A BI-HAMILTONIAN APPROACH TO HIDDEN KP EQUATIONS*

P. Casati **B. Konopelchenko** **M. Pedroni**

Abstract

Starting from a geometric approach to soliton equations, we introduce a family of dynamical systems whose reductions lead to the hidden Kadomtsev–Petviashvili (KP) equations. The example of the hidden Korteweg–de Vries (KdV) equations, obtained by means of a further reduction, is also discussed.

1. Introduction

In the theory of PDEs an important role is played by the evolutionary equations that can be solved by the inverse scattering method [11]. This class of equations is often referred to as the class of *soliton equations*, because they are likely to admit solutions in the shape of a solitary wave (see, e.g., [7,12]). Some examples of soliton equations are given by:

$$u_t - \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x = 0 \qquad \text{(Korteweg–deVries)}$$

$$u_t - \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}u^2u_x = 0 \qquad \text{(modified KdV)}$$

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u = 0 \qquad \text{(nonlinear Schrödinger)}$$

$$(u_t - \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x)_x + u_{yy} = 0 \qquad \text{(KP)}$$

These equations have an infinite sequence of 1-parameter groups of symmetries, so that they belong to a *hierarchy* of evolutionary equations, and can be considered as examples of infinite-dimensional integrable systems.

*The authors were partially supported by a research grant from the Italian M.U.R.S.T. and by the Italian National Research Council (CNR) through the G.N.F.M.
AMS Subject Classifiation: Primary 35Q53, Secondary 58F07.
Key words and frases: Integrable systems, bi-Hamiltonian manifolds, KdV and KP equations.

At the beginning of the eighties it was realized by the Kyoto school [20,6] that the KP hierarchy can be written as a family of equations for a pseudodifferential operator, and can be seen as a set of linear flows on an infinite-dimensional Grassmann manifold (the so-called Sato Grassmannian). More recently, Adler and van Moerbeke [1] showed that the geometry of the Sato Grassmannian, namely, the subsets of finite codimension called the *Birkhoff strata*, can be used to study the singularities of the solutions of the KP hierarchy.

Then, in [13,14] integrable systems have been constructed on the Birkhoff strata, by means of a restriction of the usual KP hierarchy, given by constraining the independent variables (times). This has been done via the $\overline{\partial}$-dressing method, and the resulting hierarchies of integrable systems have been called *hidden KP hierarchies*. In this setting, the hidden KdV hierarchies [18,2] have been recovered as stationary reductions.

This paper is devoted to the study of the hidden KP hierarchies and of their reductions from the point of view described in [4,8], where a family of dynamical systems (the *Central System (CS)*) has been introduced, starting from the bi-Hamiltonian approach to soliton equations [16,5]. The CS admits a lot of interesting reductions, leading to many soliton equations. In this paper we consider some variants of the CS, whose reductions are given by the hidden hierarchies mentioned above.

In Section 2 we explain the standpoint of the bi-Hamiltonian theory, in order to motivate the study of the CS. Section 3 deals with its reductions, and contains the examples of the KdV and the KP hierarchy. A "modified" version of the CS is recalled in Section 4, with the aim to present a first variant of this system. In Section 5 the simplest *hidden CS* is studied, with some of its reductions. A general description of the hidden CS is sketched in Section 6, whereas the last section is devoted to some final remarks.

We would like to thank G. Falqui, F. Magri, and L. Martínez Alonso for useful discussions and suggestions. M.P. is grateful to the organizers of WEDP99 for the opportunity to present these results there and for the nice atmosphere at the workshop. Finally, we thank the anonymous referee for suggesting an

improvement.

2. Soliton equations and bi-Hamiltonian systems

This section aims at giving some motivations for the introduction of the Central System(s) studied in this paper. We refer to [4,8] for details, further motivations, and proofs.

