SEMIGLOBAL SOLVABILITY OF A CLASS OF PLANAR VECTOR FIELDS OF INFINITE TYPE

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Abstract

We consider a class of planar vector fields having the unit circle as characteristic set. We assume that each vector field L is of infinite type on the unit circle. We study the C^ω and C^∞ solvability of the equation $Lu = f$ near the characteristic set.

Resumo

Consideramos uma classe de campos vetoriais no plano tendo a circunferência unitária como conjunto característico. Supomos que cada campo vetorial L é de tipo infinito nos pontos do conjunto característico. Estudamos a resolubilidade C^ω and C^∞ da equação $Lu = f$ perto do conjunto caracteristico.

1. Introduction

In this note we study the C^ω and C^∞ solvability of the equation

$$T_\lambda u = f \tag{1.1}$$

near the characteristic circle $\Sigma = \{0\} \times S^1 \subset \mathbb{R} \times S^1$, where

$$T_\lambda = \lambda \frac{\partial}{\partial \theta} - ir\frac{\partial}{\partial r}, \tag{1.2}$$

$\lambda = a + ib \in \mathbb{R}^* + i\mathbb{R}$, and (r, θ) are the coordinates of $\mathbb{R} \times S^1$. Note that since the change of variables $r' = r$, $\theta' = -\theta$ transforms T_λ into $T_{-\lambda}$, then it suffices to study (1.1) only for $\mathrm{R}e\lambda > 0$. The vector field T_λ is of infinite type along Σ and it satisfies the Nirenberg-Treves condition (P). It follows from classical results (see [NT], [T1] or [T2]) that for a given C^ω (resp. C^∞) function f and for every $p \in \Sigma$, there exist an open set $U \ni p$, $U \subset \mathbb{R} \times S^1$, and a function $u \in C^\omega(U)$

(resp. $u \in C^\infty(U)$) such that (1.1) holds in U. The problem is therefore relevant when one seeks solutions in a full neighborhood of Σ. Equation (1.1) has in fact been studied ([BM2]) in the C^0 category. By using Fourier series, it is easy to see that the period of f on Σ must be zero in order for a C^0 solution to exist. Thus from now on, we will assume that f satisfies

$$\int_0^{2\pi} f(0,\theta)d\theta = 0. \tag{1.3}$$

Equation (1.1) has also been studied in [M3]. It is proved, in particular, that if $\lambda \notin \mathbb{Q}$, f is C^∞ and satisfies (1.3), then for every $k \in \mathbb{Z}^+$, equation (1.1) has a C^k solution defined in a neighborhood of Σ.

In section 2, we study the problem of finding analytic solutions when f is analytic. It turns out that analytic solutions exist for every analytic function f satisfying (1.3) if and only if λ is not well approximable by rational numbers (exponential Liouville number). In section 3, we prove that for every λ, there exist C^∞ functions f satisfying (1.3) such that equation (1.1) has no C^∞ solutions. The C^∞ solvability in the region $r \geq 0$ is also addressed. The approach and the motivation for this work are related to those contained in the papers [B1,2], [BCH], [BCM], [BHS], [BM 1,2], [CH], [M1,2,3], [T1,2] and in many others.

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2. Analytic solvability and Liouville numbers

We seek analytic solvability of (1.1) when f is real analytic and satisfies (1.3). The solvability will depend on whether the invariant λ satisfies a diophantine condition. It must be noted that, when $\lambda \notin \mathbb{R}$, the analytic solvability is contained in [BM2] and that, when $\lambda = a \in \mathbb{Q}^+$, there are analytic functions f satisfying (1.3) for which (1.1) does not have analytic solutions (in fact not even formal solutions). The only remaining case is therefore when $\lambda = a \in \mathbb{R}^+ \backslash \mathbb{Q}$.

It turns out that (1.1) is solvable in the analytic category if and only if λ is not too well approximable by rationals (see [G] and [B1], [B2]). We now describe a diophantine condition for $a \in \mathbb{R}^+$

$$\begin{array}{lll} (\mathrm{DC})_1 & \exists C > 0 \ \ |\exp(i\frac{2\pi j}{a}) - 1| \geq C^{j+1} & \forall j \in \mathbb{Z}^+. \\ (\mathrm{DC})_2 & \exists C > 0 \ \ |j + ak| \geq C^{j+1} & \forall j \in \mathbb{Z}^+ \forall k \in \mathbb{Z}. \end{array}$$

