathrm{s}}(\mathrm{M}) := \sup\{\mathrm{Ind}_{\mathrm{s}}(\Omega) \bigm| \Omega \subset \mathrm{M},\ \Omega \text{ relat. compact}\}, \\ \\ \mathrm{Ind}_{\mathrm{w}}(\mathrm{M}) := \sup\{\mathrm{Ind}_{\mathrm{w}}(\Omega) \bigm| \Omega \subset \mathrm{M},\ \Omega \text{ relat. compact}\}. \end{cases} \tag{1}$$

The geometric interpretation of the index is that it measures the number of linearly independent normal deformations that decrease the volume (in the case of the weak index, the deformations under consideration leave a volume fixed).

We recall that when $H = 0$, we say that M is stable if

$$\int_M fLf\, dv_M = \int_M (|df|^2 - \{n(c + H^2) + |A^0|^2\}f^2\, dv_M \geq 0,$$

for all $f \in C_0^\infty(M)$. This means that $\mathrm{Ind}_{\mathrm{s}}(\mathrm{M}) = 0$. In the case $H \neq 0$, we call M stable if

$$\int_M fLf dv_M = \int_M (|df|^2 - \{n(c + H^2) + |A^0|^2\}f^2\, dv_M \geq 0,$$

for all $f \in C_0^\infty(M)$, with $\int_M f\, dv_M = 0$, and this means that $\mathrm{Ind}_{\mathrm{w}}(\mathrm{M}) = 0$.

Two questions are of particular interest.

Question 1. Can one find conditions (e.g. on the curvature) implying that a CMC immersion is stable (or more generally has finite Morse index) ?

Question 2. What restrictions on a CMC immersion (e.g. curvature estimates) are implied by the stability assumption (or more generally by the assumption that the Morse index is finite) ?

We shall address the first question in Section 2 and the second question in Section 3. A key proposition for our purposes is given in Section 4. Some proofs are sketched in Section 5

2. Geometric properties via curvature estimates

The first result we want to mention concerns the index. Let us first recall some earlier results.

• Let $M^2 \to \mathbb{R}^3$ be a complete minimal immersion. Then

$$\mathrm{Ind}_s(\mathrm{M}) < \infty \iff \int_{\mathrm{M}} |\mathrm{A}|^2 \mathrm{dv_M} < \infty$$

(Fischer - Colbrie 1985, [18]).

• Let $M^2 \to \mathbb{H}^3(-1)$ be a complete CMC-1 immersion. Then

$$\mathrm{Ind}_s(\mathrm{M}) < \infty \iff \int_{\mathrm{M}} |\mathrm{A}|^2 \mathrm{dv_M} < \infty$$

(do Carmo – da Silveira 1990, [13]).

• Let $M^2 \to \mathbb{H}^3(-1)$ be a complete CMC-H immersion, with $H^2 > 1$. Then

$$\mathrm{Ind}_s(\mathrm{M}) < \infty \iff \mathrm{M \text{ is closed}}$$

(da Silveira 1987, [33]).

• Let $M^n \to \mathbb{H}^{n+1}(-1)$ be a complete minimal immersion with finite total curvature, $\int_M |A|^n \, dv_M < \infty$. Then, $\mathrm{Ind}_s(\mathrm{M}) < \infty$ (Oliveira 1990, [27]).

We say that a complete CMC immersion $M^n \to M^{n+1}$ has *finite total curvature* if

$$\int_M |A^0|^n \, dv_M < \infty.$$

The following result was proved in [7].

Theorem 2.1. *Let $M^n \to \mathbb{H}^{n+1}(-1)$ be a complete CMC-H immersion with $H^2 < 1$ and finite total curvature, $\int_M |A^0|^n \, dv_M < \infty$. Then* $\mathrm{Ind}_s(\mathrm{M}) < \infty$.

Note. As shown in [33], one can not hope for a converse statement in the preceding theorem. There exist complete CMC-H immersions in the hyperbolic space, with $H^2 < 1$, which are stable and translation invariant (hence with infinite total curvature).