Let us consider the KdV equation

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x \ , \tag{1}$$

and let us suppose that the function u is periodic in x. Then (1) defines a vector field on the infinite-dimensional manifold $\mathcal{M} := C^\infty(S^1, \mathbb{R})$ of the real-valued map from the unit circle, namely $X(u) = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x$. The integral curves of X are the solutions of the KdV equation. This vector field has remarkable factorization properties. It can be written as

$$X(u) = -2\partial_x\left(\frac{1}{8}(-u_{xx} + 3u^2)\right) = P_0\left(dH_1(u)\right) \ , \tag{2}$$

where $P_0 = -2\partial_x$, $H_1(u) = \frac{1}{8}\int_{S^1}\left(\frac{1}{2}{u_x}^2 + u^3\right)dx$, and the differential $dH_1(u)$ is defined, as usual, by

$$\int_{S^1} dH_1(u)v\,dx = \frac{d}{dt}_{|t=0}H_1(u+tv) \qquad \text{for all } v \in C^\infty(S^1, \mathbb{R}). \tag{3}$$

The operator P_0 defines a composition law on functionals on $\mathcal{M}$,

$$\{F, G\}_0 = \int_{S^1} dF(u)P_0\left(dG(u)\right)dx \ , \tag{4}$$

which is $\mathbb{R}$-bilinear and skewsymmetric, and satisfies the Jacobi identity and the Leibniz rule. For this reason, $\{\cdot,\cdot\}_0$ is said to be a *Poisson bracket* on $\mathcal{M}$, and X can be thought of as an infinite-dimensional Hamiltonian system (see [10,22]).

It was also observed [15] that X admits another “Poisson factorization”, that is,

$$X(u) = \left(-\frac{1}{2}\partial_{xxx} + 2u\partial_x + u_x\right)(-\frac{1}{2}u) = P_1\left(dH_0(u)\right) \ , \tag{5}$$

where $P_1 = -\frac{1}{2}\partial_{xxx} + 2u\partial_x + u_x$ and $H_0(u) = -\frac{1}{4}\int_{S^1} u^2 dx$. The operator P_1 defines a Poisson bracket too, say $\{\cdot,\cdot\}_1$. This is *compatible* with $\{\cdot,\cdot\}_0$, in the sense that the *Poisson pencil* $\{\cdot,\cdot\}_\lambda := \{\cdot,\cdot\}_1 - \lambda\{\cdot,\cdot\}_0$ is still a Poisson bracket. Hence $\mathcal{M}$ may be called a *bi-Hamiltonian manifold* and X a *bi-Hamiltonian vector field.*

The KdV equation is well known [19] to have an infinite sequence of integrals of motion. From the bi-Hamiltonian point of view, they can be found as the coefficients of a Casimir function of the Poisson pencil, i.e., a function $H(\lambda)$ such that

$$P_\lambda\left(dH(\lambda)(u)\right) = 0 \ . \tag{6}$$

It can be shown (see, e.g., [8]) that such a function is given by $H(\lambda) = 2z\int_{S^1} h(x,z)dx$, where $\lambda = z^2$ and $h(x,z)$ is the unique solution of the Riccati equation

$$h_x + h^2 = u + z^2 \tag{7}$$

of the form $h = z + \sum_{i\geq 1} h_i z^{-i}$. Inserting this expansion in the Riccati equation, one iteratively finds that

$$h_1 = \frac{1}{2}u\ , \qquad h_2 = -\frac{1}{4}u_x\ , \qquad h_3 = \frac{1}{8}(u_{xx} - u^2)\ , \ldots \tag{8}$$

Therefore $H(\lambda) = 2\lambda + \sum_{i\geq -1} H_i\lambda^{i+1}$, where H_0 and H_1 are the functions already encountered. The coefficients H_i are easily shown to be in involution with respect to the Poisson brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$. The corresponding Hamiltonian vector fields,

$$\frac{\partial u}{\partial t_j} = X_j(u)\ , \qquad X_j = P_0\left(dH_j(u)\right)\ , \tag{9}$$

are thus symmetries of the KdV equation. They form the so-called *KdV hierarchy.*

