ON INVERSE PROBLEMS MODELED BY PDE'S

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Abstract

We investigate the iterative methods proposed by Maz'ya and Kozlov (see [3], [4]) for solving ill-posed reconstruction problems modeled by PDE's. We consider linear time dependent problems of elliptic, hyperbolic and parabolic types. Each iteration of the analyzed methods consists on the solution of a well posed boundary (or initial) value problem. The iterations are described as powers of affine operators, as in [4]. We give alternative convergence proofs for the algorithms, using spectral theory and some functional analytical results (see [5], [6]).

Resumo

Investigamos neste artigo os métodos propostos em [3] e [4] para resolver problemas mal postos de reconstrução. Consideramos problemas modelados por equações elipticas, hiperbólicas e parabólicas. Cada iteração dos métodos propostos se consiste na solução de um problema bem posto. Apresentamos demonstrações alternativas as originais, utilizando argumentos de análise funcional e teoria spectral (veja [5], [6]).

1. Introduction

1.1. Main results

We present new convergence proofs for iterative algorithms in [4] using a functional analytical approach, where each iteration is described using powers of an affine operator T. The key of the proof is to choose a correct topology for the Hilbert space where the iteration takes place, and to prove that T_l, the linear component of T, is a *regular asymptotic*, *non expansive* operator.

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Other properties of T_l such as positiveness, self-adjointness and injectivity are also verified. The ill-posed problems are presented in Section 2. In Section 3 we describe the iterative methods for each problem. The results concerning the analysis of the methods are summarized in Section 4. Some numerical results are discussed in Section 5.

The iterative procedures discussed in this paper were first presented in [4] and also treated in [1]. The iterative procedure for elliptic (stationary) Cauchy problems is discussed in [3], [5], [6] and [2]. The iterative procedure concerning parabolic problems is also treated in [8].

1.2. Preliminaries

Let H be a separable Hilbert space endowed with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The operator $T : H \to H$ is said to be *regular asymptotic* in $x \in H$ if $\|T^{k+1}(x) - T^k(x)\| \to 0$ as $k \to \infty$. If the above property holds for every $x \in H$, we say that T is regular asymptotic in H. The operator T is called *non expansive* if $\|T\| \leq 1$. The next lemma is the key of the convergence proofs presented in this article.[1]

Lemma 1 *Let* $T : H \to H$ *be a linear non expansive operator. With* Π *we denote the orthogonal projector defined on* H *onto* $\ker(I - T)$*. The following assertions are equivalent:*

a) T *is regular asymptotic in* H*;*

b) $\lim_{k\to\infty} T^k x = \Pi x.$

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary and A be a positive, self-adjoint, unbounded operator (with discrete spectrum) densely defined on the Hilbert space $H := L^2(\Omega)$. Let E_λ, $\lambda \in \mathbb{R}$, denote the resolution of the identity associated to A. We construct the family of Hilbert spaces $\mathcal{H}^s(\Omega)$, $s \geq 0$ as the domain of definition of the powers of A

$$\mathcal{H}^s(\Omega) \ := \ \{\varphi \in H \mid \|\varphi\|_s := \left(\int_0^\infty (1+\lambda^2)^s d\langle E_\lambda \varphi, \varphi\rangle\right)^{1/2} < \infty\}. \qquad (1)$$

[1] A complete proof can be found in [5].

The Hilbert spaces $\mathcal{H}^{-s}(\Omega)$ (with $s > 0$) are defined by duality:[2] $\mathcal{H}^{-s} := (\mathcal{H}^s)'$. It follows directly from the definition that $\mathcal{H}^0(\Omega) = H$.

An interesting case occurs when $A = (-\Delta)^{1/2}$, where Δ is the Laplace–Beltrami operator on Ω. In this particular case the identity $\mathcal{H}^s(\Omega) = H_0^{2s}(\Omega)$ holds, where $H_0^s(\Omega)$ is the Sobolev space of index s according to Lions and Magenes (see [7] pp. 54).

