GLOBAL ANALYTIC HYPOELLIPTICITY AND PSEUDOPERIODIC FUNCTIONS

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Abstract

We show that a class of overdetermined systems on the 3-torus associated to a closed 1-form c on the 2-torus $\mathbb{T}^2 \simeq \mathbb{R}^2/2\pi\mathbb{Z}^2$ is globally analytic hypoelliptic if and only if every $B : \mathbb{R}^2 \to \mathbb{R}$ such that $dB = \Im c$ is an open map at each point.

Resumo

Mostramos que uma classe de sistemas sobre-determinados no toro $\mathbb{T}^3$ que está associada a uma 1-forma fechada c no toro $\mathbb{T}^2 \simeq \mathbb{R}^2/2\pi\mathbb{Z}^2$ é globalmente analítica hipoelítica se, e somente se, toda $B : \mathbb{R}^2 \to \mathbb{R}$ tal que $dB = \Im c$ é uma aplicação aberta em cada ponto

1. Introduction

We consider a system $\mathbb{L} = (L_1, L_2)$ of complex vector fields on the torus $\mathbb{T}^3 \simeq \mathbb{R}^3/2\pi\mathbb{Z}^3$ of the form

$$L_j = \frac{\partial}{\partial t_j} + c_j(t)\frac{\partial}{\partial x},\ j = 1, 2, \tag{1.1}$$

where each c_j is a complex-valued, real analytic function defined on the torus $\mathbb{T}^2$.

Let $c = c_1\,dt_1 + c_2\,dt_2$ and write $c = a + ib$, $a = a_1\,dt_1 + a_2\,dt_2$, and $b = b_1\,dt_1 + b_2\,dt_2$, with each a_j, b_j real-valued.

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Define $C : \mathbb{R}^2 \to \mathbb{C}$ by $C(t) = \int_0^t c$ and write $C = A + iB$, where $A(t) = \int_0^t a$ and $B(t) = \int_0^t b$.

Recall that a system $\mathbb{L}$ as in (1.1) is formally integrable (see [T]) if $\frac{\partial c_1}{\partial t_2} = \frac{\partial c_2}{\partial t_1}$ or, equivalently, that the 1-form

$$c = c_1 dt_1 + c_2 dt_2$$

is closed. In this case, we set for $j = 1, 2$, $c_{j0} = (2\pi)^{-1} \int_0^{2\pi} c_j(t)\, dt_j$, $a_{j0} = (2\pi)^{-1} \int_0^{2\pi} a_j(t)\, dt_j$, and $b_{j0} = (2\pi)^{-1} \int_0^{2\pi} b_j(t)\, dt_j$.

We need two more definitions that are taken, along with some important results, from Arnold's article [A].

Definition 1.1. *We say that a function $H : \mathbb{R}^2 \to \mathbb{R}$ is a pseudoperiodic function if $H(t + \tau) = H(t) + \omega(\tau)$, for all $t \in \mathbb{R}^2$ and $\tau \in 2\pi\mathbb{Z}^2$, where $\omega : 2\pi\mathbb{Z}^2 \to \mathbb{R}$ is a monomorphism.*

In the case where $H = B$ as above, B is a pseudoperiodic function if, and only if, the periods b_{10}, b_{20} of the closed 1-form b are incommensurable, that is, linearly independent over $\mathbb{Q}$. In that case we may write $B(t) = P(t) + b_{10}t_1 + b_{20}t_2$ with P 2π-periodic in each variable.

Definition 1.2. *We say that a smooth function $H : \mathbb{R}^2 \to \mathbb{R}$ is in general position if it has no degenerate critical point and has all critical values distinct.*

Our main goal is to give a characterization of those systems $\mathbb{L}$ which are *globally analytic hypoelliptic*; this means that the conditions $u \in \mathcal{D}'(\mathbb{T}^3)$, $f_1, f_2 \in C^\omega(\mathbb{T}^3)$ and

$$L_j u = f_j,\ j = 1, 2 \tag{1.2}$$

imply $u \in C^\omega(\mathbb{T}^3)$.

Recall the local version of this property: $\mathbb{L}$ is *analytic hypoelliptic* if, for every open subset $U \subseteq \mathbb{T}^3$, the conditions $u \in \mathcal{D}'(U)$, $L_1u, L_2u \in C^\omega(U)$ imply $u \in C^\omega(U)$.

Our main result is as follows.

Theorem 1.3. *Let $\mathbb{L}$ be as in (1.1) a formally integrable system. We assume that $B = \int_0^t b$ is a pseudoperiodic function in general position. Then each of the following properties is equivalent to the other two:*
(i) $\mathbb{L}$ is globally analytic hypoelliptic;
(ii) $\mathbb{L}$ is analytic hypoelliptic;
(iii) B is an open map at each point.