The map $u \mapsto h(u)$ defined by (8) may be considered a moment map, since it gives us the conserved densities. Then, a natural question is: How does $h(u)$

evolve when u evolves according to the flows of the KdV hierarchy? Since h is a conserved density, we must have

$$\frac{\partial h}{\partial t_j} = \partial_x H^{(j)} \ , \tag{10}$$

for suitable currents $H^{(j)}$. The important point is that the $H^{(j)}$ can be constructed *directly* from a generic Laurent series of the form $h(x,z) = z + \sum_{i\geq 1} h_i(x) z^{-i}$, even if it does not solve the Riccati equation (7). For example, we have that

$$\begin{aligned} H^{(2)} &= h_x + h^2 - 2h_1 \\ H^{(3)} &= h_{xx} + 3hh_x + h^3 - 3h_1 h - 3(h_2 + h_{1x}) \ . \end{aligned}$$

This means that equation (10) defines a system of PDEs for the coefficients h_i. The resulting hierarchy is (equivalent to) the celebrated KP hierarchy [20,6].

Now there is a second very natural question: How do the currents $H^{(j)}$ evolve when h evolves according to the KP hierarchy? The answer is easy, provided that one observes that the $H^{(j)}$ have the form

$$H^{(j)} = z^j + \sum_{l\geq 1} H_l^j z^{-l} \tag{11}$$

and considers the space H_+ of linear combinations (with coefficients independent of z) of $H^{(0)} := 1$, $H^{(1)}$, $H^{(2)}$, $\ldots$. Indeed, the evolution of the currents is equivalent to the invariance condition

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right) H_+ \subset H_+ \ . \tag{12}$$

This implies that, for all k,

$$\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} \in H_+ \ , \tag{13}$$

that is, taking into account the asymptotic expansion of the currents,

$$\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = H^{(j+k)} + \sum_{l=1}^{k} H_l^j H^{(k-l)} + \sum_{l=1}^{j} H_l^k H^{(j-l)}. \tag{14}$$

Again, these equations make sense for general Laurent series of the form (11) and define, for each $j \geq 1$, a system of *ordinary* differential equations for the

$\{H^j_l\}_{j,l\geq 1}$. Hence, we derived from the KdV equation a remarkable hierarchy of dynamical systems. It can be shown to be strictly related to the linear flows on the Sato Grassmannian [20,21], and can be used to linearize the KdV flows, according to the Sato's point of view (see [8]).

3. The CS and its reductions

In this section we recall some of the results of [4], concerning the reductions of the Central System (14). As examples, we show how this reduction process can give us back the KdV and the KP hierarchy. The same process will be applied to more complicated cases in the next sections.

It is evident from (14) that the exactness property

$$\frac{\partial H^{(j)}}{\partial t_k} = \frac{\partial H^{(k)}}{\partial t_j} \tag{15}$$

is fulfilled along the CS flows. Moreover, one can show:

Proposition 1. *The flows of the Central System commute.*

Proof: Let us compute the action of the commutator $[X_j, X_k]$ of two vector fields of the CS on a generic current:

$$[X_j, X_k](H^{(i)}) = \frac{\partial}{\partial t_j}\frac{\partial H^{(i)}}{\partial t_k} - \frac{\partial}{\partial t_k}\frac{\partial H^{(i)}}{\partial t_j}. \tag{16}$$

Thanks to the specific form of the Laurent series $H^{(l)}$, this quantity belongs to $H_- := \langle z^{-1}, z^{-2}, \dots \rangle$. Then we observe that, using the exactness property (15), this commutator can be also written in the form

$$[X_j, X_k](H^{(i)}) = \left[\frac{\partial}{\partial t_j} + H^{(j)}, \frac{\partial}{\partial t_k} + H^{(k)}\right] H^{(i)},$$

so that the invariance property (12) entails that $[X_j, X_k](H^{(i)})$ belongs to the subspace H_+. But $H_+ \cap H_- = \{0\}$, and therefore $[X_j, X_k](H^{(i)})$ vanishes.

□

When a family $\{X_j\}$ of *commuting* vector fields is given on a manifold, and one of its members, say X_n, is fixed, two reduction processes for the whole family are naturally defined:

1. The restriction to the set of zeroes of X_n;
2. The projection along the integral curves of X_n.

Of course, we can also combine these processes in order to obtain other reductions. Let us apply this simple idea to the Central System, in order to derive the KdV hierarchy (see also [4] for the case of the so-called fractional KdV hierarchies).

First step: from CS to KP.

