A NOTE ON A THEOREM OF LAWSON AND SIMONS ON COMPACT SUBMANIFOLDS OF SPHERES

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Abstract

The purpose of this note is to extend, to lower dimensions, the theorem 1.1 below

1. Introduction

In [LS] Lawson and Simons proved the folowing

Theorem 1.1. *Let $f : M^n \to S^{n+m}$, $n \geq 5$, be an isometric immersion of a compact, connected, n-dimensional Riemannian manifold M^n into the unit $(n+m)$-dimension sphere S^{n+m}. If the square of the lenght S of the second fundamental form of the immersion satisfies*

$$S < 2\sqrt{n-1}$$

then M^n is homeomorphic to the sphere S^n.

In this note we shall prove the following.

Theorem 1.2. *Let $f : M^n \to S^{n+m}$ be an isometric immersion of a compact, connected, n-dimensional Riemannian manifold M^n. If $S < 2\sqrt{n-1}$, then the fundamental group $\pi_1(M^n)$ of M^n is finite and the universal covering $\tilde{M}^n$ of M^n is compact. Moreover*

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(a) If $n = 2$, M^2 is diffeomorphic to a sphere S^2 or to the real projective space $\mathbb{R}P^2$, according to M^2 is orientable or not.
(b) If $n = 3$ and $\pi_1(M^3) = \{0\}$, then M^3 is diffeomorphic to S^3.
(c) If $n = 4$ and M^4 is orientable, then M^4 is homeomorphic to S^4.

Corollary 1.3. *Theorems 1.1 and 1.2 are valid if M^n is complete, connected and $Sup(S) < 2\sqrt{n-1}$.*

We remark that the result in (a) is sharp, since there exists a minimal embedding of Clifford torus $S^1(1/2) \times S^1(1/2)$ in S^3 with $S = 2\sqrt{n-1}$, and the assumption of orientability is necessary, since the Veronese surface in S^4 with $S < 2\sqrt{n-1}$ (cf. 2.12 in Wei, Indiana Univ. Math. J. Vol 33, No.4, 511-529, 1984). Furthermore, if $n = 3$ and $S < 2$, then the same conclusion in (b) holds without the hypothesis on the simple connectivity (cf. [W], the second proposition on p. 535). Similarly, if $n = 4$, $S < 3$, then the same conclusion in (c) holds without the assumption on the orientability, and the same techniques can be carried over to the submanifolds in the product of spheres (cf. 2.4 in Wei, Indiana Univ. Math. J. Vol. 33, No. 4, 511-529, 1984).

NOTE ADDED ON SEPTEMBER, 13, 1998: This paper was completed in early June, 1998. Recently, we were able to extend the above results for M^n compact and $S \leq 2\sqrt{n-1}$ on M^n. Then the Ricci curvature of M^n is nonnegative and we have only two cases: (1) There exists a point x in M^n such that for all $v \neq 0$ in $T_x^{M^n}$, Ric$(v) > 0$. In this case M^n admits metric of strictly positive Ricci curvature and : (i) if n = 2, M^2 is diffeomorphic to S^2 or to $\mathbb{R}P^2$; (ii) if n = 3, M^3 is orientable with the homology group $H_2(M^3, \mathbb{Z}) = \{0\}$ and M^3 is diffeomorphic to S^3 if $\pi_1(M^3) = \{0\}$; (iii) if $n \geq 4$, and n is odd or n is even and M^n is orientable, then M^n is homeomorphic to S^n.

(2) For all points x in M^n there exists a $v \neq 0$ in $T_x^{M^n}$ such that Ric$(v) = 0$. In this case we have that : (i) if n = 2, M^2 is flat, the submanifold M^2 is minimal in S^{2+m} and M^2 is isometric to a torus $S^1(r)$ X $S^1(s)$ if M^2 is orientable and $m = 1$. (ii) If $n \geq 3$, the codimension m can be reduced to 1 and, the immersion

can be viewed as a rotational hypersurface of the a torus $S^1(r)$ x $S^{n-1}(s)$ in S^{n+1} with constant mean curvature. The proofs will appear elsewhere.

2. Notations and preliminares lemmas

In this section we introduce the basic notation and prove some preliminaries lemmas.