As we mentioned in the Introduction, there are two notions of Morse index. It follows immediately from the min-max characterization of eigenvalues that the strong and weak indices satisfy the following inequalities

$$\mathrm{Ind}_{\mathrm{w}}(\mathrm{M}) \leq \mathrm{Ind}_{\mathrm{s}}(\mathrm{M}) \leq \mathrm{Ind}_{\mathrm{w}}(\mathrm{M}) + 1.$$

The condition $\int_M f = 0$ which appears in the definition of the weak index has strong geometric significance when M is compact (with or without boundary). It does not make sense however when M is complete non-compact because CMC complete non-compact immersions have infinite volume ([31]). As a matter of fact, one has the following result ([3, 9]).

Theorem 2.2. *For complete non-compact constant mean curvature surfaces in the Euclidean or in the hyperbolic* 3*-space, the equality* $\mathrm{Ind}_{\mathrm{s}}(\mathrm{M}) = \mathrm{Ind}_{\mathrm{w}}(\mathrm{M})$ *holds.*

The finiteness of the total curvature of M also implies some topological properties of M. It is a classical result of Huber–Ossermann ([22, 29]) that a minimal surface in $\mathbb{R}^3$, with finite total curvature, must be conformally equivalent to $\overline{M} \setminus \{p_1, ..., p_k\}$, where $\overline{M}$ is a compact Riemann surface. This result has been generalized in two directions.

- Let $M^n \to \mathbb{R}^{n+p}$ be a complete minimal immersion with finite total curvature. Then M is diffeomorphic to a compact manifold minus finitely many points (Anderson 1985, [2]).
- Let $M^n \to \mathbb{H}^{n+p}$ be a complete minimal immersion with finite total curvature. Then M is diffeomorphic to the interior of a compact manifold with boundary (Oliveira 1993, [27]).

In the case of a non-zero parallel mean curvature $\vec{H}$ submanifold, we have the following:

Theorem 2.3. *Let* $M^n \to \overline{M}^{n+p}$ *be a complete isometric immersion with parallel mean curvature vector* $\vec{H}$ *into a simply-connected space form with constant sectional curvatures* $c \in \{-1, 0, 1\}$. *If* $|\vec{H}|^2 + c > 0$ *and* $\int_M |A^0|^n \, dv_M < \infty$, *then* M *is compact.*

Remarks.

(1) The above theorem has been proved in [8] (for $n = 2, p = 1$) and [16] (any n, p) in the case $c \in \{-1, 0\}$. The proofs rely on a key proposition proven in [8], see Proposition 4.1 below.

(2) The case $c = 1$ seems to be new and we give a proof in Section 5.3 below. The case $H = 0, c = 1$ is of particular interest: a complete minimal submanifold of the sphere, with finite total curvature, must be compact (a similar result is proved in [17], in dimension 2 and in higher dimensions with the additional assumption that the Ricci curvature of M is bounded from below).

(3) For a CMC H-hypersurface in $\mathbb{H}^{n+1}$, with $H < 1$, Castillon ([14]) proved that the finiteness of the total curvature implies that M is diffeomorphic to the interior of a compact manifold with boundary. With the above results, we see that for the hyperbolic space $\mathbb{H}^{n+1}$, it only remains to be decided what is the implication of finite total= curvature on the topology of M in the case $H = 1$. For $n = 2$, Huber's theorem ([22]) implies that M is conformally equivalent to $\overline{M} \setminus \{p_1, ..., p_k\}$. Based on the known examples, it is a natural question to ask whether M is diffeomorphic to a compact manifold minus finitely many points.

(4) In the proof of theorem 2.3 and also in Andeson's and Oliveira's results, a crucial step is to show that under the hypothesis, $|A^0|$ goes to zero uniformely at infinity.

3. Curvature estimates under geometric assumptions and applications

3.1. Dimension 2

It is important to have curvature estimates under certain geometric conditions. Such results have been obtained for minimal submanifolds (see [36, 34, 11]). Let us in particular mention the following theorem due (independently) to do Carmo – Peng [11], Fischer-Colbrie – Schoen [19] and Pogorelov [30].

Theorem 3.1. *A stable complete minimal immersion $M^2 \to \mathbb{R}^3$ is totally geo-*

desic, i.e. is a plane.