Given $T > 0$ we define the spaces $L^2(0,T;\mathcal{H}^s(\Omega))$ and $C(0,T;\mathcal{H}^s(\Omega))$ of functions $u : [0,T] \ni t \mapsto u(t) \in \mathcal{H}^s(\Omega)$. These are normed spaces if considered respectively with the norms

$$\|u\|_{2;0,T;s} := \left(\int_0^T \|u(t)\|_s^2 \, dt \right)^{1/2} \quad \text{and} \quad \|u\|_{\infty;0,T;s} := \sup_{t\in[0,T]} \|u(t)\|_s.$$

2. The ill-posed problems

2.1. An elliptic problem

Given functions $(f,g) \in \mathcal{H}^{1/2}(\Omega) \times \mathcal{H}^{-1/2}(\Omega)$, find $u \in (V_e, \|\cdot\|_{V_e})$, where

$$\begin{aligned} V_e &:= \{v \in L_2(0,T;\mathcal{H}^1(\Omega)) \mid (\partial_t^2 - A^2)u = 0 \text{ in } (0,T)\times\Omega\} \\ \|u\|_{V_e} &:= \left(\int_0^T (\|u(t)\|_1^2 + \|\partial_t u(t)\|_0^2) \, dt \right)^{1/2}, \end{aligned}$$

that satisfies

$$(P_e) \qquad \begin{cases} (\partial_t^2 - A^2)u = 0\,, \text{ in } (0,T)\times\Omega \\ u(0) = f\,, \quad \partial_t u(0) = g\,. \end{cases}$$

Note that if $u \in V_e$, then $\partial_t u \in L_2(0,T;H)$ and adequate trace theorems (see [7]) guarantee that $u(0), u(T) \in \mathcal{H}^{1/2}(\Omega)$ and $\partial_t u(0), \partial_t u(T) \in \mathcal{H}^{-1/2}(\Omega)$. The ill-posedness of problem (P_e) can be easily verified from the explicit representation of its solution:

$$u(t,x) = \cosh(At)f(x) + \sinh(At)A^{-1}g(x). \tag{2}$$

[2] Alternatively one can define $\mathcal{H}^{-s}(\Omega)$ as the completion of H in the (-s)-norm defined in (1).

2.2. A hyperbolic problem

Given functions $f, g \in \mathcal{H}^1(\Omega)$ find $u \in (V_h, \|\cdot\|_{V_h})$, where

$$\begin{aligned} V_h &:= \{v \in C(0,T;\mathcal{H}^1(\Omega)) \mid \partial_t u \in C(0,T;H) \text{ and } (\partial_t^2 + A^2)u = 0\} \\ \|u\|_{V_h} &:= \sup_{t\in[0,T]} \left(\|u(t)\|_1^2 + \|\partial_t u(t)\|_0^2\right)^{1/2}, \end{aligned}$$

which satisfies

$$(P_h) \qquad \left\{ \begin{array}{l} (\partial_t^2 + A^2)u \;=\; 0\,, \text{ in } (0,T)\times\Omega \\ u(0) \;=\; f\,, \quad u(T) \;=\; g\,. \end{array} \right.$$

Note that if $u \in V_h$, then $u(0), u(T) \in \mathcal{H}^1(\Omega)$ and $\partial_t u(0), \partial_t u(T) \in H$. We assume further the numbers $k\pi/T$, $k = 1, 2, \dots$ are not eigenvalues of A.[3] This hyperbolic (Dirichlet) boundary value problem is ill-posed if the distance from the set $M := \{k\pi/T; k \in \mathbb{N}\}$ to $\sigma(A)$ (the spectrum of A) is zero. This follows from the explicit representation of the solution of (P_h)

$$u(t,x) \;=\; \sin(A(T-t))\sin(AT)^{-1}f(x) \;+\; \sin(At)\sin(AT)^{-1}g(x). \qquad (3)$$