Let $M = M^n$, $n \geq 2$, be a connected n-dimensional Riemannian manifold. We denote by $< \ >$ the metric and by $\| \ \|$ the respective norm. If R denotes the curvature tensor of M, then the Ricci tensor (at $x \in M$) is defined by

$$\mathrm{Ric}(v,w) = \sum_{i=1}^{n} < R(v_i, v)w, v_i >,$$

where v, w lie in the tangent space T_x^M of M at x, and $\{v_i\}_{i=1}^n$ is any orthonormal basis of T_x^M. The Ricci curvature $\mathrm{Ric}(v)$ in the unit direction $v \in T_x^M$ and the scalar curvature τ of M in x are given respectively by

$$\mathrm{Ric}(v) =< Qv, v > \ , \quad \tau = \mathrm{tr}\ Q, \tag{0}$$

where $Q : T_x^M \to T_x^M$ is given by

$$< Qv, w > = \ \mathrm{Ric}(v,w), \ v, w \in T_x^M.$$

Let $f : M^n \to Q_c^{n+m}$, $m \geq 1$, be an isometric immersion, where Q_c^{n+m} is a complete, simply connected $(n+m)$-dimensional manifold with constant sectional curvature c. For each $x \in M$, $(T_x^M)^\perp$ will denote the normal space of f at $x \in M$ and $\alpha : T_x^M \times T_x^M \to (T_x^M)^\perp$ will denote the second fundamental form of f at x.

If $\{e_\beta\}_{\beta=1}^m$ is any orthonormal basis of $(T_x^M)^\perp$, then the Weingarten operator $A_\beta = A_{e_\beta}$ in the normal direction e_β, is defined by

$$< A_\beta v, w > = < \alpha(v,w), e_\beta >, \ v, w \in T_x^M.$$

The mean curvature vector $\vec{H} = \vec{H}(x)$ at x and its norm are defined (respec.) by

$$\vec{H} = \frac{1}{n} \ \sum_{\beta=1}^{m} (\mathrm{tr}\ A_\beta) e_\beta \tag{1}$$

and

$$H = \| \vec{H} \| \tag{2}$$

The square of the lenght of the second fundamental form of f at x is defined by

$$S = \sum_{\beta=1}^{m} \mathrm{tr}\ A_\beta^2 \qquad (3).$$

We then have the following relations (see [E], p. 141):

$$\sum_{\beta=1}^{m} A_\beta^2 - \sum_{\beta=1}^{m} (\mathrm{tr}\ A_\beta) A_\beta = -Q + (n-1)cI, \tag{4}$$

where $I : T_x^M \to T_x^M$ is the identity map, and

$$S = -\tau + n^2 H^2 + n(n-1)c. \tag{5}$$

Firstly we prove the following:

Lemma 2.1. *Let V be a real vector space of dimension $n \geq 2$, and let $A : V \to V$ be a symmetric linear map, with $tr\ A = nH$. Denote by $< , >$ and $\| \ \|$ the inner product and respective norm of V.*

If $v \in V$, $\| v \| = 1$ and if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of A, then
(i) $< A^2 v, v > \leq \left(\frac{n-1}{n}\right) [tr\ A^2 - nH^2] + 2H < Av, v > -H^2$;
(ii) $< Av, v > \geq \lambda_1 \geq H - \sqrt{\left(\frac{n-1}{n}\right) [tr\ A^2 - nH^2]}$.

Proof. Let $\{v_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors of A, where $Av_i = \lambda_i v_i$, for all i. Assume first that tr $A = nH = 0$, and let λ_j be such that

$$\lambda_j^2 = \max_i \{\lambda_i^2\}.$$

Since

$$\lambda_j = -\sum_{i \neq j} \lambda_i,$$

then

$$\lambda_j^2 = \left(\sum_{i \neq j} \lambda_i \right)^2 \leq (n-1) \sum_{i \neq j} \lambda_i^2.$$

Therefore

$$\lambda_j^2 \leq \left(\frac{n-1}{n}\right) \text{tr } A^2.$$

If $v = \sum_{i=1}^{n} a_i v_i$, where $\sum_{i=1}^{n} a_i^2 = 1$, it is clear that

$$< A^2 v, v > \leq \lambda_j^2 \leq \left(\frac{n-1}{n}\right) \text{tr } A^2$$

and then (i) follows for $H = 0$. For (ii), note that $< Av, v > \geq \lambda_1$. Since $\lambda_1 \leq H = 0$ and

$$\lambda_1^2 = \left(\sum_{i \neq 1} \lambda_i\right)^2 \ \leq \ (n-1) \sum_{i \neq 1} \lambda_i^2,$$

then

$$\lambda_1^2 \leq \left(\frac{n-1}{n}\right) \text{tr } A^2$$

and

$$\lambda_1 \geq -\sqrt{\left(\frac{n-1}{n}\right) \text{tr } A^2},$$

which is the desired result (ii) for $H = 0$.