One way to prove this result is to make use of a curvature estimate due to Schoen [34]. Let $M \to \mathbb{R}^3$ be a minimal immersion. Then, the norm of the second fundamental form satisfies an estimate of the form $|A|(x_0) \leq CR^{-1}$ where C is a universal constant, provided that the ball $B(x_0, R)$ is relatively compact in M and stable. In order to prove the theorem, it suffices to apply this estimate with R going to infinity.

This estimate of the second fundamental form has been generalized in [9] to the CMC case as follows (let us recall, at this stage, that finite index implies that the stability operator is bounded from below, see [18]).

Theorem 3.2. *Let (M, g) be an oriented Riemannian surface. Let $i : (M, g) \to (\overline{M}, \overline{g})$ be an isometric immersion with constant mean curvature H of (M, g) into a simply-connected 3-manifold with constant sectional curvatures c. Assume there are positive numbers $R' > R$ such that the geodesic g-ball $B(R')$ centered at some x_0 in M, with radius R', is relatively compact in (M, g) and that the stability operator L_g, with Dirichlet boundary conditions, is bounded from below by some number $-2\ell \geq 0$ on $B(R')$. Let A^0 and K_g denote respectively the traceless second fundamental form and the Gaussian curvature of the immersion.*
Given $\Lambda > 0$, there exist positive constants $C(\Lambda), c(\Lambda)$, which only depend on Λ, such that

$$|A^0|^2(x) \leq C(\Lambda)R^{-2} \text{ and } |K_g(x| \leq C(\Lambda)R^{-2}$$

for all $x \in B(c(\Lambda)R)$, provided one of the following conditions holds:

$$(A) \qquad c + H^2 \leq 0 \text{ and } 4R^2\Big((c + H^2) + \ell\Big) \leq \Lambda,$$

or

$$(B) \qquad c + H^2 > 0,\ 4(c + H^2)R^2 \leq \pi^2 \text{ and } 4\ell R^2 \leq \Lambda.$$

Comment. It can be shown (see [20]) that an estimate of the form $(c + H^2)R^2 \leq$ Const. is necessary in case (B).

Applications. Let us briefly indicate two applications of this theorem.

(1) For complete non-compact CMC surfaces in Euclidean or hyperbolic space, the notions of weak and strong indices coincide (see Theorem 2.2).

(2) Let $M^2 \to \mathbb{H}^3(-1)$ be a complete non-compact stable CMC-H surface with $H^2 = 1$. Then, M is totally umbilic (see [33]).

3.2. Higher dimensions.

Several authors investigated the possible generalization of Theorem 3.1 to higher dimensions. For that purpose, one needs to make some extra assumption.

Schoen, Simon and Yau proved the following result (we state a particular case for simplicity, see Theorem 2 in [36]).

Theorem 3.3. *Let $M^n \to \mathbb{R}^{n+1}$ be a complete minimal immersion and let $B(R)$ denote the intersection of M with the Euclidean R-ball centered at some $x_0 \in M$. Assume that M is stable and that $\lim_{R\to\infty} R^{-p}\mathrm{Vol(R)} = 0$ for some $p \in]0, 4 + \sqrt{8/n}[$. Then M is totally geodesic.*

This result applies in particular when $n \leq 5$ and hence implies Bernstein's theorem for minimal graphs in $\mathbb{R}^{n+1}$ when $n \leq 5$.

Other generalizations assume some control on the curvature of M. Let us mention the following results.

Theorem 3.4. *Let $M^n \to \mathbb{R}^{n+1}$ be a stable complete minimal immersion. Let K denote the scalar curvature of M and let $B(r)$ denote the geodesic ball of radius r (for the induced metric on M), with center some fixed point $x_0 \in M$.*

Asume that

$$\lim_{R\to\infty} R^{-2-2q} \int_{B(R)} |K|\, dv_M = 0$$

for some $q < \sqrt{2/n}$. *Then,* M *is a hyperplane. In particular, if* $\int_M |K|\, dv_M < \infty$, M *is a hyperplane.*

A generalization of this result to CMC hypersurfaces in spaces forms appears in [1].

Theorem 3.5. *Let* $M^n \to \mathbb{R}^{n+1}$ *be a complete minimal immersion with second fundamental form* A. *Assume that* $n \leq 5$ *and that* $\int_M |A|^n\, dv_M < \infty$. *Then* M *is a totally geodesic* n*-space.*

Simons' equation and the first steps in the dGMN method play a central role in the proofs of the preceding results. As shown in [37] one can actually remove the assumption $n \leq 5$ in Theorem 3.5; the proof is difficult and involves Simons' equation, interior curvature estimates (via the dGMN method) and compactness theorems.