2.1. A parabolic problem

Given a function $f \in H = L_2(\Omega)$ find $u \in (V_p, \|\cdot\|_{V_p})$, where

$$\begin{aligned} V_p &:= \{v \in L_2(0,T;\mathcal{H}^1(\Omega)) \mid (\partial_t + A^2)u = 0 \text{ in } (0,T)\times\Omega\} \\ \|u\|_{V_p} &:= \left(\int_0^T (\|u(t)\|_1^2 + \|\partial_t u(t)\|_{-1}^2)\, dt\right)^{1/2}, \end{aligned}$$

which satisfies

$$(P_p) \qquad \left\{ \begin{array}{l} (\partial_t + A^2)u \;=\; 0\,, \text{ in } (0,T)\times\Omega \\ u(T) \;=\; f\,. \end{array} \right.$$

Note that if $u \in V_p$, then $u(0), u(T) \in H$. This corresponds to the well known problem of inverse heat transport, which is known to be severely ill-posed. The solution of (P_p) has the explicit representation

$$u(t,x) \;=\; \exp(A^2(T-t))f(x). \qquad (4)$$

[3]If this condition is not satisfied, one can easily see that problem (P_h) is not uniquely solvable.

3. Description of the Methods

3.1. An iterative procedure for the elliptic problem

Consider problem (P_e) with data $(f,g) \in \mathcal{H}^{1/2}(\Omega) \times \mathcal{H}^{-1/2}(\Omega)$. Given any initial guess $\varphi_0 \in \mathcal{H}^{-1/2}(\Omega)$ for $\partial_t u(T)$ we try to improve it by solving the following mixed boundary value problems (BVP) of elliptic type

$$\left\{\begin{array}{l} (\partial_t^2 - A^2)v \;=\; 0\,,\ \text{in } (0,T)\times\Omega \\ v(0) \;=\; f\,,\ \ \partial_t v(T) \;=\; \varphi \end{array}\right. \qquad \left\{\begin{array}{l} (\partial_t^2 - A^2)w \;=\; 0\,,\ \text{in } (0,T)\times\Omega \\ \partial_t w(0) \;=\; g\,,\ \ w(T) \;=\; v(T) \end{array}\right.$$

and defining $\varphi_1 := \partial_t w(T)$. Each one of the mixed BVP's above has a solution in V_e and consequently $\varphi_1 \in \mathcal{H}^{-1/2}(\Omega)$. Repeating this procedure we can construct a sequence $\{\varphi_k\}$ in $\mathcal{H}^{-1/2}(\Omega)$. Using the explicit representation of the solutions v and w of the above problems, one obtains

$$\varphi_1(x) \;=\; \tanh(AT)^2\varphi(x) \;+\; \sinh(At)\cosh(AT)^{-2}\,Af(x) \;+\; \cosh(AT)^{-1}\,g(x).$$

Defining the affine operator $T_e : \mathcal{H}^{-1/2} \to \mathcal{H}^{-1/2}$, $T_e(\varphi) := \tanh(AT)^2\varphi + h_{f,g}$, where $h_{f,g} := \sinh(At)\cosh(AT)^{-2}Af + \cosh(AT)^{-1}g$, the iterative algorithm can be rewritten as

$$\varphi_k \;=\; T_e(\varphi_{k-1}) \;=\; T_e^k(\varphi_0) \;=\; \tanh(AT)^{2k}\varphi_0 + \sum_{j=0}^{k-1}\tanh(AT)^{2j}h_{f,g}. \tag{5}$$

3.2. An iterative procedure for the hyperbolic problem

Let's now consider problem (P_h) with data $f,g \in \mathcal{H}^1(\Omega)$. Given any initial guess $\varphi_0 \in H$ for $\partial_t u(0)$ we try to improve it by solving the following initial value problems (IVP) of hyperbolic type[4]

$$\left\{\begin{array}{l} (\partial_t^2 + A^2)v \;=\; 0\,,\ \text{in } (0,T)\times\Omega \\ v(0) \;=\; f\,,\ \ \partial_t v(0) \;=\; \varphi \end{array}\right. \qquad \left\{\begin{array}{l} (\partial_t^2 + A^2)w \;=\; 0\,,\ \text{in } (0,T)\times\Omega \\ w(T) \;=\; g\,,\ \ \partial_t w(T) \;=\; \partial_t v(T) \end{array}\right.$$

and defining $\varphi_1 := \partial_t w(0)$. Each one of the IVP's above has a solution in V_h and consequently $\varphi_1 \in H$. Repeating this procedure we can construct a sequence

[4]The second problem is considered with reversed time.