In fact (ii) is equivalent to (iii) by [BT] and these are also equivalent to the hypocomplexity of $\mathbb{L}$ (see [T]). Note that the implication $(ii) \Rightarrow (i)$ is trivial. Therefore all we have to do is to prove that $(i) \Rightarrow (iii)$.

In order to prove $(i) \Rightarrow (iii)$ we assume that (iii) does not hold and show that (1.2) has singular solutions, that is, there exist $u \in \mathcal{D}'(\mathbb{T}^3) \setminus C^\omega(\mathbb{T}^3)$ and $f_1, f_2 \in C^\omega(\mathbb{T}^3)$ such that $L_j u = f_j$, $j = 1, 2$.

In section 2 we show how to do this in a special case; we use the method of stationary phase. In section 3 we prove that the general case can be reduced to the special case of section 2, by means of diffeomorphisms of the torus; we use in a essential way results of [A], especially the existence of a transversal to the level sets of B.

2. A special case

Proposition 2.1. *Let $\mathbb{L}$ be as in (1.1) a formally integrable system. We assume that $B = \int_0^t b$ is a pseudoperiodic function in general position. In addition suppose that $b_{20} < b_{10} < 0$, $b(0,0) = 0$, $C(0,0) = 0$, and the maximum of B over $[0, 2\pi]^2$ is not attained at the boundary. Then $\mathbb{L}$ is not globally analytic hypoelliptic.*

Proof: Let

$$M \doteq \max_{[0,2\pi]^2} B = B(t^*) > 0 \tag{2.1}$$

where $t^* = (t_1^*, t_2^*) \in (0, 2\pi)^2$.

Let $M' = \max_{t_1 \in [0,2\pi]} B(t_1, 0)$ and $M'' = \max_{t_2 \in [0,2\pi]} B(0, t_2)$. Then we have $M' < M$ and $M'' < M$.

If $u \in \mathcal{D}'(\mathbb{T}^3)$ is a solution to (1.2) then the compatibility condition $L_1 f_2 = L_2 f_1$ must be satisfied; set

$$h \doteq L_1 f_2 = L_2 f_1. \tag{2.2}$$

Our plan is to choose $h \in C^\omega(\mathbb{T}^3)$ and obtain f_1, f_2 from (2.2) and then obtain u from (1.2). More precisely we will look for h, f_1, f_2 and u in the form of a partial Fourier series in the x-variable, as follows:

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi}\sum_{n=1}^{\infty} \hat{u}(t,n)e^{inx} \\ f_j(t,x) &= \frac{1}{2\pi}\sum_{n=1}^{\infty} \hat{f}_j(t,n)e^{inx},\ j=1,2 \\ h(t,x) &= \frac{1}{2\pi}\sum_{n=1}^{\infty} \hat{h}(t,n)e^{inx}. \end{aligned}$$

If we set

$$L_{jn} = \frac{\partial}{\partial t_j} + inc_j(t),\ j = 1,2$$

we are led to the equations

$$L_{1n}\hat{u}(t,n) = \hat{f}_1(t,n),\ n \geq 1, \tag{2.3}$$

$$L_{2n}\hat{u}(t,n) = \hat{f}_2(t,n),\ n \geq 1, \tag{2.4}$$

$$L_{1n}\hat{f}_2(t,n) = \hat{h}(t,n),\ n \geq 1, \text{ and} \tag{2.5}$$

$$L_{2n}\hat{f}_1(t,n) = \hat{h}(t,n),\ n \geq 1. \tag{2.6}$$

Since $b_{10} \neq 0$ and $b_{20} \neq 0$ we can conclude that for every $n \geq 1$, each of the equations (2.3)-(2.6) has a unique 2π-periodic solution for arbitrary 2π-periodic right-hand sides. Furthermore if the right-hand side is real analytic so is the solution. In fact we can write formulas for such solutions, namely

$$\hat{u}(t,n) \doteq d_{1n}\int_0^{2\pi} e^{-in\{C(t_1,t_2)-C(t_1-s_1,t_2)\}}\hat{f}_1(t_1-s_1,t_2,n)\,ds_1$$

$$\hat{u}(t,n) = d_{2n}\int_0^{2\pi} e^{-in\{C(t_1,t_2)-C(t_1,t_2-s_2)\}}\hat{f}_2(t_1,t_2-s_2,n)\,ds_2$$

$$\hat{f}_1(t,n) \doteq d_{2n}\int_0^{2\pi} e^{-in\{C(t_1,t_2)-C(t_1,t_2-s_2)\}}\hat{h}(t_1,t_2-s_2,n)\,ds_2 \tag{2.7}$$

$$\hat{f}_2(t,n) \doteq d_{1n}\int_0^{2\pi} e^{-in\{C(t_1,t_2)-C(t_1-s_1,t_2)\}}\hat{h}(t_1-s_1,t_2,n)\,ds_1, \tag{2.8}$$

where $d_{jn} = (1 - e^{-2\pi i n c_{j0}})^{-1}$, $j = 1, 2$, for $n \geq 1$.