Lemma 2.1. *For $a \in \mathbb{R}^+ \backslash \mathbb{Q}$, the diophantine conditions (DC)$_1$ and (DC)$_2$ are equivalent.*

Proof: Suppose that (DC_2) does not hold. Then,

$$\forall l \in \mathbb{Z}^+, \ \exists j_l \in \mathbb{Z}^+, \ k_l \in \mathbb{Z} : \ \ |j_l + ak_l| < l^{-(j_l+1)}.$$

For each l we have then

$$\begin{aligned} \left|\exp(i\tfrac{2\pi j_l}{a}) - 1\right|^2 &= \left|\exp(i\tfrac{2\pi(j_l+ak_l)}{a}) - 1\right|^2 \\ &= 2\left(1 - \cos\tfrac{2\pi(j_l+ak_l)}{a}\right) \\ &= 2\tfrac{2\pi(j_l+ak_l)}{a}\sin\theta_l \end{aligned}$$

for some θ_l. It follows immediately that (DC_1) does not hold.

Conversely, suppose that (DC_1) does not hold. Then

$$\forall l \in \mathbb{Z}^+, \ \exists j_l \in \mathbb{Z}^+ \ : \ \ \left|\exp(i\frac{2\pi j_l}{a}) - 1\right| < l^{-(j_l+1)}.$$

For each l, let $k_l = [\frac{j_l}{a}]$ be the integral part of $\frac{j_l}{a}$. It follows from the above assumption that

$$\left|\exp(i\frac{2\pi(j_l - ak_l)}{a}) - 1\right| < l^{-(j_l+1)}.$$

Since $0 \leq \frac{j_l}{a} - k_l < 1$, we have $\lim(j_l - ak_l) = 0$. Consequently,

$$\left|\exp(i\frac{2\pi(j_l - ak_l)}{a}) - 1\right|^2 = 2\left(1 - \cos(\frac{2\pi(j_l - ak_l)}{a})\right) \geq (\frac{2\pi(j_l - ak_l)}{a})^2,$$

for l large, and thus (DC_2) does not hold.

$\square$

As in [B2], we make the following definition for an irrational number. An irrational number α is said to be an exponential Liouville number if there exists $\epsilon > 0$ such that the inequality

$$|\alpha - \frac{p}{q}| \leq \exp(-\epsilon q) \tag{2.1}$$

has infinitely many rational solutions p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$. The following lemmas are easy to prove.

Lemma 2.2. *The number $a \in \mathbb{R}^+ \backslash \mathbb{Q}$ is an exponential Liouville number if and only if the diophantine condition (DC) does not hold.*

Lemma 2.3. *The number $a \in \mathbb{R}^+ \backslash \mathbb{Q}$ satisfies (DC) if and only if $\frac{1}{a}$ satisfies (DC).*

For a construction of exponential Liouville numbers by means of continued fractions see [B2] and [GPY]. In [G] one finds an example of a Liouville number which is not exponential Liouville; recall that an irrational number α is said to be a Liouville number if for every positive integer N there exists $K \geq 0$ such that the inequality $|\alpha - p/q| \leq Kq^{-N}$ has infinitely many rational solutions p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$

We are now ready to state the main result of this section.

Theorem 2.1. *Let $\lambda = a \in \mathbb{R}^+ \backslash \mathbb{Q}$. Equation (1.1) has a real analytic solution u defined near Σ for every real analytic function f satisfying (1.3) if and only if the invariant a is not an exponential Liouville number.*

Proof: Let $f(r, \theta)$ be a real analytic function defined near Σ. We use the Taylor expansion about $r = 0$ and write

$$f(r, \theta) = \sum_{j \geq 0} f_j(\theta) \frac{r^j}{j!}, \tag{2.2}$$

where each $f_j \in C^\omega(S^1)$. If $u(r, \theta)$ is a real analytic solution of (1.1), then

$$u(r, \theta) = \sum_{j \geq 0} u_j(\theta) \frac{r^j}{j!}, \tag{2.3}$$

where each $u_j \in C^\omega(S^1)$ satisfies the ode

$$a\frac{du_j}{d\theta} - iju_j = f_j. \tag{2.4$_j$}$$

Note that since f satisfies (1.3), i.e.,

$$\int_0^{2\pi} f_0(\theta)d\theta = 0, \tag{2.5}$$

each equation $(2.4)_j$ can be solved with

$$u_0(\theta) = \frac{1}{a}\int_0^\theta f_0(s)ds + K_0, \tag{2.6$_0$}$$

where $K_0 \in \mathbb{C}$ is arbitrary, and for $j > 0$, the solution u_j is uniquely determined by the formula

$$u_j(\theta) = \frac{1}{a}\int_0^\theta \exp[i\frac{j(\theta - s)}{a}]f_j(s)ds \,+\, K_j \exp(i\frac{j\theta}{a}), \tag{2.6$_j$}$$

where

$$K_j = \left[1 - \exp(i\frac{2\pi j}{a})\right]^{-1}\frac{1}{a}\int_0^{2\pi} \exp[i\frac{j(2\pi - s)}{a}]f_j(s)ds. \tag{2.7$_j$}$$

Note that K_j is well defined since $a \notin \mathbb{Q}$.