We project the CS vector fields along the t_1–flow,

$$\frac{\partial H^{(k)}}{\partial t_1} + H^{(1)}H^{(k)} = H^{(k+1)} + \sum_{l=1}^{k} H_l^1 H^{(k-l)} + H_1^k \, . \tag{17}$$

This amounts to the following "spatialization" procedure. Let us put $x := t_1$ and use (17) to express $H^{(k)}$, for $k \geq 2$, in terms of $h := H^{(1)}$ and its x–derivatives. It is not difficult to see that this way we find the same expressions of the previous section. Hence, the CS, which is a system of ODEs for the $\{H_j^k\}_{j,k\geq 1}$, reduces to the KP hierarchy, a system of PDEs for the coefficients of h.

Second step: from KP to KdV.

It is well–known from the early works on soliton equations that the KdV hierarchy can be obtained from the KP hierarchy by means of a stationary reduction. The zeroes of the second KP flow are given by $\partial_x H^{(2)} = 0$. It is clear from (15) that the smaller set where $H^{(2)} = z^2$ is also invariant with respect to the KP hierarchy. This invariant set is explicitly given by the constraint

$$h_x + h^2 - 2h_1 = z^2 \ ,$$

allowing one to write all the coefficients h_i, for $i \geq 2$, in terms of h_1:

$$h_2 = -\frac{1}{2}h_{1x} \ , \qquad h_3 = \frac{1}{4}h_{1xx} - \frac{1}{2}{h_1}^2 \ , \qquad \dots$$

Therefore, we have obtained a hierarchy for h_1, which is the KdV hierarchy. One can check that

$$\frac{\partial h_1}{\partial t_3} = \frac{1}{4} h_{1xxx} - 3h_1 h_{1x} \ , \tag{18}$$

which is the KdV equation (1) after putting $u = 2h_1$.

Remark 2. The second step is the restriction to the Laurent series h such that $H^{(2)} = z^2$. This is clearly equivalent to say that $z^2(H_+) \subset H_+$. But this condition always leads to a restriction of the CS, because

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right)(z^2 H^{(k)}) = z^2 \left(\frac{\partial}{\partial t_j} + H^{(j)}\right)(H^{(k)}) \in H_+$$

at the points where $z^2(H_+) \subset H_+$. In the following sections we will often use this kind of restriction, and we will denote by $\mathcal{S}_2$ the subset of the currents satifying this constraint.

4. The modified CS and its reductions

It turns out that all the important properties of the CS follow from the invariance relation (12), and do not depend on the specific form of the currents $H^{(j)}$. Hence, we can change the definition of these currents and, via the condition (12), we can define new dynamical systems with the same properties of the CS. In this section we consider the modified CS introduced in [17], and we will present its simplest reductions, namely, the modified KP and the modified KdV hierarchy.

We consider currents of the form

$$H^{(k)} = z^k + \sum_{l \geq 0} H_l^k z^{-l} \ , \qquad \text{for } k \geq 1 \ , \tag{19}$$

and we suppose H_+ to be the span of $(H^{(1)}, H^{(2)}, \ldots)$. Then the invariance condition (12) still defines a system of ODEs for the coefficients $\{H_l^k\}_{k,l \geq 0}$, explicitly given by

$$\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = H^{(j+k)} + \sum_{l=0}^{k-1} H_l^j H^{(k-l)} + \sum_{l=0}^{j-1} H_l^k H^{(j-l)} \ . \tag{20}$$

We call this hierarchy the *modified Central System* (mCS). Since in the proof of Proposition 1 we used only the invariance condition (12), we can assert that the mCS flows commute. Hence, the same reduction processes can be applied to these equations. Let us see some examples.

First step: from mCS to mKP.