Note that there exists $\kappa > 1$ such that

$$\kappa^{-1} < |d_{jn}| < \kappa, \; j = 1, 2, \; n \geq 1. \tag{2.9}$$

We can also write

$$\hat{u}(t, n) = d_{1n} d_{2n} \int_0^{2\pi} \int_0^{2\pi} e^{-in\{C(t) - C(t-s)\}} \, \hat{h}(t - s, n) \, ds_1 \, ds_2. \tag{2.10}$$

Set $\psi(t) \doteq \{M + K[2 - \cos t_1 - \cos t_2]\} + i\{a_1(0) \sin t_1 + a_2(0) \sin t_2 - A(t^*)\}$ where M is as in (2.1) and $K > 0$ will be chosen later on.

We choose $\hat{h}(t, n) = e^{-n\psi(t)}$, $n \geq 1$.

We have, for any $K > 0$, $|\hat{h}(t, n)| \leq e^{-Mn}$, $n \geq 1$, $t \in [0, 2\pi]^2$, which implies, for some $0 < M_1 < M$ and some $\delta > 0$, $|\hat{h}(t + it', n)| \leq e^{-M_1 n}$, $n \geq 1$, $t \in [0, 2\pi]^2$, and $|t'| < \delta$. Therefore $h \in C^\omega(\mathbb{T}^3)$.

Formulas (2.7), (2.8), (2.10) become

$$\hat{f}_1(t, n) = d_{2n} \int_0^{2\pi} e^{-n\{i[C(t_1, t_2) - C(t_1, t_2 - s_2)] + \psi(t_1, t_2 - s_2)\}} \, ds_2, \tag{2.11}$$

$$\hat{f}_2(t, n) = d_{1n} \int_0^{2\pi} e^{-n\{i[C(t_1, t_2) - C(t_1 - s_1, t_2)] + \psi(t_1 - s_1, t_2)\}} \, ds_1, \tag{2.12}$$

$$\hat{u}(t, n) = d_{1n} d_{2n} \int_0^{2\pi} \int_0^{2\pi} e^{-n\{i[C(t) - C(t-s)] + \psi(t-s)\}} ds_1 \, ds_2. \tag{2.13}$$

Let $\phi(t, s) \doteq i[C(t) - C(t - s)] + \psi(t - s)$ be the factor appearing in the exponent in the integral, that is,

$$\begin{aligned} \phi(t, s) =& i[C(t) - C(t - s)] + M + K[2 - \cos(t_1 - s_1) - \cos(t_2 - s_2)] + \\ & + i[a_1(0) \sin(t_1 - s_1) + a_2(0) \sin(t_2 - s_2) - A(t^*)]. \end{aligned}$$

Let $\varphi(t, s) \doteq -\Re(\phi(t, s)) = B(t) - B(t - s) - M - K[2 - \cos(t_1 - s_1) - \cos(t_2 - s_2)]$.

We now make a detailed analysis of the values of $\varphi(t, s)$. We claim that $\varphi(t, s) \leq 0$, for t, $s \in [0, 2\pi]^2$.

Provided $K > 0$ is large, the main contribution to the value of (2.13), as $n \to \infty$, comes from small neighborhoods of points where $2 - \cos(t_1 - s_1) - \cos(t_2 - s_2) = 0$; thus we proceed to look at such points.

We claim that there exists $\delta_1 > 0$ such that $\varphi(t,s) \leq 0$ for all $t, s \in [0,2\pi]^2$ with $|t-s| < \delta_1$.

First observe that, for $t \in [0,2\pi]^2$, the function

$$\vartheta(u) \doteq \frac{B(t) - B(u) - M}{2 - \cos u_1 - \cos u_2}, \qquad u = (u_1, u_2)$$

has an upper bound when u is near 0, $u \neq 0$.