Suppose now that $H \neq 0$ and let $B = A - HI$, where $I : V \to V$ is the identity map. Then tr $B = 0$, $B^2 = A^2 - 2HA + H^2 I$ and tr $B^2 =$ tr $A^2 - nH^2$. The result (i) follows immediately by applying the case $H = 0$ to B. Since $< Av, v > \geq \lambda_1$ and $H \geq \lambda_1$, choose $v = v_1$ in (i) for $B = A - HI$. Then

$$(\lambda_1 - H)^2 \ \leq \ \left(\frac{n-1}{n}\right) [\text{tr } A^2 - nH^2],$$

and (ii) follows for $H \neq 0$.

Lemma 2.2. *Let $f : M^n \to Q_c^{n+m}$ be an isometric immersion and let $x \in M^n$. For $\vec{H}(x) \neq 0$, denote by $e_1 = \frac{1}{H}\vec{H}(x)$, $A_1 = A_{e_1}$ the Weingarten operator in direction e_1 and $\lambda_1 = min \{\lambda \ / \ \lambda$ is eigenvalue of $A_1\}$. For $\vec{H}(x) = 0$, let $\lambda_1 = 0$. If $v \in T_x^M$, $\| v \| = 1$ then*

$$Ric(v) \geq \left(\frac{n-1}{n}\right)(nc - S) + (n-2)H\lambda_1 + nH^2. \tag{6}$$

Proof. Let $\{e_\beta\}_{\beta=1}^m$ be an orthonormal basis of $(T_x^{M^n})^\perp$ such that $e_1 = \frac{1}{H}\vec{H}(x)$, when $\vec{H}(x) \neq 0$. Since tr $A_1 = nH$ and tr $A_\beta = 0$, for all $\beta > 1$, where $A_\beta = A_{e_\beta}$, it follows from (4) that

$$\sum_{\beta=2}^{m} < A_\beta^2 v, v > + < A_1^2 v, v > - nH < A_1 v, v > = -\mathrm{Ric}\,(v) + (n-1)c \quad (7)$$

For $\vec{H}(x) = 0$, choose $\{e_\beta\}_{\beta=1}^n$ be an orthonormal basis of $(T_x^{M^n})^\perp$, where $A_\beta = A_{e_\beta}$, tr $A_\beta = 0$, for all β and then (7) also follows from (4).Now, applying the lemma 2.1 for each A_β in (7), we get (6).

Lemma 2.3. *Let $M = M^n$, $n \geq 4$, be a connected, compact, n-dimensional Riemannian manifold such that M is orientable if n is even, $\tilde{M} = \tilde{M}^n$ the universal covering of M and $\pi_1(M)$ the fundamental group of M. Denote by $H_p(M, \mathbb{Z})$ and by $H_p(\tilde{M}, \mathbb{Z})$ the p-dimensional homology groups, with integer coefficients, of M and $\tilde{M}$, respectively. If $H_1(M, \mathbb{Z})$ is finite, $\tilde{M}$ is compact and $H_p(M, \mathbb{Z}) = H_p(\tilde{M}, \mathbb{Z}) = \{0\}$, for all $p = 2, 3, \cdots, n-2$ then M^n is homeomorphic to a sphere S^n.*