$\{\varphi_k\}$ in H. Determining the solutions v and w of the above problems, one obtains

$$\varphi_1(x) \;=\; \partial_t w(0,x) \;=\; \cos(AT)^2\varphi(x) - \cos(AT)\sin(AT)\,Af(x) + \sin(AT)\,g(x).$$

Now defining the affine operator $T_h : H \to H$, $T_h(\varphi) := \cos(AT)^2\varphi \;+\; h_{f,g}$, where $h_{f,g} := -\cos(AT)\sin(AT)Af + \sin(AT)g$, the iteration can be written as

$$\varphi_k \;=\; T_h(\varphi_{k-1}) \;=\; T_h^k(\varphi_0) \;=\; \cos(AT)^{2k}\varphi_0 + \sum_{j=0}^{k-1}\cos(AT)^{2j}h_{f,g}. \tag{6}$$

3.3. An iterative procedure for the parabolic problem

We consider problem (P_p) with data $f \in H$. Let $\varphi_0 \in H$ be an initial guess for $u(0)$ and define $\bar\lambda := \inf\{\lambda; \lambda \in \sigma(A)\}$. Now choose a positive parameter γ such that $\gamma < 2\exp(\bar\lambda^2 T)$. The method consists in solving the initial value problems of parabolic type

$$\begin{cases} (\partial_t + A^2)v_0 \;=\; 0\,, \text{ in } (0,T)\times\Omega \\ v_0(0) \;=\; \varphi_0 \end{cases} \qquad \begin{cases} (\partial_t + A^2)v_k \;=\; 0\,, \text{ in } (0,T)\times\Omega \\ v_k(0) \;=\; v_{k-1}(0) - \gamma(v_{k-1}(T) - f) \end{cases}$$

for $k \geq 1$. The sequence $\{\varphi_k\}$ is now defined by $\varphi_k := v_k(0) \in H$. Determining the solutions v_k of the above problems, we have $\varphi_{k+1}(x) = (I - \gamma\exp(-A^2T))\varphi_k(x) + \gamma f(x)$. Define the affine operator $T_p : H \to H$, $T_p(\varphi) := (I - \gamma\exp(-A^2T))\,\varphi \;+\; h_f$, where $h_f := \gamma f$, we can write the iteration as

$$\begin{aligned} \varphi_k \;&=\; T_p(\varphi_{k-1}) \;=\; T_p^k(\varphi_0) \\ &=\; (I - \gamma\exp(-A^2T))^k\varphi_0 \;+\; \sum_{j=0}^{k-1}(I - \gamma\exp(-A^2T))^j h_f. \end{aligned} \tag{7}$$

4. Analysis of the Methods

4.1. The elliptic case

We start presenting a result, which is a generalization of the Cauchy–Kowalewski theorem. A complete proof can be found in [5].

Lemma 2 *Given $(f, g) \in \mathcal{H}^{1/2} \times \mathcal{H}^{-1/2}$, the problem (P_e) has at most one solution in V_e.*

In the next theorem we verify some properties of the operator $T_{l,e}$, which will be needed for the convergence proof of the algorithm.