We project the mCS vector fields along the t_1–flow,

$$\frac{\partial H^{(k)}}{\partial t_1} + H^{(1)}H^{(k)} = H^{(k+1)} + \sum_{l=0}^{k-1} H_l^1 H^{(k-l)} + H_0^k H^{(1)} \ . \tag{21}$$

This reduction can be described by putting $x := t_1$ and by using (21) to express $H^{(k)}$, for $k \geq 2$, in terms of $h := H^{(1)}$ and its x–derivatives. For example,

$$\begin{aligned} H^{(2)} &= h_x + h^2 - 2h_0 h \\ H^{(3)} &= h_{xx} + 3hh_x + h^3 - 3h_0(h_x + h^2) - 3(h_{0x} - {h_0}^2 + h_1)h \ . \end{aligned}$$

Thus, the mCS reduces to the *modified KP (mKP) hierarchy*, a system of PDEs for the coefficients $h_0, h_1, \ldots$ of h. This system still has the conservation laws form

$$\frac{\partial h}{\partial t_j} = \frac{\partial H^{(j)}}{\partial t_1} = \partial_x H^{(j)} \ .$$

In particular,

$$\begin{aligned} \frac{\partial h}{\partial t_2} &= \partial_x \left(h_x + h^2 - 2h_0 h\right) \\ \frac{\partial h}{\partial t_3} &= \partial_x \Big(h_{xx} + 3hh_x + h^3 - 3h_0(h_x + h^2) - 3(h_{0x} - {h_0}^2 + h_1)h\Big) \ . \end{aligned}$$

Second step: from mKP to mKdV.

This is another standard stationary reduction. The zeroes of the second mKP flow are given by $\partial_x H^{(2)} = 0$, and the subset where $H^{(2)} = z^2$ is also invariant with respect to the mKP hierarchy. It is given by the constraint

$$h_x + h^2 - 2h_0 h = z^2 \ ,$$

allowing one to write all the coefficients h_i, for $i \geq 1$, in terms of h_0:

$$h_1 = -\frac{1}{2}(-h_{0x} + {h_0}^2) \ , \qquad h_2 = \frac{1}{4}h_{0xx} - \frac{1}{2}h_0 h_{0x} \ , \qquad \ldots$$

This way we obtain a hierarchy (called the mKdV hierarchy) for the single field h_0. The first nontrivial equation is

$$\frac{\partial h_0}{\partial t_3} = \frac{1}{4}(h_{0xxx} - 6{h_0}^2 h_{0x}),$$

which is the mKdV equation written in the introduction.

5. A hidden CS and its reductions

In this section we perform a more substantial change of the Central System (14). The currents have the form

$$H^{(k)} = z^k + H^k_{-1} z + \sum_{l \geq 1} H^k_l z^{-l}, \qquad k \geq 2\ .$$

The equations of motion are always defined via the invariance condition

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right) H_+ \subset H_+\ , \qquad j \geq 2\ ,$$

where $H_+ = \langle H^{(0)}, H^{(2)},\ H^{(3)}, \dots \rangle$ and $H^{(0)} = 1$. They have the explicit form

$$\begin{aligned}\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} =& H^{(j+k)} + \sum_{l=1}^{k-2} H^j_l H^{(k-l)} + \sum_{l=1}^{j-2} H^k_l H^{(j-l)} + H^j_{-1} H^{(k+1)} \\ &+ H^k_{-1} H^{(j+1)} + H^k_{-1} H^j_{-1} H^{(2)} + H^k_{-1} H^j_1 + H^j_{-1} H^k_1 + H^j_k + H^k_j \end{aligned} \tag{22}$$

Once more, the flows commute, and we can reduce the hierarchy as we did in the last two sections.

First step: spatialization with respect to $x := t_3$.

For $k \geq 2$ we have

$$\begin{aligned}\frac{\partial H^{(k)}}{\partial x} + H^{(3)} H^{(k)} =& H^{(k+3)} + \sum_{l=1}^{k-2} H^3_l H^{(k-l)} + H^k_1 H^{(2)} + H^3_{-1} H^{(k+1)} \\ &+ H^k_{-1} H^{(4)} + H^k_{-1} H^3_{-1} H^{(2)} + H^k_{-1} H^3_1 + H^3_{-1} H^k_1 + H^3_k + H^k_3 \end{aligned}$$

Hence $H^{(i)}$, for $i \geq 5$, can be written in terms of $H^{(2)}$, $H^{(3)}$, $H^{(4)}$ and their x-derivatives.

Second step: restriction to $\mathcal{S}_2$.

