UNIQUENESS OF THE CAUCHY PROBLEM FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

We consider the uniqueness of the Cauchy problem for semilinear 2×2 systems of partial differential equations. The results are similar to the ones obtained for a single equation.

Resumo

Nós consideramos a unicidade do problema de Cauchy para sistemas 2×2 semilineares de equações diferenciais parciais. Os resultados são similares aqueles obtidos para uma equação escalar.

1. Introduction

There are two basic theorems for the initial value problem, that is, the Cauchy problem for linear partial differential operators with analytic coefficients in $\mathbb{R}^n$. The first one is the Cauchy-Kovalevsky theorem which says that the non-characteristic Cauchy problem has one, and just one, solution when the data are analytic. The second one, known as Hölmgren's theorem, guarantees that the solution of the Cauchy problem is unique even if it is assumed to be only in C^m, where m is the order of the operator; in fact, it is known that this theorem holds even if the solution is a hyperfunction. For scalar operators, non-uniqueness results for non-characteristic Cauchy problems have been extensively studied in the late 70's and early 80's. The spirit of the uniqueness

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theorems is that the result should not depend on the lower order terms; see [1] and [3] and the references therein for more details.

The present paper was motivated by the work of G. Métivier [2] who discussed counterexamples to the validity of Hölmgren's theorem for systems of partial differential equations. Métivier starts from non-uniqueness results of the non-characteristic Cauchy problem for a class of smooth linear partial differential operators of the second order. He adds an additional variable and then, by writing the operator in system form, he obtains examples for non-uniqueness of the Cauchy problem for analytic systems. In this way, he arrives at some very simple first order semilinear systems of operators with constant coefficients for which the conclusion of Hölmgren's theorem does not hold. In principle, his construction can be carried out when the principal symbol of the operator, for which we have non-uniqueness for the Cauchy problem, is independent of the non-characteristic direction.

Basically, we start from the known results for a single partial differential equation for which the uniqueness property holds independently of lower order coefficients; see [1] and [4]. Then we extend these results to semilinear systems. The main step, similar to the scalar case, is to prove a Carleman estimate for systems.

Before stating the basic definitions and results we mention that, in the context of linear systems, there is a very general uniqueness result due to A. Caldéron, who dealt with hyperbolic and with some cases of elliptic systems, see [3].

The general setting for extensions of the Hölmgren's theorem can be written as

$$I\ \partial_t \mathcal{U} + \Sigma_{j=1}^{n} A_j(x,t) D_{x_j} \mathcal{U} + f(x,t,\mathcal{U}) = 0, \tag{1.1}$$

where $\mathcal{U}$ is a vector valued C^1 function defined in a neighbourhood of $(x_0, 0) \in \mathbb{R}^n \times \mathbb{R}$ with values on $\mathbb{R}^k$, and I is the identity matrix on $\mathbb{R}^k$. The A_j's are $k \times k$ analytic matrices and f is a vector valued analytic mapping of its arguments with values in $\mathbb{R}^k$ and $f|_{\mathcal{U}=0} = 0$.

Assume that

$$\mathcal{U}(x,t) = 0 \quad \forall \quad t \leq 0. \tag{1.2}$$

The general question is: for which n, k and A_j's, the solution $\mathcal{U} = 0$ is the only one such that (1.1) and (1.2) hold near $t = 0$?

In this paper we narrowed the gap between Métivier's counterexamples and the uniqueness results. For the sake of completeness we recall the construction of a basic counterexample for uniqueness of solutions found in [2]:

Let $P = \partial_t^2 - \partial_{x_1}^2 + \partial_{x_2}^2$; then there exist a neighbourhood , V, of the origin and functions $a, \psi \in C^\infty(V)$ such that:

$$(P - a\partial_{x_2})(\psi) = 0, \quad \psi, a|_{t<0} = 0 \quad on \quad V \quad and \quad 0 \in supp(a) \cap supp(\psi).$$

See [1](or [3]).