It is a well-known fact (due to Chern and Osserman [15, 28]) that the total curvature $\int_M K\, dv_M$ of a minimal surface in Euclidean space has values in $2\pi\mathbb{Z}$. In particular, if $\int_M |K|\, dv_M$ is small enough, M must be a plane. This gap phenomenon occurs in higher dimension as well (the proof relies on Simons' equation).

Theorem 3.6. *Let* $M^n \to \mathbb{R}^N$ *be a complete minimal immersion with second fundamental form* A. *There exist explicit constants* $c(n) < c(n, N)$, *which only depend on the dimension (resp. on the dimension and on the codimension) such that* $\int_M |A|^n\, dv_M \leq c(n)$ *(resp.* $\leq c(n, N)$*) implies that* M *is a totally geodesic* n*-space.*

See [23] for gap phenomena under different assumptions. We finally would like to point out [26] for other recent results in this circle of ideas.

Let us close this section by mentioning the following curvature estimate of Schoen, Simon and Yau [36] whose proofs involves an improved Simons' inequality and the dGMN method.

Theorem 3.7. *Let $M^n \to \mathbb{R}^{n+1}$ be a minimal immersion (not necessarily complete). Assume that M is stable and that $n \leq 5$. Then there exists a constant β such that $|A(x_0)| \leq \beta R^{-1}$ (with the notations of theorem).*

4. A key proposition

The proofs of the above results rely on a key proposition whose proof uses analytic arguments and of course our main assumption, namely that we have an immersion $i : (M, g) \to (\overline{M}, \overline{g})$ with parallel mean curvature vector $\vec{H}$ into a simply-connected space form with curvature $c \in \{-1, 0, 1\}$. Let $H = |\vec{H}|$.

The main analytic ingredients are Simons' inequality and the Hoffman – Spruck Sobolev inequality. We review them now.

• Simons' equation

Under the assumption that $\overline{M}$ has constant sectional curvatures and that the immersion $M \to \overline{M}$ has parallel mean curvature vector, the norm $u := |A^0|$ of the traceless second fundamental form satisfies the following inequality (in the sense of distributions)

$$\Delta_g u \leq a u^3 + b u \tag{2}$$

where Δ_g is the non-negative Laplacian on (M, g) and where a, b are positive constants which depend on $\mathrm{dim\,M}, \mathrm{codim\,M}, \mathrm{H}$ (see [6, 32]).

• Sobolev inequality

There exists a universal constant $c(n)$ such that for any $f \in C_0^1(M)$, with $f \geq 0$,

$$\|f\|_{n/(n-1)} \leq c(n)\{\|df\|_1 + \|Hf\|_1\} \tag{3}$$

provided the volume of the support of the function f satisfies

$$\mathrm{Vol}\,(\mathrm{Supp}\,(\mathrm{f})) \leq \mathrm{C}(\mathrm{n}, \mathrm{inj}(\mathrm{M}, \overline{\mathrm{M}})) \tag{4}$$

where $C(n, inj(M, \overline{M}))$ only depends on the dimension n and on the minimum distance in $\overline{M}$ of points in M to their cut-locus in $\overline{M}$. This function can be chosen to be infinite when $\overline{M}$ is simply-connected, with non-positive sectional curvatures, $c \leq 0$.

The main technical fact in order to prove Theorems 2.1 and 2.3 is the following proposition ([8], Theorem 4.1 and the note following Corollary 3.2).

Proposition 4.1. *Assume M^n is complete and satisfies the Sobolev inequality (3). Assume there is given a non-negative function u which satisfies inequality (2) in the sense of distributions and such that $\int_M u^n\, dv_M < \infty$. Then, u tends to zero uniformly at infinity.*

5. The dGMN method applied to geometry

5.1. Sketch of the proof of Proposition 4.1 in dimension 2

We shall limit ourselves to the two dimensional case for simplicity. Given some positive numbers $0 < R < S$ and fixing some point $x_0 \in M$, we denote by $E(R)$ the exterior of the geodesic ball with radius R and center at x_0, $E(R) = M \setminus \overline{B}(x_0, R)$, and we denote by $A(R, S)$ the annulus with center x_0 and radii S, R, $A(R, S) := B(x_0, S) \setminus \overline{B}(x_0, R)$.