Theorem 3 *The linear operator $T_{l,e}$ is positive, self-adjoint, injective, regular asymptotic, non-expansive and $1 \notin \sigma_p(T_{l,e})$.*

Proof: The injectivity follows from Lemma 2. The properties: positiveness, self-adjointness and $1 \notin \sigma_p(T_{l,e})$ follow from the assumptions on A. The last two properties follow from the inequality

$$\|(I - T_{l,e})x\|^2 \ \leq \ (\|x\|^2 - \|T_{l,e}x\|^2), \ \forall x \in \mathcal{H}^{-1/2}. \tag{8}$$

□

Theorem 4 *If problem (P_e) is consistent for the data (f, g), then the sequence φ_k converges to $\partial_t u(T)$ in the norm of $\mathcal{H}^{-1/2}(\Omega)$.*

Proof: Follows from Theorem 3 and Lemma 1. □

The converse of Theorem 4 is also true. This is discussed in

Theorem 5 *If the sequence φ_k converges, say to $\bar{\varphi}$, then problem (P_e) is consistent for the data (f, g) and it's solution $u \in V_e$ satisfies $\partial_t u(T) = \bar{\varphi}$.*

Proof: Follows from Lemma 2 and the definition of T_e. □

4.2. The hyperbolic case

Theorem 6 *The linear operator $T_{l,h} : H \to H$ is positive, self-adjoint, injective, non-expansive, regular asymptotic and 1 is not an eigenvalue of $T_{l,h}$.*

Proof: Analogous to the proof of Theorem 3. □

Theorem 7 *If problem (P_h) is consistent for the data (f, g), then the sequence φ_k converges to $\partial_t u(0)$ in the norm of H.*

Proof: Follows from Theorem 6 and Lemma 1. □

Theorem 8 *If the sequence φ_k converges, say to $\bar{\varphi}$, then problem (P_h) is consistent for the Cauchy data (f, g) and it's solution $u \in V_h$ satisfies $\partial_t u(0) = \bar{\varphi}$.*
Proof: Analogous to the proof of Theorem 5. ◻

4.3. The parabolic case

Lemma 9 *Given $f \in H$, the problem (P_p) has exactly one solution in V_p.*
Proof: This result is suggested by the general representation of the solution given in (4). A complete proof can be found in [7], Chapter 3. ◻

Theorem 10 *The linear operator $T_{l,p} : H \to H$ is self-adjoint, non-expansive, regular asymptotic and 1 is not an eigenvalue of $T_{l,p}$. Further, if it is possible to choose $\gamma < 2\exp(\tilde{\lambda}^2 T)$, where $\tilde{\lambda} := (\bar{\lambda}^2 - T^{-1}\ln 2)^{1/2}$, then $T_{l,p}$ is also injective.*
Proof: Analogous to the proof of Theorem 3. The injectivity under the extra assumption on γ follows from an inequality similar to (8). ◻

Theorem 11 *Given $f \in H$, let $u \in V_p$ be the uniquely determined solution of problem (P_p). Then the sequence φ_k converges to $u(0)$ in the norm of H.*
Proof: Follows from Theorem 10 and Lemma 1. ◻

5. Numerical results

Example 12 *Consider the problem of finding $u(0) \in L^2(\Omega)$, where u solves*

$$\begin{cases} a^2\,\partial_t u \,-\, \Delta u = 0 \\ \qquad\quad u(T) = f \end{cases}$$

In this example $\Omega = [0,1] \times [0,1]$, $a^2 = 2$ and the final time is $T = 0.625$. We choose the parameter $\gamma = 2$ for the iteration. In Figure 1 one can see the problem data f and the corresponding solution $u(0)$.

In Figure 2 the error $|\varphi_k - u(0)|$ is is shown after 10, 10^4, 10^5 and 10^6 iterations. One should note that the reconstruction error is smaller at the part of the domain where $u(0)$ is smooth. In Table 1 the evolution of the relative error in the L^2–norm of the iteration is shown. Note that the convergence speed decays exponentially as we iterate.