Indeed, since $\vartheta(u) \leq -B(u)/(2 - \cos u_1 - \cos u_2)$, $B(0) = 0$, $dB(0) = 0$, and $(u_1^2 + u_2^2)/\pi \leq 2 - \cos u_1 - \cos u_2 \leq (u_1^2 + u_2^2)/2$, for $|u_1|$, $|u_2| \leq \pi/2$, we see that

$$\vartheta(u) \leq -\frac{u_1^2 \partial_{t_1^2}^2 B(0,0) + 2u_1u_2\partial_{t_1t_2}^2 B(0,0) + u_2^2\partial_{t_2^2}^2 B(0,0) + R_3(u)}{2 - \cos u_1 - \cos u_2}$$

where $\lim\limits_{u_1, u_2 \to 0} \dfrac{R_3(u_1,u_2)}{u_1^2 + u_2^2} = 0$. Thus

$$\lim_{u_1, u_2 \to 0} \frac{R_3(u_1,u_2)}{2 - \cos u_1 - \cos u_2} = \lim_{u_1, u_2 \to 0} \frac{R_3(u_1,u_2)}{u_1^2 + u_2^2} \frac{u_1^2 + u_2^2}{2 - \cos u_1 - \cos u_2} = 0$$

and

$$\frac{-u_1^2 \partial_{t_1^2}^2 B(0,0) - 2u_1u_2\partial_{t_1t_2}^2 B(0,0) - u_2^2\partial_{t_2^2} B(0,0)}{2 - \cos u_1 - \cos u_2} \leq C$$

where $C = 2\pi \max\{|\partial_{t_1^2}^2 B(0)|, |\partial_{t_1t_2}^2 B(0)|, |\partial_{t_2^2}^2 B(0)|\}$.

Therefore, there exists $\delta_1 > 0$ and $K_1 > 0$ such that $\varphi(t,s) \leq 0$ whenever $|t_1 - s_1|, |t_2 - s_2| < \delta_1$ and $K \geq K_1$.

Note that when $|t_1 - s_1| = 2\pi$ and $|t_2 - s_2| = 0$ then, obviously, $t_2 = s_2$, and $t_1 = 2\pi$ and $s_1 = 0$ or else $t_1 = 0$ and $s_1 = 2\pi$. We have, for $t_2 \in [0,2\pi]$,

$$\begin{aligned}
\varphi(2\pi, t_2, 0, t_2) &= B(2\pi, t_2) - B(2\pi, 0) - M \\
&= P(2\pi, t_2) + 2\pi b_{10} + b_{20}t_2 - P(2\pi, 0) - 2\pi b_{10} - M \\
&= P(2\pi, t_2) + b_{20}t_2 - M = P(0, t_2) + b_{20}t_2 - M \\
&= B(0, t_2) - M \leq M'' - M < 0
\end{aligned}$$

and also

$$\begin{aligned}
\varphi(0, t_2, 2\pi, t_2) &= B(0, t_2) - B(-2\pi, 0) - M \\
&= P(0, t_2) + b_{20}t_2 - P(-2\pi, 0) + 2\pi b_{10} - M \\
&= P(0, t_2) + 2\pi b_{10} + b_{20}t_2 - M \\
&\leq P(0, t_2) + b_{20}t_2 - M = B(0, t_2) - M \leq M'' - M < 0.
\end{aligned}$$

When $|t_1 - s_1| = 0$ and $|t_2 - s_2| = 2\pi$ we have $t_1 = s_1$, $t_2 = 2\pi$ and $s_2 = 0$ or else $t_2 = 0$ and $s_2 = 2\pi$, and for $t_1 \in [0, 2\pi]$ it follows

$$\begin{aligned}
\varphi(t_1, 2\pi, t_1, 0) &= B(t_1, 2\pi) - B(0, 2\pi) - M \\
&= P(t_1, 2\pi) + b_{10}t_1 + b_{20}2\pi - 2\pi b_{20} - M \\
&= P(t_1, 0) + b_{10}t_1 - M = B(t_1, 0) - M \leq M' - M < 0, \\
\varphi(t_1, 0, t_1, 2\pi) &= B(t_1, 0) - B(0, -2\pi) - M \\
&= B(t_1, 0) + 2\pi b_{20} - M \leq B(t_1, 0) - M \leq M' - M < 0.
\end{aligned}$$

When $|t_2 - s_2| = |t_1 - s_1| = 2\pi$ we have

$$\begin{gathered}
t_1 = 0,\ s_1 = 2\pi,\ t_2 = 0,\ s_2 = 2\pi \\
t_1 = 2\pi,\ s_1 = 0,\ t_2 = 0,\ s_2 = 2\pi \\
t_1 = 0,\ s_1 = 2\pi,\ t_2 = 2\pi,\ s_2 = 0 \\
t_1 = 2\pi,\ s_1 = 0,\ t_2 = 2\pi,\ s_2 = 0
\end{gathered}$$

hence

$$\begin{aligned}
\varphi(0, 2\pi, 0, 2\pi) &= B(0, 2\pi) - B(0, 0) - M \leq M'' - M < 0, \\
\varphi(2\pi, 0, 0, 2\pi) &= B(2\pi, 0) - B(2\pi, -2\pi) - M \\
&= 2\pi b_{10} - 2\pi b_{10} + 2\pi b_{20} - M \leq -M < 0, \\
\varphi(0, 2\pi, 2\pi, 0) &= B(0, 2\pi) - B(-2\pi, 2\pi) - M \\
&= 2\pi b_{20} + 2\pi b_{10} - 2\pi b_{20} - M \leq -M < 0, \text{ and} \\
\varphi(2\pi, 0, 2\pi, 0) &= B(2\pi, 0) - B(0, 0) - M \leq 2\pi b_{10} - M \leq -M < 0.
\end{aligned}$$