In dimension 2, the Sobolev inequality can be written

$$\int_M f^2\, dv_M \leq c(M)\{(\int_M |df|\, dv_M)^2 + (\int_M f\, dv_M)^2\} \tag{5}$$

for all $f \in C_0^1(M), f \geq 0$.

Recall also our basic assumption on the function u,

$$\int_M u^2 \, dv_M < \infty. \tag{6}$$

For the sake of simplicity, we shall now only give a formal argument, integrating by parts over sets without taking the boundary terms into account (the correct proof involves using cut-off functions at this stage, see [8]). Since our computations are formal, we shall also forget the measure in the integrals.

Take $f = u^2 \mathbb{1}_{A(R,S)}$ in the Sobolev inequality (5). Then (in the sequel, we shall denote by c_i constants which only depend on M),

$$\int_{A(R,S)} u^4 \leq c_1 \{ (\int_{A(R,S)} u|du|)^2 + (\int_{A(R,S)} u^2)^2 \}$$

which implies that

$$\int_{A(R,S)} u^4 \leq c_1 \{ (\int_{A(R,S)} |du|^2)(\int_{A(R,S)} u^2) + (\int_{A(R,S)} u^2)^2 \}.$$

Multiplying inequality (2) by u and integrating over $A(R,S)$ gives

$$\int_{A(R,S)} |du|^2 \leq a \int_{A(R,S)} u^4 + b \int_{A(R,S)} u^2$$

Plugging this last inequality into the preceding one, we obtain

$$\int_{A(R,S)} u^4 (1 - ac_1 \int_{A(R,S)} u^2) \leq c_1 (b+1) (\int_{A(R,S)} u^2)^2.$$

Since $\int_{E(R)} u^2$ tends to zero when R tends to infinity, taking R large and letting S tend to infinity in the preceding inequality it follows that $\int_{E(R)} u^4$, $\int_{E(R)} |du|^2$ are finite and controled in terms of $\int_{E(R)} u^2$.

As a matter of fact, making the above argument rigorous by using cut-off functions, we obtain the following lemma

Lemma 5.1. *There exists a number R_0 and a constant $c_2(M)$ such that the function u satisfies the inequalities*

$$\Big(\int_{E(R+2)} u^4\, dv_M\Big)^{1/4} \le C_2(M)\Big(\int_{E(R)} u^2\, dv_M\Big)^{1/2},$$

$$\Big(\int_{E(R+2)} |du|^2\, dv_M\Big)^{1/2} \le C_2(M)\Big(\int_{E(R)} u^2\, dv_M\Big)^{1/2}.$$

These estimates are our initial estimates in order to be able to apply the dGMN method. We now use Simons' equation again, multiply it by u^{2k-1}, for any $k \ge 1$ and integrate by parts (formally) on the set $A(R,S)$ to get

$$\int_{A(R,S)} |du^k|^2 \le \frac{k^2}{2k-1}\{a\int_{A(R,S)} u^{2k+2} + b\int_{A(R,S)} u^{2k}\}$$

and we formally apply Sobolev inequality (5) to the function $u^{2k}\mathbb{1}_{A(R,S)}$ to obtain

$$(\int_{A(R,S)} u^{4k})^{1/2} \le C\{2\int_{A(R,S)} u^k|du^k| + \int_{A(R,S)} u^{2k}\},$$

which gives

$$\|u^k\|_4^2 \le c_1\{\|u^k\|_2^2 + \|\,|du^k|\,\|_2^2\}.$$

where the L^p norms are taken on the set $A(R,S)$. We arrive at the inequality

$$\|u^k\|_4^2 \le c_3\{(1+k)\|u^k\|_2^2 + k\|u^{k+1}\|_2^2\} \tag{7}$$

for all $k \ge 1$ and L^p norms taken on $A(R,S)$. Using Lemma 5.1, inequality (7), the inequality

$$\int_{A(R,S)} u^6 \le (\int_{A(R,S)} u^8)^{1/2}(\int_{A(R,S)} u^4)^{1/2},$$

and letting S go to infinity, we can control the integrals $\int_{E(R)} u^8$ and $\int_{E(R)} u^6$ in terms of $\int_{E(R)} u^2$.