For a real analytic function $f(r,\theta)$ as in (2.3) and satisfying (2.5), the series (2.3), where u_j is given by (2.6_j), defines a formal solution of (1.1). We will show that when the invariant a satisfies (DC), the series (2.3) defines a real analytic function near Σ. Before we proceed with the estimation of $|u_j(\theta)|$, recall that since f is real analytic, then there exists $C_1 > 0$ such that

$$|f_j(\theta)| \leq C_1^{j+1}, \quad \forall j \in \mathbb{Z}^+,\ \forall\theta \in S^1. \tag{2.8$_j$}$$

In view of $(DC)_1$, we have

$$\left|\left[1 - \exp(i\frac{2\pi j}{a})\right]^{-1}\right| \leq C^{-(j+1)} \quad \forall j \in \mathbb{Z}^+. \tag{2.9$_j$}$$

We also have

$$\left|\frac{1}{a}\int_0^{2\pi} \exp[i\frac{j(2\pi - s)}{a}]f_j(s)ds \exp(i\frac{j\theta}{a})\right| \leq \frac{2\pi}{a}C_1^{j+1} \leq C_2^{j+1}, \quad \forall j \in \mathbb{Z}^+, \tag{2.10$_j$}$$

for some $C_2 > 0$. Similarly, we can show that

$$\left| \frac{1}{a} \int_0^\theta \exp[i \frac{j(\theta - s)}{a}] f_j(s) ds \right| \leq C_3^{j+1}, \quad \forall j \in \mathbb{Z}^+, \tag{2.11_j}$$

for some $C_3 > 0$. Hence, it follows at once from (2.6), (2.9), (2.10), and (2.11), that

$$|u_j(\theta)| \leq C^{-(j+1)} C_2^{j+1} + C_3^{j+1} \leq C_4^{j+1}, \tag{2.12_j}$$

for some $C_4 > 0$. If we let $\hat{\theta} = \theta + i\theta'$ and $\hat{r} = r + ir'$ with $\theta', r' \in \mathbb{R}$, $|\theta'| < \epsilon$, $|r'| < \epsilon$, then it follows, after complexifying f and the formal series (2.3), that estimates (2.12_j) remain valid for the complexified functions $u_j(\hat{\theta})$ (with possibly different constants and small ϵ). It follows then that the series $\sum u_j(\hat{\theta})\hat{r}^j/j!$ converges uniformly to a holomorphic function in the variables $(\hat{\theta}, \hat{r})$. The restriction of this holomorphic function to $\theta' = r' = 0$ defines a real analytic function near Σ.

We prove the necessity of (DC) by contradiction. When (DC) fails, we construct a real analytic function f satisfying (1.3) such that equation (1.1) has no real analytic solution. Suppose then that $(DC)_2$ does not hold. We can find a sequence

$$\{(j_l, k_l)\}_{l \in \mathbb{Z}^+} \subset \mathbb{Z}^+ \times \mathbb{Z}^+$$

such that

$$|j_l - ak_l| < l^{-(j_l+1)} \quad \forall l \in \mathbb{Z}^+. \tag{2.13}$$

We have then $\lim(j_l/k_l) = a$ and this implies that we may take j_l to be an increasing sequence with $\lim j_l = \infty$. We can also assert the existence of constants $c_1, c_2 > 0$ such that

$$c_1 j_l \leq k_l \leq c_2 j_l, \quad \forall l \in \mathbb{Z}^+. \tag{2.14}$$

Now set

$$f(r, \theta) = \sum_{l \geq 0} \mathrm{e}^{ik_l\theta} r^{j_l}. \tag{2.15}$$

Condition (1.3) is trivially satisfied and f is continuous near Σ since $|\exp(ik_l\theta)| = 1$ for each l. Condition (2.14) guarantees that f is also real analytic. To see why, we can complexify r and θ into $\hat{r} = r + ir'$, $\hat{\theta} = \theta + i\theta'$ with $|r'| < \epsilon$, $|\theta'| < \epsilon$ and $\epsilon > 0$ small enough. The series