Let $\chi \in C^\infty(\mathbb{R})$ be a real valued function such that $0 \in supp(\chi) = [0, \infty)$. We take $u_1(x_1, x_2, x_3, t) = \chi(x_3)\partial_{x_2}\psi(x_1, x_2, t)$, $u_2(x_1, x_2, x_3, t) = \chi(x_3)(\partial_t + \partial_{x_1})\psi(x_1, x_2, t)$ and $u_3(x_1, x_2, x_3, t) = a(x_1, x_2, t - x_3)$; then (1.1) and (1.2) hold for

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$f(\mathcal{U}) = \begin{pmatrix} 0 \\ u_1 u_3 \\ 0 \end{pmatrix}.$$

This example of Métivier has motivated us to ask the following question: If n=1 and k=2, is there a unique $\mathcal{U}$ satisfying (1.1) and (1.2)?

To answer this question, we point out that in the plane there are linear partial differential equations of first order of the form $P_0\psi = \partial_t\psi + a\partial_x\psi = 0$, where a and ψ have the same properties as above. Taking real and complex parts of this equation we have that the solution is not always unique for systems with C^∞ coefficients. See [1](or [4]).

Also, we observe that even for simple hyperbolic systems sometimes we cannot uncouple. For example, consider

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$f(x,t,\mathcal{U}) = \begin{pmatrix} f_1(u_1,u_2) \\ f_2(u_1,u_2) \end{pmatrix}$$

where $f_i(u_1,u_2)$ is a polynomial of degree five on u_i, for each i.
This last observation explains why uniqueness for systems is not a direct consequence of the results for the scalar case. To deal with this problem we consider an extension of a technique used for the scalar case, namely Carleman's estimates.
Let $M_2(\mathbb{R})$ be the space of real 2×2 matrices, $M_2^1 = C^1(\mathbb{R}^2, M_2(\mathbb{R}))$ and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 1. *If $A \in M_2^1$, we say that:*
i) A is symmetrizable if there is an invertible matrix $B \in M_2^1$ such that BAB^{-1} is symmetric.
ii) A is elliptic if the imaginary part of the eigenvalues of A never vanishes.
iii) A satisfies condition $\mathcal{P}$ if there is an invertible matrix $B \in M_2^1$ such that $BAB^{-1} = aI + bJ$, where a and b are real valued C^2 functions and $b(x,.)$ does not change sign.
iv) A is of finite order if there is an invertible matrix $B \in M_2^1$ such that $BAB^{-1} = aI + bJ$, where a and b are smooth and $b(x,.)$ has only zeroes of finite order.

Using the above notation, we will prove the

Theorem. *Let $\mathcal{U} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Assume that one of the following conditions holds:*
1) A is a symmetrizable C^1 matrix valued.
2) A is an elliptic C^1 matrix valued.

3) A satisfies $\mathcal{P}$ condition.

4) A is of finite order.

Then $\mathcal{U} = 0$ is the only solution of (1.1) and (1.2), near $t = 0$.

It is easy to see that the proofs given below allow us to consider f as a continuous mapping which is C^1 with respect to $\mathcal{U}$. The corresponding for the linear case were proved by Caldéron [3] (under assumptions 1) or 2)) and by Strauss and Treves [4] (under assumptions 3) and 4)). Here we extend these theorems to the semilinear case. The proof is organized as follows. In section 2, we prove that the theorem is true if A satisfies condition (i). In sections 3, 4 and 5, we prove the same for conditions (ii), (iii) and (iv), respectively.

2. The Symmetric Case

This is a well known result. The technique is called Energy Method. We include it here for our very simple situation for the sake of completeness.

Step 1

We start by considering a special case with two restrictions, namely: (a) the matrix A_1 is symmetric, and (b) *supp* $\mathcal{U}$ is contained in $Q = (-r, r) \times (-T, T) \cup [-r, r] \times \{T\}$.

We may write

$$A_1 = -\begin{pmatrix} a(x,t) & b(x,t) \\ b(x,t) & c(x,t) \end{pmatrix},$$

where $a, b, c \in C^1$ are real valued functions.