Thus, there exists $\delta_2 > 0$ such that $\varphi(t, s) \leq 0$ whenever $|(t - s) - p_j| < \delta_2$ where $p_1 = (2\pi, 0)$, $p_2 = (0, 2\pi)$, $p_3 = (2\pi, 2\pi)$, $p_4 = (-2\pi, 0)$, $p_5 = (0, -2\pi)$, $p_6 = (-2\pi, -2\pi)$, $p_7 = (-2\pi, 2\pi)$ and $p_8 = (2\pi, -2\pi)$.

Finally, let

$$m \doteq \min\{2 - \cos u_1 - \cos u_2\,;\ |u| \geq \delta_1\,,\ |u - p_j| \geq \delta_2\ j = 1, \cdots, 8\} > 0$$

and choose $K > K_1$ large enough such that for $t, s \in [0, 2\pi]^2$ with $|t - s| \geq \delta_1$ and $|t - s - p_j| \geq \delta_2$, $j = 1, \cdots, 8$ we have

$$\varphi(t, s) \leq \max_{t,s \in [0,2\pi]^2} \{B(t) - B(t - s) - M\} - Km \leq 0.$$

We have shown that $\varphi(t, s) \leq 0,\ t, s \in [0, 2\pi]^2$ which implies, for some $C > 0$,

$$|\hat{u}(t, n)| \leq C,\ t, s \in [0, 2\pi]^2,\ n \geq 1$$

hence $u \in \mathcal{D}'(\mathbb{T}^3)$.

In (2.11) the relevant function is

$$\varphi(t_1, t_2, 0, s_2) = B(t) - B(t_1, t_2 - s_2) - M - K[2 - \cos t_1 - \cos(t_2 - s_2)].$$

It is easy to see that $\varphi(t', s') \leq M' - M$ for each point (t', s') such that $2 - \cos t'_1 - \cos(t'_2 - s'_2) = 0$. Furthermore, an argument similar to the one used above implies, for large $K > 0$, the existence of $\delta > 0$ such that $\varphi(t, s) \leq M' - M$ for each point (t, s) such that $|(t - s) - (t' - s')| < \delta$. Finally, if $|(t - s) - (t' - s')| \geq \delta$ we get $2 - \cos t_1 - \cos(t_2 - s_2) \geq 2 - 2\cos\delta > 0$; if we take $K > 0$ large we get $\varphi(t, s) \leq -K[1 - \cos\delta]$ for these values of t and s.

We conclude that

$$\varphi(t, s) \leq M' - M, \text{ for all } t, s \in [0, 2\pi]^2, \text{ with } s_1 = 0$$

which implies

$$|\hat{f}_1(t, n)| \leq e^{(M' - M)n},\ t \in [0, 2\pi]^2,\ n \geq 1$$

whence $f_1 \in C^\omega(\mathbb{T}^3)$.

Similarly, from (2.12), we get

$$|\hat{f}_2(t, n)| \leq e^{(M'' - M)n},\ t \in [0, 2\pi]^2,\ n \geq 1$$

which shows that $f_2 \in C^\omega(\mathbb{T}^3)$.

We now analyze the behavior of $\hat{u}(t^*, n)$, where $B(t^*) = M$ and so $\phi(t^*, t^*) = 0$. We are going to use the method of stationary phase (see [Sj]).

We have $\Re(\phi(t^*, s)) > 0$ if $s \neq t^*$ and $\Re(\phi(t^*, t^*)) = 0$.

Let

$$\hat{u}(t^*, n) = d_{1n} d_{2n} (I_n + J_n) \tag{2.14}$$

where

$$I_n \doteq \int\int_{|\sigma|<\delta} e^{-n\beta(\sigma)} d\sigma_1\, d\sigma_2, \quad J_n \doteq \int_{s\in[0,2\pi]^2,\, |s-t^*|\geq\delta} e^{-n\phi(t^*,s)} ds_1\, ds_2,$$

$\beta(\sigma) \doteq \phi(t^*, t^* - \sigma)$, $\sigma = t^* - s$, and $d\sigma_1\, d\sigma_2 = ds_1\, ds_2$.

It is clear that $|J_n|$ is exponentially decaying for any choice of $\delta > 0$ provided $K \geq K_1$ as before.