$$\sum_{l \geq 0} \mathrm{e}^{ik_l \hat{\theta}} \hat{r}^{j_l}$$

defines a continuous function near $\Sigma \subset \mathbb{C}^2$ and it is holomorphic with respect to $(\hat{r}, \hat{\theta})$. The holomorphy of the series follows from the estimates

$$\begin{aligned} \left| \sum_{l \geq 0} \mathrm{e}^{ik_l \hat{\theta}} \hat{r}^{j_l} \right| &\leq \sum_{l \geq 0} \mathrm{e}^{-\theta' k_l} |\hat{r}^{j_l}| \leq \sum_{l \geq 0} \mathrm{e}^{\epsilon k_l} |2\epsilon|^{j_l} \\ &\leq \sum_{l \geq 0} \mathrm{e}^{\epsilon(k_l - a^{-1} j_l)} |2\epsilon \mathrm{e}^{a^{-1}\epsilon}|^{j_l} \leq C \sum_{l \geq 0} |2\epsilon \mathrm{e}^{a^{-1}\epsilon}|^{j_l}, \end{aligned}$$

for some positive constant C (since $\lim(ak_l - j_l) = 0$).

We claim that for such a function f, equation (1.1) has no real analytic solution. Indeed, if $u(r, \theta)$ were such a C^ω solution, then a straightforward computation would give

$$u(r, \theta) = K + i \sum_{l \geq 0} \frac{1}{j_l - ak_l} \mathrm{e}^{ik_l \theta} r^{j_l}, \tag{2.16}$$

with $K \in \mathbb{C}$. But the above series has radius of convergence equal to 0 since it follows from (2.13) that

$$\left| \sum_{l \geq 0} \frac{1}{j_l - ak_l} \mathrm{e}^{ik_l \theta} r^{j_l} \right| \geq \sum_{l \geq 0} l^{j_l + 1} |r|^{j_l}. \tag{2.17}$$

3. Nonexistence of C^∞ solutions

We prove here that in general equation (1.1) does not have C^∞ solutions. More precisely, we have the following theorem.

Theorem 3.1. *Let $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$. Then there exists C^∞ functions f satisfying (1.3) so that equation (1.1) does not have C^∞ solutions in any neighborhood of Σ.*

Proof: For $\epsilon > 0$, let

$$\begin{aligned} A_\epsilon &= \{(r,\theta) \in \mathbb{R} \times S^1 :\ |r| < \epsilon\}; \\ A_\epsilon^+ &= \{(r,\theta) \in A_\epsilon :\ r > 0\}; \\ A_\epsilon^- &= \{(r,\theta) \in A_\epsilon :\ r < 0\}; \end{aligned} \tag{3.1}$$

The function $z = r^\lambda \mathrm{e}^{i\theta}$ is a first integral of T_λ in A_ϵ^+ and in A_ϵ^-. It is continuous on A_ϵ and maps $\overline{A_\epsilon^+}$ and $\overline{A_\epsilon^-}$ onto the disc $\overline{D(0,\epsilon^a)}$, sending Σ into 0. Since

$$T_\lambda \overline{z} = -2ia\overline{z}, \tag{3.2}$$

then the pushforward of the equations

$$T_\lambda u = f \quad \text{in} \quad A_\epsilon^\pm \tag{3.3}$$

give rise to the inhomogeneous CR equations

$$-2ia\overline{z}\frac{\partial \hat{u}^\pm}{\partial \overline{z}} = \hat{f}^\pm(z) \quad \text{in} \quad D(0,\epsilon^a)\backslash 0, \tag{3.4}$$

where $\hat{f}^\pm$ and $\hat{u}^\pm$ are the pushforwards of f and u via the map z. Let

$$\sum_{j\geq 0} \alpha_j z^j \tag{3.5}$$

be a series in one variable, with coefficients in $\mathbb{C}$, and with radius of convergence equal to zero. Let $g(z)$ be a C^∞ function in the disc $D(0,\epsilon^a)$ whose Taylor series at 0 coincides with the series (3.5). The function $\frac{\partial g(z)}{\partial \overline{z}}$ is then a C^∞ function in $D(0,\epsilon^a)$ and is flat at 0 (its partial derivatives of all orders vanish at 0). Define a function $f(r,\theta)$ in A_ϵ by

$$\begin{aligned} f(r,\theta) &= -2iar^{\overline{\lambda}}\mathrm{e}^{-i\theta}\tfrac{\partial g(z)}{\partial \overline{z}}(r^\lambda \mathrm{e}^{i\theta}) \quad \text{for } r \geq 0 \quad \text{and} \\ f(r,\theta) &= 0 \quad \text{for } r < 0. \end{aligned} \tag{3.6}$$

The function f is C^∞ in A_ϵ and is flat along the circle Σ. Condition (1.3) is trivially satisfied for this function.