Multiplying equation (1.1) on the left by $2\,\mathcal{U}^t$ we obtain:

$$2\,\mathcal{U}^t(I\partial_t + A_1\partial_x)\mathcal{U} = -2\,\mathcal{U}^t f(x,t,\mathcal{U}),$$

that is,

$$\partial_t(u_1^2 + u_2^2) - a\partial_x u_1^2 - c\partial_x u_2^2 - 2b\partial_x(u_1u_2) = -2u_1f_1 - 2u_2f_2.$$

Integration with respect to x yields

$$\partial_t \int (u_1^2 + u_2^2)dx = -\int (a_x u_1^2 + c_x u_2^2 + 2b_x u_1 u_2)dx - 2\int (u_1 f_1 + u_2 f_2)dx,$$

hence

$$\partial_t ||\mathcal{U}(.,t)||^2 \le M||\mathcal{U}(.,t)||^2.$$

Here, $||\mathcal{U}(.,t)||^2 = \int (u_1^2 + u_2^2)dx$ and M depends only on $||a_x||\infty$, $||b_x||_\infty$, $||c_x||_\infty$, $||(f_1)_{u_1}||_\infty$, $||(f_2)_{u_1}||_\infty$, $||(f_1)_{u_2}||_\infty$ and $||(f_2)_{u_2}||_\infty$. Integrating this inequality with respect to t, and using $\mathcal{U}(x,0) = 0$, we obtain

$$||\mathcal{U}(.,t)||^2 \le M \int_0^t ||\mathcal{U}(.,s)||^2 ds.$$

With the notation $\phi(t) = \int_0^t ||\mathcal{U}(.,s)||^2 ds$, this can be recast into the form

$$\phi'(t) \le M\phi(t).$$

Now, because of the assumption that $\mathcal{U} \in C^1 \cap L^\infty$, there is $h > 0$ such that $\phi(h) \ge \phi(t)$, $\forall t \in [0,h]$. So $\phi(h) \le Mh\phi(h)$, therefore $\phi(h) = 0$, if $0 < h < \frac{1}{M}$.

Step 2

Now we consider the general case, where A is symmetrizable and there is no restriction on *supp* $\mathcal{U}$. By the usual parabolic change of variables, (see [4]), this reduces to the case that *supp* $\mathcal{U}(.,t)$ is a compact subset of $(-r,r)$ for each $t \in (-T,T)$. Also, by hypothesis there is an invertible $B \in M_2^1$ such that $A_1 = B^{-1}AB$ is a symmetric matrix. Thus, there exists a vector valued C^1 function $\mathcal{V}$ such $\mathcal{U} = B\mathcal{V}$. Therefore,
$(I\partial_t + A\partial_x)\mathcal{U} = (I\partial_t + A\partial_x)B\mathcal{V} = (B\partial_t + AB\partial_x)\mathcal{V} + (B_t + AB_x)\mathcal{V} = B(\partial_t + A_1\partial_x)\mathcal{V} + (B_t + AB_x)\mathcal{V}$.
Substituting this result in (1.1) and multiplying it by B^{-1} we obtain

$$[I\partial_t + A_1\partial_x + (B^{-1}B_t + B^{-1}AB_x)]\mathcal{V} + B^{-1}f(x,t,B\mathcal{V}) = 0.$$

Then, the conclusion of the proof follows from the special case considered in Step 1.

3. The Elliptic Case

Step 1

We start by considering the special case with three restrictions. These are:

$$(a)\quad A = \alpha I + \beta J,$$

where $\alpha, \beta \in C^1$ are real valued functions and β never vanishes, (b) $\mathcal{U}$ has compact support and is of class $C^1(Q)$, where Q is a rectangle of the form $(-r, r) \times (-T, T)$, and (c) $f(x, t, u_1, u_2) = 0$. Without loss of generality we may assume that $(x_0, 0) = (0, 0)$.

Let $\phi \in C^2(\mathbb{R})$ be a real valued function and consider

$$\mathcal{V}(x, t) = e^{\frac{k\phi}{2}}\mathcal{U}(x, t).$$

Then, (1.1) reduces to:

$$(I\partial_t + \alpha I\partial_x + \beta J\partial_x - \frac{k\phi_t}{2}I)\mathcal{V} = 0,$$

where $k > 0$ is a large parameter to be chosen later.