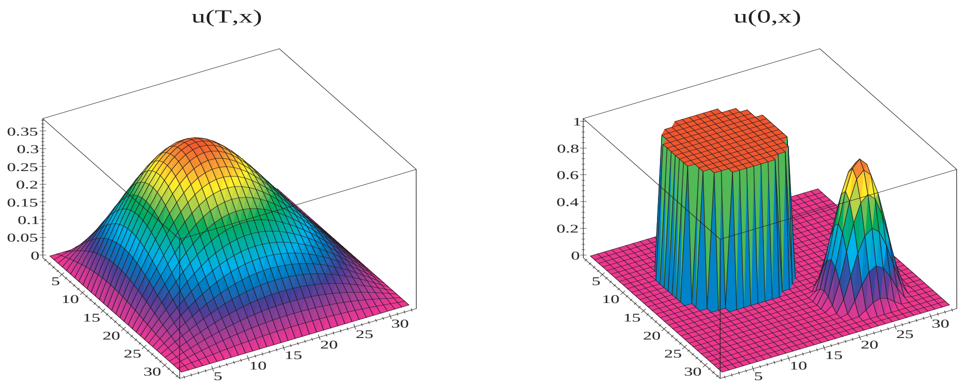

Figure 1: *Problem data* $u(T) = f$ *and corresponding* $u(0)$.

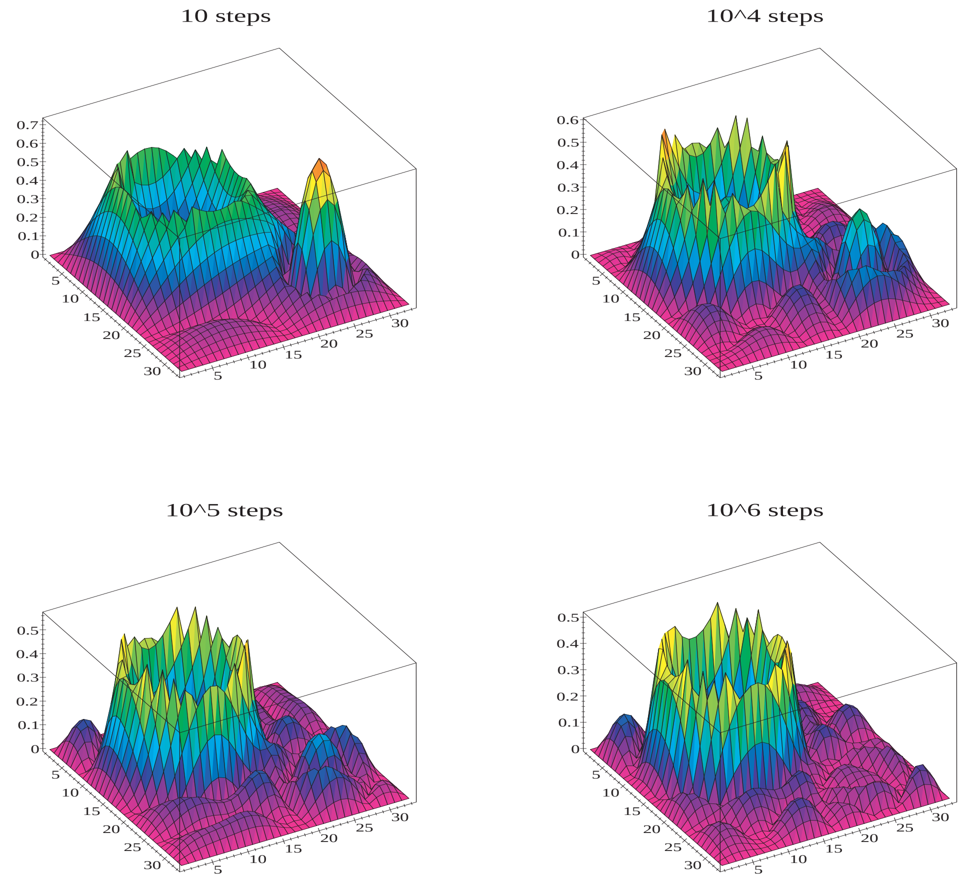

Figure 2: *Evolution of the error* $|\varphi_k - u(0)|$.