Now we claim that for such a function f, equation (1.1) does not have a C^∞ solution in any neighborhood of Σ. To see why, we transfer equations (3.3) via the first integral z and get the CR equations

$$\frac{\partial \hat{u}^+}{\partial \overline{z}} = \frac{\partial g(z)}{\partial \overline{z}} \quad \text{and} \quad \frac{\partial \hat{u}^-}{\partial \overline{z}} = 0 \quad \text{in } D(0,\epsilon^a). \tag{3.7}$$

It follows at once that

$$\hat{u}^+(z) = g(z) + h^+(z) \quad \text{and} \quad \hat{u}^-(z) = h^-(z),$$

where $h^\pm$ are holomorphic functions in the disc. Hence any solution of (1.1) in A_ϵ has the form

$$\begin{aligned} u(r,\theta) &= g(r^\lambda \mathrm{e}^{i\theta}) + h^+(r^\lambda \mathrm{e}^{i\theta}) \qquad \text{for } r \geq 0 \\ u(r,\theta) &= h^-(r^\lambda \mathrm{e}^{i\theta}) \qquad \text{for } r < 0 \end{aligned} \tag{3.8}$$

But such a function cannot be C^∞ on Σ since the Taylor series at 0 of g diverges and thus cannot be equal to that of the holomorphic function $(h^- - h^+)$

□

The above construction can be used to show the nonexistence of C^∞ solutions in $\overline{A}_\epsilon^+$ when $\lambda \notin \mathbb{Z}$.

Theorem 3.2. *If $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$ and $\lambda \notin \mathbb{Z}$, then there exist C^∞ functions f in A_ϵ satisfying (1.3) such that there is no C^∞ function u defined in A_ϵ and satisfying*

$$T_\lambda u = f \quad \text{in } A_\epsilon^+. \tag{3.9}$$

Proof: First note that if $\lambda \notin \mathbb{Q}$, then

$$r^{\lambda j} \mathrm{e}^{ij\theta} \notin C^\infty(A_\epsilon) \quad \forall j \in \mathbb{Z}^+, \tag{3.10}$$

and if $\lambda = \frac{p}{q} \in \mathbb{Q}^+$ with p, q relatively prime and $q > 1$, then

$$r^{\lambda j} \mathrm{e}^{ij\theta} \notin C^\infty(A_\epsilon) \quad \forall j \in \mathbb{Z}^+ \backslash q\mathbb{Z}^+. \tag{3.11}$$

Let f and g be as in the proof of Theorem 3.1. If $u \in C^\infty(A_\epsilon)$ satisfies (3.9), then

$$u(r,\theta) = g(r^\lambda \mathrm{e}^{i\theta}) + h^+(r^\lambda \mathrm{e}^{i\theta}) \quad \text{in } A_\epsilon^+ \tag{3.12}$$

with h^+ holomorphic.

Now suppose that $\lambda \notin \mathbb{Q}$. We show that any function u defined by (3.12) cannot be C^∞ on Σ. For this, it suffices to notice that since the series (3.5) has radius of convergence 0, then for every holomorphic function

$$h^+(z) = \sum_{j \geq 0} c_j z^j \tag{3.13}$$

there exists $n > 0$ such that

$$\alpha_n + c_n \neq 0. \tag{3.14}$$

Let $N \in \mathbb{Z}^+$ such that $N > n$. We have

$$u(r,\theta) = \sum_{j=0}^{N} (\alpha_j + c_j) r^{\lambda j} \mathrm{e}^{ij\theta} \; + \; O(|r|^{Na}). \tag{3.15}$$

It follows at once from (3.14) and (3.10) that

$$u \notin C^{[an]+1}(\overline{A}_\epsilon^+), \tag{3.16}$$

where $[x]$ denotes the integral part of the real number x. When $\lambda = p/q \in \mathbb{Q}^+ \backslash \mathbb{Z}$, the above argument works as well if one replaces the divergent series $\sum \alpha_j z^j$ by the series $\sum \alpha_j z^{qj+1}$ for example.

$\square$

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