Setting

$$M = I\partial_t + \alpha I\partial_x \quad \text{and} \quad N = -\frac{k\phi_t}{2}I + \beta J\partial_x,$$

we can thus write

$$\int_0^T ||M\mathcal{V} + N\mathcal{V}||^2 dt = \int_0^T (||M\mathcal{V}||^2 + ||N\mathcal{V}||^2 + 2 < M\mathcal{V}, N\mathcal{V} >)\, dt. \tag{3.1}$$

First we study the last integral:

$$\begin{aligned}\int_0^T < M\mathcal{V}, N\mathcal{V} > dt &= -\frac{k}{4}\int_0^T\int_{-r}^r \phi_t\partial_t(v_1^2+v_2^2)dxdt - \frac{k}{4}\int_0^T\int_{-r}^r \alpha\phi_t\partial_x(v_1^2+v_2^2)dxdt \\ &\quad - \int_0^T\int_{-r}^r \beta\partial_t v_1\partial_x v_2 dxdt + \int_0^T\int_{-r}^r \beta\partial_x v_1\partial_t v_2 dxdt \\ &= \frac{k}{4}\int_0^T\int_{-r}^r (\phi_{tt} + (\alpha\phi_t)_x)(v_1^2 + v_2^2)dxdt \\ &\quad + \int_0^T\int_{-r}^r (-\beta\partial_t v_1\partial_x v_2 + \beta\partial_x v_1\partial_t v_2)dxdt. \end{aligned} \tag{3.2}$$

Call Γ the last term on the right hand side of (3.2). We have
$2\Gamma = 2\int_0^T\int_{-r}^r(-\beta\partial_t v_1\partial_x v_2 + \beta\partial_x v_1\partial_t v_2)dxdt = \int_0^T\int_{-r}^r\beta_t(v_1\partial_x v_2 - v_2\partial_x v_1)dxdt$
$+\int_0^T\int_{-r}^r(-\beta_x v_1\partial_t v_2 + \beta_x v_2\partial_t v_1)dxdt = \int_0^T\int_{-r}^r(\frac{\beta_t}{\beta}v_1\beta\partial_x v_2 - \frac{\beta_t}{\beta}v_2\beta\partial_x v_1)dxdt$
$+\int_0^T\int_{-r}^r(-v_1\beta_x\partial_t v_2 + v_2\beta_x\partial_t v_1)dxdt = -\int_0^T < \frac{\beta_t}{\beta}I\mathcal{V}, N\mathcal{V} > dt$
$-\int_0^T < \frac{\beta_t}{\beta}I\mathcal{V}, \frac{k\phi_t}{2}I\mathcal{V} > dt - \int_0^T < M\mathcal{V}, \beta_x J\mathcal{V} > dt + \int_0^T < \alpha I\partial_x\mathcal{V}, \beta_x J\mathcal{V} > dt.$
By substituting Γ in (3.2), we get
$\int_0^T < M\mathcal{V}, N\mathcal{V} > dt = \frac{k}{4}\int_0^T\int_{-r}^r[\phi_{tt}+(\alpha\phi_t)_x](v_1^2+v_2^2)dxdt - \frac{1}{2}\int_0^T < \frac{\beta_t}{\beta}I\mathcal{V}, N\mathcal{V} > dt$
$-\frac{1}{2}\int_0^T < \frac{\beta_t}{\beta}I\mathcal{V}, \frac{k\phi_t}{2}I\mathcal{V} > dt - \frac{1}{2}\int_0^T < M\mathcal{V}, \beta_x J\mathcal{V} > dt + \frac{1}{2}\int_0^T < \alpha\frac{\beta_x}{\beta}I\mathcal{V}, N\mathcal{V} > dt$
$+\frac{1}{2}\int_0^T < \alpha\frac{\beta_x}{\beta}I\mathcal{V}, \frac{k\phi_t}{2}I\mathcal{V} > dt.$

Taking absolute value in the last identity we have, for each $\epsilon > 0$,
$2|\int_0^T < M\mathcal{V}, N\mathcal{V} > dt| \le \frac{k}{2}Max_Q\{|\phi_{tt}+(\alpha\phi_t)_x|\}\int_0^T||\mathcal{V}||^2dt + \frac{1}{\epsilon}Max_Q\{|\frac{\beta_t}{\beta}|\}\int_0^T||\mathcal{V}||^2dt$
$+\epsilon\int_0^T||N\mathcal{V}||^2dt + \frac{k}{2}Max_Q\{|\frac{\beta_t}{\beta}\phi_t|\}\int_0^T||\mathcal{V}||^2dt + \epsilon\int_0^T||M\mathcal{V}||^2dt + \frac{1}{\epsilon}Max_Q\{|\beta_x|\}\int_0^T||\mathcal{V}||^2dt$
$+\frac{1}{\epsilon}Max_Q\{|\alpha\frac{\beta_x}{\beta}|\}\int_0^T||\mathcal{V}||^2dt + \epsilon\int_0^T||N\mathcal{V}||^2dt + \frac{k}{2}Max_Q\{|\alpha\frac{\beta_x}{\beta}\phi_t|\}\int_0^T||\mathcal{V}||^2dt.$
Therefore,