Proof. Firstly we prove that M is orientable if n is odd. In fact, if M is not orientable then by coroll.7.12 of ([B], p.346), $H_n(M, \mathbb{Z}) = \{0\}$. But the Euler characteristic $\chi(M)$ of M is zero.For other side $\chi(M) = b_0 - b_1 + ... - b_n$, where b_i = rank $H_i(M, \mathbb{Z})$.Then $\chi(M) = 1 + b_{n-1}$ (contradiction !).Since $H_1(M, \mathbb{Z})$ is finite, the torsion part of $H_1(M, \mathbb{Z})$ is $H_1(M, \mathbb{Z})$. But by universal coefficient theorem ([B], p.282, corollary 7.3) the cohomology group $H^i(M, \mathbb{Z})$ is isomorphic to direct sum of F_i and T_{i-1}, where F_i and T_i are the free part and torsion part,respectively, of $H_i(M, \mathbb{Z})$ and then $H^{n-1}(M, \mathbb{Z}) =$ is isomorphic to a F_{n-1}. Now by the Poincare duality ([B], p.339) $H_1(M, \mathbb{Z})$ is isomorphic to a $H^{n-1}(M, \mathbb{Z})$ and so $H_1(M, \mathbb{Z}) = \{0\}$. Again, by the universal coefficient theorem $H^1(M, \mathbb{Z}) = \{0\}$ and by the Poincare duality $H_{n-1}(M, \mathbb{Z}) = \{0\}$. Then M is a homology sphere. The above arguments applied to $\tilde{M}$ tells us that it is a homology sphere. Since $\pi_1(\tilde{M}) = \{0\}$, by the Hurewicz isomorphism theorem([S],p.398, theorem 5) the i-th homotopy group of $\tilde{M}^n$ $\pi_i(\tilde{M}^n) = \{0\}$, for all $i = 1, 2, \cdots, n-1$ and $\pi_n(\tilde{M}^n)$ is isomorphic a $\mathbb{Z}$. Let now $g : S^n \to \tilde{M}^n$ be a representative for a generator of $\pi_n(\tilde{M}^n)$. g is a mapping between simply connected CW complex which induces isomorphisms of homology groups. Hence by Whitehead theorem ([S],p.399,theorem 9) and ([S], p.406, theorem 25) g is a weak homotopy equivalence.Then by ([S],p.405,corollary 24) g is a homotopy equivalence and $\tilde{M}^n$ is indeed a homotopy sphere. By the generalized Poincaré conjecture (Smale, $n \geq 5$,Freedman $n = 4$), we have that $\tilde{M}$ is homeomorphic to a sphere. Therefore we have a homology sphere M which is covered by a sphere $\tilde{M}$ and so by a theorem of Sjerve [S1], $\pi_1(M) = \{0\}$ and hence M is also homeomorphic to a sphere. This proves the lemma 2.3.

3. Prof of Theorem 1.2

Let $x \in M^n$ and v a unity vector in $T_x M$. Define $S_H = 0$ if $H(x) = 0$ and $S_H = trA_{e_1}^2$, where $\frac{e_1 = vec(H)}{H}$ if $H(x) \neq 0$. Choosing $c = 1$ in lemma 2.2 and applying lemma 2.1 (ii) to (6) we obtain

$$\text{Ric }(v) \geq \left(\tfrac{n-1}{n}\right)\left[nc + 2nH^2 - S - (n-2)H\sqrt{\left(\tfrac{n}{n-1}\right)(S_H - nH^2)}\right]. \tag{8}$$

Since $S \geq S_H$, then

$$\text{Ric }(v) \geq \left(\tfrac{n-1}{n}\right)\left[nc + 2nH^2 - S - (n-2)H\sqrt{\left(\tfrac{n}{n-1}\right)(S - nH^2)}\right]. \tag{9}$$

or

$$\text{Ric }(v) \geq (n-1)\left[1 - \frac{S}{2\sqrt{n-1}} + \frac{a^2}{\sqrt{n-1}}\right], \tag{10}$$

where $a = \frac{1}{\sqrt{2n}}\left[\left(\sqrt{n-1} - 1\right)\sqrt{S - nH^2} - \left(\sqrt{n-1} + 1\right)\sqrt{nH^2}\right]$. It follows from (10) that Ric $(v) > 0$, for all $x \in M^n$, $v \in T_x^M$, $\| v \| = 1$ if $S < 2\sqrt{n-1}$ on M^n and then by Myers' theorem, $\tilde{M}^n$ is compact and $\pi_1(M^n)$ is finite.

If $n = 2$, applying the Gauss-Bonnet formula, we obtain part (a) of Theorem 1.2.

If $n = 3$ and $\pi_1(M^3) = 0$, (b) follows by employing Hamilton's theorem [H].

If $n = 4$, since $S < 2\sqrt{3} < 4$, by using theorem 4 of [LS], we obtain that $H_2(M^4, \mathbb{Z}) = 0$. Let now $\pi : \tilde{M}^4 \to M^4$ be the covering map. The above arguments applied to the isometric immersion $f \circ \pi$ to S^{4+m}, tells us that $H_2(\tilde{M}^4, \mathbb{Z}) = 0$. But since that $\pi_1(M^4)$ is finite then $H_1(M^4, \mathbb{Z})$ is finite ([B], p.174, theorem 3.4) and the case $n = 4$ follows directly from lemma 2.3.

Remark 3.1. *Formula (9) was obtained by Leung [L].*

Proof of Corollary 1.3. If Sup $S < 2\sqrt{n-1}$ then by using (10), we obtain that Ric $(v) \geq \delta > 0$, where

$$\delta = \frac{\sqrt{n-1}}{2}(2\sqrt{n-1} - \text{Sup } S).$$

Then, the result follows from Bonnet-Myers' theorem.

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