We now write

$$\|u^{k+1}\|_2^2 \le (\int_{A(R,S)} u^{3k})^{2/3}(\int_{A(R,S)} u^6)^{1/3} \le c_4\|u^k\|_3^2$$

and we finally obtain, using interpolation,

$$\|u^k\|_4^2 \leq c_5 k(1+k^2)\|u^k\|_2^2 \tag{8}$$

for all $k \geq 1$ and L^p norms take over $A(R,S)$.

When $k=1$, the right hand side is $\|u\|_2$ which is finite by assumption. We write inequality (8) for $k_i = 2^i, i \geq 1$ and we multiply these inequalities to obtain the final inequality

$$\sup_{2R \leq d(x_0,x) \leq S} u(x) \leq C_M\Big(\int_{A(R,2S)} u^2 \, dv_M$$

and hence, letting S tend to infinity,

$$\sup_{E(2R)} u(x) \leq C_M\Big(\int_{E(R)} u^2 \, dv_M\Big)^{1/2}. \tag{9}$$

In particular, since $\int_M u^2 \, dv_M < \infty$, the function u tends to zero uniformly at infinity. This is precisely the statement of Proposition 4.1 in dimension 2.

5.2. Sketch of the proof of Theorem 2.1

It is not difficult to show that the estimates in Lemma 5.1 imply that there exists a function $b(R)$, which tends to zero when R tends to infinity, such that

$$\int_M (|df|^2 - u^2 f^2) \, dv_M \geq (1 - b(R)) \int_M |df|^2 \, dv_M - b(R) \int_M f^2 \, dv_M \tag{10}$$

for all functions $f \in C_0^\infty(E(R))$. Inequality (10) imples that the operator $\Delta - u^2$ is bounded from below and that its essential spectrum in bounded from below by 0. Since $H^2 < 1$, this imples that the stability operator $L = \Delta - u^2 + 2(1-H^2)$ is bounded from below and has essential spectrum bounded from below by $2(1-H^2) > 0$. It follows that L has finite index.

□

5.3. Sketch of the proof of Theorem 2.3

Case 1, $c \leq 0$ (8, 16]). One can apply Proposition 4.1 with our immersion M with parallel mean curvature vector $\vec{H}$ and $u := |A^0|$. It follows from the Gauss equation ([24]) that the Ricci curvature of M is bounded from below by $(c + |\vec{H}|^2)/2 > 0$, outside a compact set and hence, using the same arguments as in the proof of Bonnet-Myers theorem, we= conclude that M is compact.

Case 2, $c = 1$. We are given a complete immersion $M \to S^{n+p}$ with parallel mean curvature vector $\vec{H}$. Compose the immersion $i_1 : M^n \to S^{n+p}$ with the standard embedding $j : S^{n+p} \to \mathbb{R}^{n+p+1}$. This gives an isometric immersion $i_2 = j \circ i_1 : M \to \mathbb{R}^{n+p+1}$. Given $x \in S^{n+p}$ let us denote by $\nu(x) = -x$ the unit normal to S^{n+p} at x. We then have the following lemma.

Lemma 5.2. *Assume the immersion i_1 has parallel mean curvature vector $\vec{H}$ and traceless second fundamental form A^0 (with values in the normal bundle of M in S^{n+p}). Then the immersion i_2 has parallel mean curvature vector $\vec{\mathcal{H}}$ and traceless second fundamental form $\mathcal{A}^0$ (with values in the normal bundle of M in $\mathbb{R}^{n+p+1}$). They satisfy*

$$\vec{\mathcal{H}} = \vec{H} + \nu \text{ and } \mathcal{A}^0 = A^0.$$

In particular, $|\vec{\mathcal{H}}|^2 = |\vec{H}|^2 + 1$.

Since $|\vec{H}|^2 + 1 > 0$ always, and since $\mathcal{A}^0 = A^0$, we can apply Theorem 2.3, case $c = 0$ which we have proved earlier, to the immersion i_2 and conclude that M must me compact.