10 *steps*	10^3 *steps*	10^4 *steps*	10^5 *steps*	10^6 *steps*
49.8%	*42.2%*	*40.1%*	*36.2%*	*31.4%*

Table 1: *Evolution of the relative error in the L^2–norm.*

Example 13 *Consider the problem of finding $w(0) \in L^2(\Omega)$, where w solves:*

$$\begin{cases} \partial_t w \; - \; \Delta w = 0 \\ \qquad\quad w(T) = f \end{cases}$$

The set Ω is the same as in the previous example, the final time is $T = 0.625$ and we chose the parameter $\gamma = 2$ for the iteration. We solve the direct Cauchy-problem for two different initial conditions, which are shown in Figures 3 and 4 respectively.

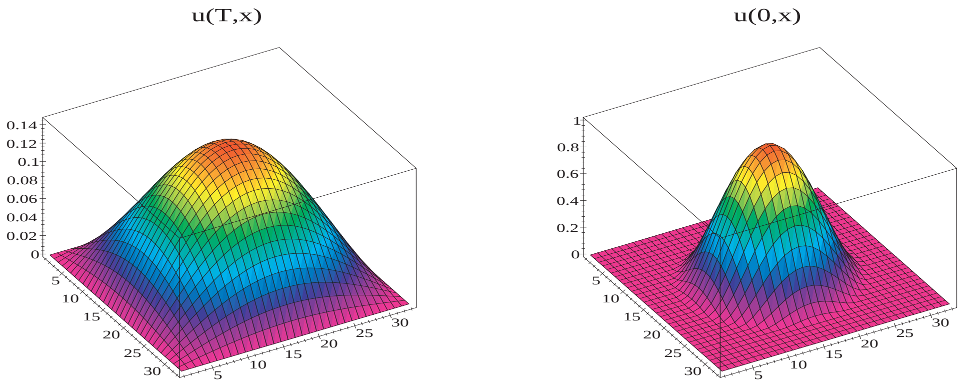

Figure 3: *First choice of $f = u(T)$ and corresponding $u(0)$.*

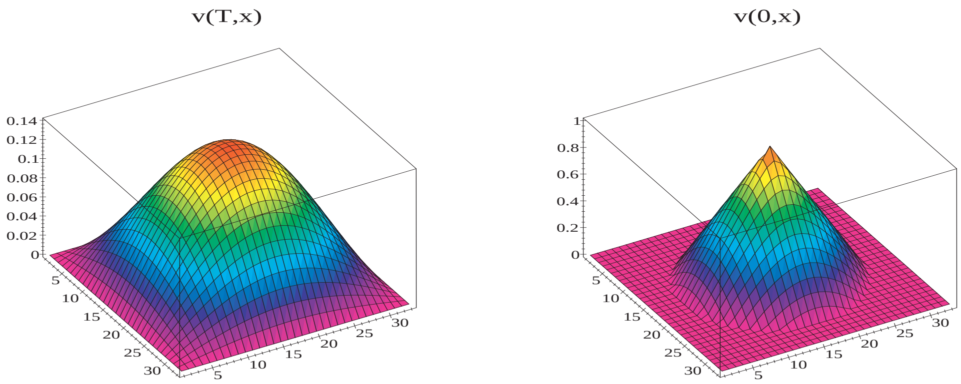

Figure 4: *Second choice of $f = v(T)$ and corresponding $v(0)$.*

In Figure 5 the evolution of the iteration error for both problems is shown after 10, 10^3 *and* 10^5 *steps.*