We observe that

$$\begin{aligned}\beta(\sigma) =& \phi(t^*, t^* - \sigma) = B(\sigma) + K[2 - \cos\sigma_1 - \cos\sigma_2] \\ & + i\{-A(\sigma) + a_1(0)\sin\sigma_1 + a_2(0)\sin\sigma_2\} \\ =& - iC(\sigma) + K[2 - \cos\sigma_1 - \cos\sigma_2] + i\{a_1(0)\sin\sigma_1 + a_2(0)\sin\sigma_2\}.\end{aligned}$$

Take $\delta > 0$ small so that β has a holomorphic extension to a neighborhood of

$$\{z = (z_1, z_2) = (\sigma_1 + i\tau_1, \sigma_2 + i\tau_2); |\sigma_j| \leq \delta,\ |\tau_j| \leq \delta,\ j = 1, 2\}$$

given by

$$\widetilde{\beta}(z) = \widetilde{\beta}(\sigma + i\tau) = -iC(z) + K[2 - \cos z_1 - \cos z_2] + i\{a_1(0)\sin z_1 + a_2(0)\sin z_2\}.$$

We observe that $\beta(0) = 0$, $\nabla\beta(0) = 0$, and

$$\left(\tfrac{\partial^2\beta}{\partial\sigma_j\partial\sigma_k}(0) \right)_{1\leq j,k\leq 2} = KI - i \left(\tfrac{\partial c_j}{\partial\sigma_k}(0) \right)_{1\leq j,k\leq 2}.$$

For $K > 0$ sufficiently large and for $\delta > 0$ sufficiently small it is clear that the origin is the only critical point of β; it is also clear that $z = 0$ is a nondegenerate critical point since

$$\begin{aligned}det \left(\tfrac{\partial^2\beta}{\partial\sigma_j\partial\sigma_k}(0) \right)_{1\leq j,k\leq 2} &= (K - i\frac{\partial c_1}{\partial\sigma_1}(0))(K - i\frac{\partial c_2}{\partial\sigma_2}(0)) + (\frac{\partial c_1}{\partial\sigma_2}(0))^2 \\ &= K^2\{1 + O(1/K)\}.\end{aligned}$$

We also have $\Re\beta(\sigma) > 0$ if $|\sigma| = \delta$.

We conclude that

$$I_n = (2\pi)^{-1}[det(KI - i(\frac{\partial c_j}{\partial \sigma_k}(0)))]^{-1} n^{-1}(1 + O(n^{-1})), \text{ as } n \to \infty,$$

which, together with (2.9), and (2.14) shows that $\hat{u}(t^*, n)$ is not exponentially decaying as $n \to \infty$, hence t^* is indeed in the t-projection of $ss_a(u)$.

3. Reduction to the special case

In this section we show that by means of real analytic diffeomorphisms, B as in theorem () can be taken as in proposition ().

We note first that we may assume $b_{20} < b_{10} < 0$ by using simple diffeomorphism of the torus.

We now recall some terminology and quote results from [A].

Proposition 3.1. (Arnold) *Suppose that H is a pseudoperiodic function in general position. Then we have:*

1. *Any superlevel $\{t; H(t) > c\}$ has exactly one unbounded component, denoted by N_c and this component contains a half-plane;*
2. *Any connected component of a level set of H passing through a critical point is either bounded (a point or a lemniscate-like curve) or it has the shape of a folium of Descartes.*

Note that in the unbounded case, a critical level set of H separates the plane into two unbounded components and a disk; the closure of the disk is called a *trap*. Thus, a trap is homeomorphic to a closed disk and has a critical point on the boundary, called the *vertex* of the trap.

Proposition 3.2. (Arnold) *Suppose that H is a pseudoperiodic function in general position. Then traps with distinct vertices are disjoint.*

A *normal curve* is a component of a nonsingular level set of H that does not intersect any trap.

Proposition 3.3. **(Arnold)** *Suppose that H is a pseudoperiodic function in general position. Then any normal curve is unbounded. Therefore each critical point lies in one, and only one, trap.*

Proposition 3.4. **(Arnold)** *Suppose that H is a pseudoperiodic function in general position. Then there exists a closed, smooth, non-selfintersecting curve σ on $\mathbb{T}^2$ such that the lifting, $\tilde{\sigma}$, does not intersect any trap and $H \circ \tilde{\sigma}$ is strictly monotone. Furthermore, we may assume that $H \circ \tilde{\sigma}$ is decreasing and has no singular points and $\sigma \sim \sigma_2$.*