□

Proof of the Lemma. It might be useful, at this stage to recall that $\vec{H}$ parallel means parallel with respect to the normal connection. Call D the connection on M, ∇ the connection on S^{n+p} and $\overline{\nabla}$ the connection on $\mathbb{R}^{n+p+1}$ and denote by g the metric on M. Let X, Y be tangent vector fields to M. Since j is totally umbilic and taking into account the choice made for ν, we have

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)\nu \text{ and } \overline{\nabla}_X Y = D_X Y + \vec{A}(X,Y) + g(X,Y)\nu$$

where $\vec{A}$ is the second fundamental form of i_1. It follows that the second fundamental form $\vec{\mathcal{A}}$ of i_2 is given by $\vec{\mathcal{A}} = \vec{A} + g\nu$. This gives the formula for the mean curvature vectors and the fact that $\mathcal{A}^0 = A^0$.

The fact that the normal component of $\overline{\nabla}_X \vec{\mathcal{H}}$ is zero easily follows from the fact that the normal component of $\nabla_X \vec{H}$ (in the sphere) is zero and from $\overline{\nabla}_X \nu = -X$. We have proved that i_2 has parallel mean curvature vector.

□

5.4. About the proofs of Theorems 2.2 and 3.2

The proof of Theorem 2.2 makes use of the estimates in Theorem 3.2 and of a general result concerning the weak and strong indices of Schrödinger operators with bounded potential on Riemannian manifolds with infinite volume and controled volume growth (see [3]).

The proof of Theorem 3.2 follows the lines of [34], with arguments adapted to the CMC case. There are two main steps in the proof:

Step 1: one reduces the problem to the case of a simply-connected disk;

Step 2: one gives an estimate comparing the induced metric and the Euclidean metric on the disk.

The proof is quite technical and uses the dGMN method repeatedly, we refer to [9] for more details.

6. Concluding remarks

Remark 1. The exponent n (n is the dimension of M) is critical in Theorem 2.3 and Proposition 4.1 in the sense that the assumptions $\int_M |A^0|^n \, dv_M < \infty$ and $\int_M u^n \, dv_M < \infty$ can be replaced by the assumptions $\int_M |A^0|^q \, dv_M < \infty$ and $\int_M u^q \, dv_M < \infty$ respectively, for any $q \geq n$ leading to the following proposition

Proposition 6.1. *Assume M^n is complete and satisfies the Sobolev inequality (3). Assume there is given a non-negative function u which satisfies inequality (2) in the sense of distributions and such that $\int_M u^q \, dv_M < \infty$ for some $q \geq n$. Then, u tends to zero uniformly at infinity.*

and its corollary

Theorem 6.2. *Let $M^n \to \overline{M}^{n+p}$ be a complete isometric immersion with parallel mean curvature vector $\vec{H}$ into a simply-connected space form with constant sectional curvatures $c \in \{-1, 0, 1\}$. If $|\vec{H}|^2 + c > 0$ and $\int_M |A^0|^q \, dv_M < \infty$, for some $q \geq n$, then M is compact.*

The proof of Proposition 6.1 follows exactly the same lines as that of Proposition 4.1. Let us point out that Theorem 6.2 has also been obtained in [17], in the particular case in which $c = 1, H = 0$.

Remark 2. Two natural questions arise from the above mentioned results. To what extent is it necessary to assume that the ambient space is simply-connected necessary ? To what extent can one generalize the above results to the case of an ambient space with variable sectional curvatures ?

As pointed out in Section 4, the Hoffman – Spruck Sobolev inequality is only valid for functions satisfying a condition involving the injectivity radius of the ambient space. It is therefore not clear how to generalize Theorem 2.3 to immersion into non simply-connected spaces.

On the other-hand, looking carefully at the proof of Theorem 3.2, one can see that it is possible to remove the assumption that the ambient space is simply-connected, provided one reduces the size of R in Conditions (A) and (B), taking the injectivity radius of $\overline{M}$ into account.

It is reasonnable to expect that the above results can be generalized to the case in which the ambient space has varying sectional curvatures (as in [36, 34]). One could indeed use the computations in [36] and obtain an inequality à la Simons, with an ambient space with varying sectional curvatures. One could also use the fact that the Hoffman – Spruck Sobolev inequality holds for such ambient spaces. We have however not checked the details.

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