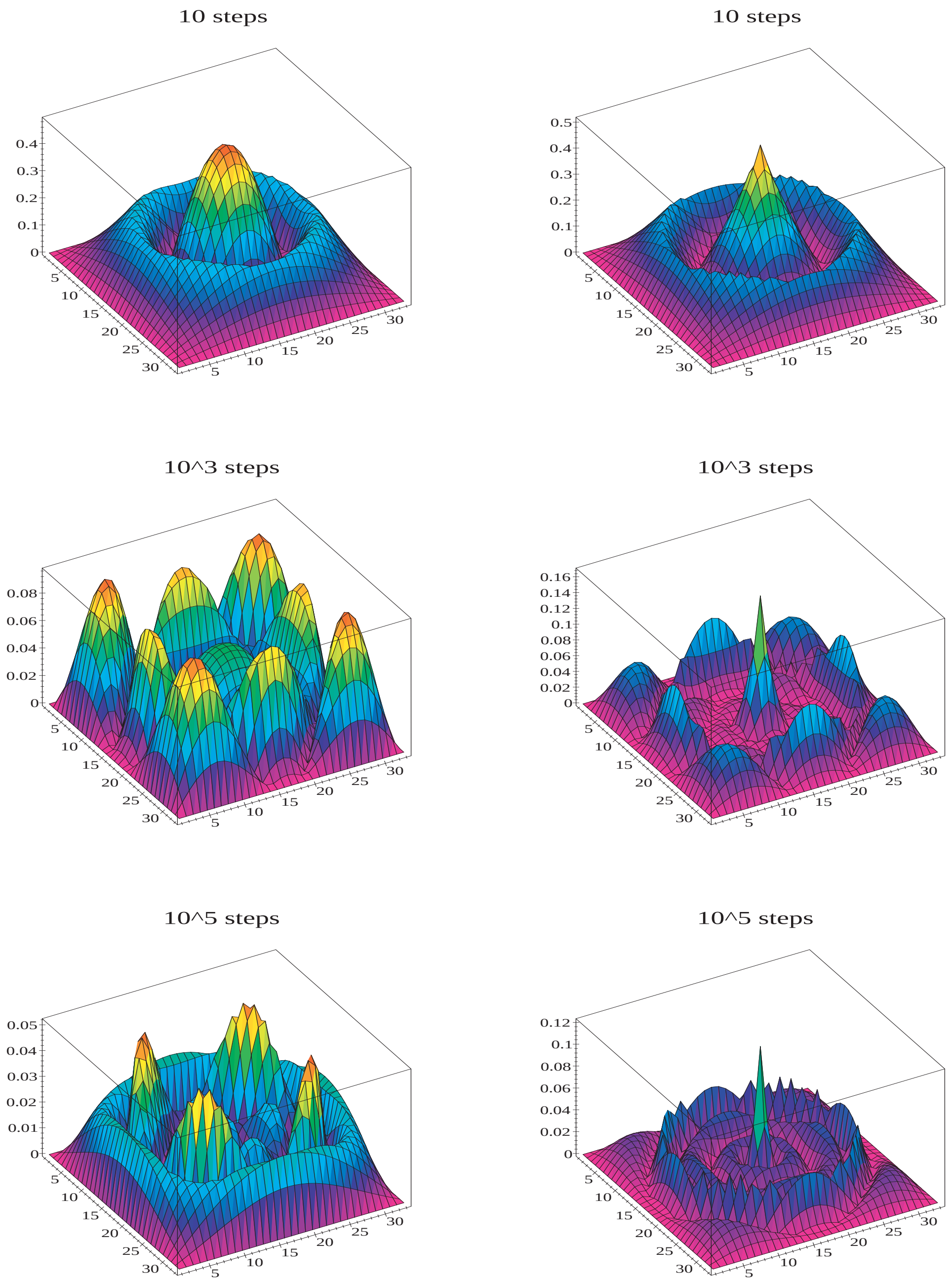

Figure 5: *Evolution of the error for the boundary conditions $u(T)$ and $v(T)$ respectively.*

In Table 2 we present the evolution of the relative error in the L^2–norm for both boundary conditions $u(T)$ and $v(T)$.

	10 *steps*	10^2 *steps*	10^3 *steps*	10^4 *steps*	10^5 *steps*
$f = u(T)$	*53.5%*	*37.9%*	*15.0%*	*12.1%*	*6.6%*
$f = v(T)$	*46.1%*	*32.2%*	*10.6%*	*9.1%*	*6.7%*

Table 2: *Evolution of the relative error in the L^2–norm.*

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