$$2\,|\int_0^T < M\mathcal{V}, N\mathcal{V} > dt| \le C_\epsilon\int_0^T||\mathcal{V}||^2dt + 2\epsilon\int_0^T(||M\mathcal{V}||^2 + ||N\mathcal{V}||^2)dt.$$

Here $C_\epsilon = \frac{k}{2}C_1 + \frac{1}{\epsilon}C_2$ with $C_1 = Max_Q\{|\phi_{tt} + (\alpha\phi_t)_x|\} + 2Max_Q\{|\frac{\beta_t}{\beta}\phi_t|\} + Max_Q\{|\alpha\frac{\beta_x}{\beta}\phi_t|\}$ and $C_2 = Max_Q\{|\alpha\frac{\beta_x}{\beta}|\} + Max_Q\{|\frac{\beta_t}{\beta}|\} + Max_Q\{|\beta_x|\}$. Observe that C_1 and C_2 are independent of ϵ and k.
We obtain,

$$\int_0^T(||M\mathcal{V}||^2 + ||N\mathcal{V}||^2)dt + \frac{k}{2}\int_0^T||\mathcal{V}||^2dt \le$$
$$\le \int_0^T e^{k\phi}||\mathcal{L}\mathcal{U}||^2dt + 2\epsilon\int_0^T(||M\mathcal{V}||^2 + ||N\mathcal{V}||^2)dt + C_\epsilon\int_0^T||\mathcal{V}||^2dt.$$

Where $\mathcal{L} = I\partial_t + A\partial_x$. Taking $\epsilon \le \frac{1}{4}$ we can therefore conclude that

$$\frac{k}{2}\int_Q e^{k\phi}|\mathcal{U}|^2dxdt \le \int_Q e^{k\phi}|\mathcal{L}\mathcal{U}|^2dxdt + C_\epsilon\int_Q e^{k\phi}|\mathcal{U}|^2dxdt. \qquad (3.3)$$

Step 2

Let us now consider the more general case where we pose no restrictions on A and on the support of $\mathcal{U}$, but retain the restriction that $f(x, tu_1, u_2) = 0$.

Then, again by the usual parabolic change of variables, we can assume that *supp* $\mathcal{U}(.,t)$ is compact in $(-r,r)$ for each $t \in (-T,T)$.
From linear algebra we know that there is an invertible $B \in M_2^1(\mathbb{R})$ such that

$$AB = B(\alpha I + \beta J)$$

where $\alpha, \beta \in C^1$ and β never vanishes. As in step 2 of section 2, there exists a vector valued C^1 function $\mathcal{V}$ such that $\mathcal{U} = B\mathcal{V}$. Therefore
$(I\partial_t + A\partial_x)\mathcal{U} = (I\partial_t + A\partial_x)B\mathcal{V} = (B\partial_t + AB\partial_x)\mathcal{V} + (B_t + AB_x)\mathcal{V} = B(\partial_t + (\alpha I + \beta J)\partial_x)\mathcal{V} + (B_t + AB_x)\mathcal{V}$.
Multiplying by B^{-1} we obtain:

$$[I\partial_t + (\alpha I + \beta J)\partial_x + (B^{-1}B_t + B^{-1}AB_x)]\mathcal{V} = 0.$$

By repeating the procedure of Step 1 we conclude that inequality (3.3) also holds in this situation.

Step 3

We now consider the general case, i.e., we also drop the condition on f. We expand a nonlinear f into a first order Taylor expansion at $(x,t,0,0)$. In this way, we write f as a linear operator $F(x,t)$ obtaining that f is written as a linear operator $F(x,t)$ in (u_1,u_2) with C^1 entries. Now adding this term to $\mathcal{L}$ we have a new N in the above discussion, namely

$$N = -\frac{k\phi_t}{2} + \beta J\partial_x + F.$$

By inspection of the arguments used in Step 1, we see that the presence of this modified N will only imply different constants in equation (3.3).