We must impose that $z^2(H_+) \subset H_+$. This implies $H^{(2)} = z^2$, $H^{(4)} = z^4$, and the following constraint on $h := H^{(3)}$,

$$\partial_x h + h^2 = z^6 + 2h_{-1}z^4 + (2h_1 + h_{-1}^2)z^2 + 2(h_3 + h_{-1}h_1) \ .$$

It allows us to compute all the coefficients of h in terms of h_{-1}, h_1, and h_3. The first nontrivial equations of the reduced hierarchy are

$$\begin{aligned}
\frac{\partial h_{-1}}{\partial t_5} &= h_{1x} - 2h_{-1}h_{-1x} \\
\frac{\partial h_1}{\partial t_5} &= h_{3x} - 2h_{-1x}h_1 - h_{-1}h_{1x} \\
\frac{\partial h_3}{\partial t_5} &= \frac{1}{4}h_{-1xxx} - h_1h_{1x} - 2h_{-1x}h_3 - 2h_{-1}h_{3x}
\end{aligned}$$

It can be shown that this hierarchy coincides with the simplest hidden KdV hierarchy of [18], whose associated linear problem is a Schrödinger equation with energy-dependent potential.

6. General examples of hidden CS

The aim of this section is to give a general class of examples of hidden Central Systems. Particular cases are given by the "standard" CS presented in Section 3, by the mCS of Section 4, and by the system (22) studied in the previous section. We will also obtain, by means of a reduction, the hidden KdV hierarchies introduced in [18].

Let us fix a strictly increasing sequence of nonnegative integers,

$$S = \{s_0, s_1, \dots\} \ ,$$

with the property that there exists $n_0 \in \mathbb{N}$ such that $s_{n+1} = s_n$ for $n \geq n_0$. Let us consider currents $H^{(s_k)}$, for $s_k \in S$, of the form

$$H^{(s_k)} = z^{s_k} + \sum_{l \notin S} H^{s_k}_{-l} z^l \ .$$

Then we define the *hidden Central System (hCS)* associated with S to be the family of equations defined by the invariance condition

$$\left(\frac{\partial}{\partial t_{s_k}} + H^{(s_k)}\right) H_+ \subset H_+ \ , \qquad s_k \in S \ , \tag{23}$$

where H_+ is the span of $\{H^{(s_k)}\}_{s_k\in S}$. It can be easily checked that this gives, for each s_k, a system of ODEs for the coefficients $\{H^{s_k}_{-l}\}_{s_k\in S, l\notin S}$. The standard CS corresponds to the choice $S = \{0,1,2,\dots\}$, after the restriction to $H^{(0)} = 1$. The mCS is obtained for $S = \{1,2,3,\dots\}$, whereas the system (22) of the previous section comes from the choice $S = \{0,2,3,\dots\}$, again after the restriction to $H^{(0)} = 1$.

It is not difficult to mimic the proofs of Section 3 in order to show that the flows of the hCS commute, and that the exactness condition

$$\frac{\partial H^{(s_j)}}{\partial t_{s_k}} = \frac{\partial H^{(s_k)}}{\partial t_{s_j}} \ , \qquad s_j, s_k \in S \ , \tag{24}$$

holds. Thus we can perform also for this system the reduction processes discussed in the previous sections.

Before giving some examples of reductions, we observe that (24) implies the existence of a function $\psi(t_{s_0}, t_{s_1}, \dots)$ such that

$$\frac{\partial}{\partial t_{s_k}} \log\psi = H^{(s_k)} \ .$$

The asymptotic expansion of the $H^{(s_k)}$ entails that

$$\psi = (1 + w_1 z^{-1} + \dots) \exp\left(\sum_{s_k\in S} t_{s_k} z^{s_k} + \sum_{j\notin S, j\geq 0} b_j z^j\right) \ ,$$

where $b_j(t_{s_0}, t_{s_1}, \dots)$ satisfies

$$\frac{\partial b_j}{\partial t_{s_k}} = H^{s_k}_{-j} \ .$$

These remarks could be used to make a comparison between the results presented here and those of [13,14]. In particular, one could show that ψ is a (desingularized) wave-function, and the b_j give constraints on the times of the KP hierarchy.