By a theorem of Grauert and Remmert (theorem (5.1) of chapter 2 in [H]) we may replace σ, as in proposition (), by a real analytic loop while keeping transversality and $\sigma \sim \sigma_2$. By theorem (2.1) in [E], σ is isotopic to σ_2 and by theorem (1.3) of chapter 8 in [H], σ is diffeotopic to σ_2, that is, exists a smooth diffeomorphism of the torus sending σ onto σ_2. Applying Grauert-Remmert to this diffeomorphism we get a real analytic diffeomorphism which sends σ onto a real analytic loop σ', so close to σ_2, that it may be represented by the graph of a function of t_2. A further C^ω diffeomorphism straightens out this graph, and we finally have obtained a real analytic diffeomorphism of the torus sending σ onto σ_2; in other words, in the new coordinates — still denoted (t_1, t_2) — $\{t_1 = 0\}$ is transversal.

There exists $\delta > 0$ such that each unbounded connected component of a level set hits each vertical line $\{t_1 = 2k\pi + \delta'\}$, $k \in \mathbb{Z}$, $|\delta'| < \delta$, exactly once, always from the same side, which we may assume to be the left side.

Suppose that $P = (t_1, t_2)$ is a point of local extremum of B. There exists a unique unbounded connected component of a level set of B, denoted by $\mathcal{F}'$, such that P belongs to the trap associated to $\mathcal{F}'$, having t_0 as its vertex. By replacing P by one of its translates, $P_{kj} \doteq P + (2\pi k, 2\pi j)$, we may assume that $0 < t_1 < 2\pi$ and that $\mathcal{F}'$ crosses $t_1 = 0$ at a point $(0, \bar{t_2})$ with $0 \le \bar{t_2} < 2\pi$.

By means of the vertical translation $(t_1, t_2) \mapsto (t_1, t_2 - \bar{t_2})$ which sends $(0, \bar{t_2})$ to the origin and preserves the monotonicity of $s_2 \mapsto B(2\pi j, s_2)$, we may assume that $0 \in \mathcal{F}'$. We also assume that $B(0) = 0$ and, since B is in general position, $B(P) \neq 0$. Replacing B by $\tilde{B}(t) = -B(-t)$, (i.e., $(x,t) \mapsto (-x,-t)$), if necessary, we may assume that $\tilde{M} \doteq B(P) > 0$. Observe that $\tilde{B}$ enjoys all the relevant properties of B, namely, $\tilde{B}$ is in general position, decreasing, and the averages b_{10} and b_{20} remain unchanged.

It is worth noting that by the monotonicity of $B_{|\{t_1=2\pi\}}$ there exists a unique t_2^* such that $B(2\pi, t_2^*) = 0$ which satisfies $-2\pi < t_2^* < 0$ in view of $B(2\pi, 0) = 2\pi b_{10} < 0 = B(2\pi, t_2^*) < 2\pi(b_{10} - b_{20}) = B(2\pi, -2\pi)$.

Let $\delta_0 > 0$ be such that $|B| < \tilde{M}/2$ over the square $(-\delta_0, \delta_0)^2$, $t_2 \mapsto B(\delta', t_2)$ and $t_2 \mapsto B(2\pi - \delta', t_2)$ are decreasing for each $|\delta'| \leq \delta_0$. Since B is in general position, taking a smaller $\delta_0 > 0$, if necessary, we may assume that the disk $D(t_0, \delta_0)$ contains only one critical point (the vertex of the trap), $D(t_0, \delta_0) \setminus \mathcal{F}'$ consists of four sectors, and $|B| < \tilde{M}/2$ on $D(t_0, \delta_0)$. Note that at least one of the sectors of $D(t_0, \delta_0) \setminus \mathcal{F}'$ is contained in N_0.

We now take a normal curve η lying in N_0 satisfying the following properties with $c \doteq B_{|\eta}$:

(i) $N \doteq (N_0 \setminus N_c) \cap ([0, 2\pi] \times \mathbb{R})$ contains no traps (there is only a finite number of traps inside a bounded region);

(ii) $B_{|\bar{N}} < \tilde{M}/2$;

(iii) η crosses the set $\Delta_0 \doteq ((0, \delta_0) \times \{-\delta_0\}) \cup (\{\delta_0\} \times [-\delta_0, \delta_0]) \cup ((0, \delta_0) \times \{\delta_0\})$ exactly once (the origin is a regular point);

(iv) η intersects one of the sectors of $D(t_0, \delta_0) \setminus \mathcal{F}'$ that lies in N_0. Let us denote this sector by S_0 (any point of S_0 is regular);

(v) $\eta(s) \in [0, 2\pi] \times \mathbb{R}$, $0 \leq s \leq 1$, $\eta(0) = (0, y)$, where $-\delta_0 < y < 0$, and $\eta(1) = (2\pi, y')$, for some $y' \in (-2\pi, t_2^*)$.

Let $s_1 \in [0,1]$ such that $(x_1, y_1) \doteq \eta(s_1) \in \Delta_0$, $s_2 = \sup\{s \in [0,1]; \eta(t) \notin S_0, \forall t \leq s\}$, $s_3 = \inf\{s \in [0,1]; \eta(t) \notin S_0, \forall t \geq s\}$, and $s_4 \in [0,1]$ such that $\eta(s_4) = (2\pi - \delta_0, y_2)$, for some y_2.

We define a new curve γ_0 which agrees with η for $s \in [s_1, s_2] \cup [s_3, s_4]$. For $s \in [0, s_1]$, γ_0 is defined as the juxtaposition of the segments joining $(0,0)$ to $(x_1, 0)$ and $(x_1, 0)$ to $\eta(s_1)$. For $s \in [s_2, s_3]$, γ_0 is the juxtaposition of the segments joining $\eta(s_2)$ to t_0 and t_0 to $\eta(s_3)$. For $s \in [s_4, 1]$, γ_0 is the juxtaposition of the segments joining $\eta(s_4)$ to $(2\pi - \delta_0, 0)$ and $(2\pi - \delta_0, 0)$ to $(2\pi, 0)$. Finally, γ_0 is extended periodically and we may assume that it is smooth.

By means of a smooth diffeomorphism, say Ψ_0, which equals the identity along the strips $(2\pi k - \delta, 2\pi k + \delta) \times \mathbb{R}$, we can send γ_0 to the loop $t_2 = 0$. The monotonicity of $B(0, \cdot)$ is preserved and we have $B(t_1, t_2) < \tilde{M}/2$, for $(t_1, t_2) \in [0, 2\pi] \times (-\delta_1, \delta_1)$, for some $\delta_1 > 0$.

Select an analytic curve close to γ_0 (in the C^1 sense), passing through t_0, whose image under Ψ_0 lies on $\mathbb{R} \times (-\delta_1/2, \delta_1/2)$. Now, take a C^ω diffeomorphism close to the graph of an analytic function of t_2 and sending t_0 to a point of the form $t^* \doteq (\tau, 0)$, $0 \leq \tau < 2\pi$. This diffeomorphism can be taken so that $B(0, \cdot)$ is still decreasing.

A further C^ω diffeomorphism straightens out this graph, and, hence, we have obtained a real analytic diffeomorphism such that in the new variables B has the following properties: $B(0, \cdot)$ is decreasing, $|B(t_1, t_2)| < \tilde{M}/2$ on $[0, 2\pi] \times (-\delta_2, \delta_2)$, for some $\delta_2 > 0$. Note that there exists $\delta_3 > 0$ such that $B(t_1, \cdot)$ is still decreasing for all $t_1 \in (-\delta_3, \delta_3)$.

Let $\Theta : \mathbb{R} \to \mathbb{R}$ be a real analytic, $2\pi-$periodic function such that $\Theta(0) = \tau$, $\Theta(t_2) \in [0, \tau]$ for $t_2 \in [0, \delta_2] \cup [2\pi - \delta_2, 2\pi]$, and $\Theta(t_2) \in [-\delta_3, \delta_3]$ for $t_2 \in [\delta_2, 2\pi - \delta_2]$.

We claim that $B(t_1, 0) < \tilde{M}/2$ for $\tau \leq t_1 \leq 2\pi + \tau$. Indeed, for $\pi \leq t_1 \leq 2\pi$ it is obvious and for $0 \leq t_1 - 2\pi \leq \tau$, we have

$$\begin{aligned} B(t_1, 0) &= P(t_1, 0) + b_{10} t_1 = P(t_1 - 2\pi, 0) + b_{10} t_1 \\ &\leq P(t_1 - 2\pi, 0) + b_{10}(t_1 - 2\pi) = B(t_1 - 2\pi, 0) < \tilde{M}/2. \end{aligned}$$

Note that $B(\Theta(t_2), t_2) < \tilde{M}/2$ when $t_2 \in [0, 2\pi]$. This follows immediately once one notes that:

(i) the graph of $\Theta_{|[0,2\pi]}$ lies on the union of $[0, 2\pi] \times [0, \delta_2]$, $[-\delta_3, \delta_3] \times [0, 2\pi]$, and $[0, 2\pi] \times [2\pi - \delta_2, 2\pi]$,

(ii) $B(t_1, \cdot)$ is decreasing for $t_1 \in [-\delta_3, \delta_3]$, and

(iii) $b_{20} < 0$.

Finally, the map $\Phi(t_1, t_2) = (t_1 - \Theta(t_2), t_2)$ sends $(\tau, 0)$ to the origin and the graph of Θ to the new $t_1 = 0$, and reduces the problem to the special case.

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