Now we will explain how to recover the hKdV hierarchies in this formalism. We must consider the hCS associated with $S = \{0, 2, 4, \ldots, 2n, 2n+1, 2n+2, \ldots\}$. Then the currents have the form

$$H^{(s_k)} = z^{s_k} + \sum_{i=1}^{n} H^{s_k}_{-2i+1} z^{2i-1} + \sum_{l \leq -1} H^{s_k}_{-l} z^l \ , \qquad s_k \in S \ ,$$

and we can restrict to $H^{(0)} = 1$.

As usual, the first step is a spatialization, in this case with respect to $x := t_{2n+1}$. The projected hierarchy is a system of PDEs for the currents

$$H^{(2)}, H^{(4)}, \ldots, H^{(2n)}, H^{(2n+1)}, H^{(2n+2)}, H^{(2n+4)}, \ldots, H^{(4n)} \ .$$

The second step is the restriction to the subset $\mathcal{S}_2$ of the set of the stationary points of the second flow. This subset is defined by the condition $z^2(H_+) \subset H_+$, which implies $H^{(2k)} = z^{2k}$ and the following constraint on $h := H^{(2n+1)}$,

$$z^{4n+2} = h_x + h^2 - p(z^2) \ , \tag{25}$$

where $p(z^2) = \sum_{i\geq 0}^{2n} u_i z^{2i}$ is a polynomial in z^2 whose coefficients are polynomials in the coefficients of h. In the case $n = 1$, already discussed in the previous section, we have seen that $p(z^2) = u_2 z^4 + u_1 z^2 + u_0$, with

$$u_2 = 2h_{-1} \ , \qquad u_1 = 2h_1 + h_{-1}^2 \ , \qquad u_0 = 2(h_3 + h_{-1}h_1) \ . \tag{26}$$

Equation (25) shows that the associated linear problem is a Schrödinger equation with energy-dependent potential. Indeed, in terms of the wave-function ψ it reads

$$\psi_{xx} = \left(z^{4n+2} + p(z^2)\right) \psi \ .$$

It can be checked that the constraint (25) gives all the coefficients of h in terms of

$$h_{-2n+1}, h_{-2n+3}, \ldots, h_{-1}, h_1, h_3, \ldots, h_{2n+1} \ .$$

The resulting hierarchy can be shown to be a reformulation of the hidden KdV hierarchy of [18]. In the case $n = 1$ the isomorphism is explicitly given by equations (26).

7. Final comments

1. In [9] the well-known stationary reductions of the KdV hierarchy are studied as suitable reductions of the CS, and the separability of these finite-dimensional integrable systems is recovered in the framework of bi-Hamiltonian manifolds. It seems worthwhile to investigate the corresponding reductions of the *hidden* CS, and their separability as well.

2. A first relation between the approach presented here and the $\overline{\partial}$-method of [13] to study the hidden KP hierarchies has been shown in the previous section. It would be interesting to understand better this link, and to show the explicit relation with the desingularization procedure of [1] and with the geometry of the Sato Grassmannian.

3. In this paper we have given the simplest examples of reductions of hCS, but it is clear from what we said in Section 3 that one can choose an arbitrary time t_{s_j} for the spatialization, and then another time t_{s_k} for the restriction to $z^{s_k}(H_+) \subset H_+$. In [3] the hCS of Section 5 is reduced according to the choice $s_j = 3$ and $s_k = 4$. The resulting hierarchy seems not to have been ever considered in the literature.

4. As already mentioned at the end of Section 2, a linearization of the "standard" CS has been performed in [8]. It should be investigated whether this procedure can be applied also to the hidden CS.

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Paolo Casati
Dip. di Matematica e Applicazioni
Univ. di Milano-Bicocca
Via degli Arcimboldi 8
I-20126 Milano (Italy)
e-mail: casati@mat.unimi.it

Boris Konopelchenko
Dip. di Fisica
Univ. di Lecce
Via Arnesano
I-73100 Lecce (Italy)
e-mail: konopel@le.infn.it

Marco Pedroni
Dip. di Matematica
Univ. di Genova
Via Dodecaneso 35
I-16146 Genova (Italy)
e-mail: pedroni@dima.unige.it