Step 4

Now we use the inequality (3.3) to prove the uniqueness of the solution of (1.1)-(1.2). For that purpose, we take $\phi(t) = \frac{(t-T)^2}{4}$, $\epsilon = \frac{1}{4}$, $T > 0$ small such that $C_1 < 1$, and $k_0 > \frac{8C_2}{1-C_1}$. The proof of the following proposition follows directly from (3.3).

Proposition 3.1. *There is a constant $C > 0$ such that $\forall k > k_0$, we have*

$$\frac{k}{2}\int_Q e^{\frac{k(t-T)^2}{4}}|\mathcal{U}|^2 dxdt \leq C\int_Q e^{\frac{k(t-T)^2}{4}}|\mathcal{L}\mathcal{U}|^2 dxdt.$$

for all $\mathcal{U} \in C^1(Q)$ *of compact support.*

We now multiply $\mathcal{U}$ by a cut off function $\theta \in C^\infty(\mathbb{R})$ satisfying $\theta(t) = 0$ for $t > T$, $\theta(t) = 1$ for $t < T_1 < T$. Then, from Proposition 3.1, it follows that

$$\int_Q e^{\frac{k(t-T)^2}{4}} |\theta \mathcal{U}|^2 dxdt \leq \frac{2C}{k} \int_Q e^{\frac{k(t-T)^2}{4}} |\mathcal{L}(\theta \mathcal{U})|^2 dxdt.$$

Let $\mathcal{M}_\theta$ denote the operator describing multiplication by θ. We then express the right hand side of the above inequality in terms of the commutator of $\mathcal{L}$ with $\mathcal{M}_\theta$ to obtain

$$\int_0^{T_2} e^{\frac{k(t-T)^2}{4}} ||\mathcal{U}||^2 dt \leq \frac{C'}{k} \int_{T_1}^{T} e^{\frac{k(t-T)^2}{4}} ||[\mathcal{L}, \mathcal{M}_\theta]\mathcal{U}|^2 dt,$$

for sufficiently large k and $0 < T_2 < T_1$, where C' is a new constant. From the fact that $e^{k(t-T)^2/4}$ is a decreasing function in [0,T], we have

$$\int_0^{T_2} ||\mathcal{U}||^2 dt \leq \frac{C'}{k} e^{k\{(T_1-T)^2-(T_2-T)^2\}/4} \int_{T_1}^{T} ||[\mathcal{L}, \mathcal{M}_\theta]\mathcal{U}||^2 dt.$$

Finally, letting $k \to \infty$, we obtain $(0,0) \notin$ *supp* $\mathcal{U}$, from which the uniqueness follows.

4. The Condition P Case

Step 1

We start by considering a special case with two restrictions, namely that the matrix A_1 can be expressed as

$$A_1 = b(x,t)J$$

and that

$$f(x,t,\mathcal{U}) = c \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $b, c \in C^2(\mathbb{R}^2)$ are real valued functions and $b \geq 0$.

As before, we work in a neighbourhood of the origin with $(x_0, 0) = (0,0)$. Then

$$\mathcal{L} = I\partial_t + bJ\partial_x + cI. \tag{4.1}$$

Let $\mathcal{U}$ be of class $C^1(Q)$, where Q is a rectangle of the form $(-r,r)\times(-T,T)$, such that $supp\,\mathcal{U}(.,t)$ is compact in $(-r,r)$ for each $t\in(-T,T)$.
The content of the following proposition is a Carleman's estimate for this case.

Proposition 4.1. *Suppose that there are $f_0, f_1, \theta \in C^2(I\!R)$ real valued functions defined in a neighbourhood Q of the origin such that*

$$\partial_t f_1 + \partial_x(b\partial_x f) > 0, \tag{4.2}$$

where $f(x,t) = f_0(x) + \int_{\theta(x)}^t b(x,s) f_1(s)ds$.

Then there are $C, k_0 > 0$ such that

$$k\int_Q e^{2kf}|\mathcal{U}|^2 dxdt \le C\int_Q e^{2kf}\{|<\mathcal{L}\mathcal{U}, J\partial_x\mathcal{U}>| + k|<\mathcal{L}\mathcal{U}, J\mathcal{U}>|\}dxdt \tag{4.3}$$

$\forall k > k_0$.

Proof: With $c_1(x,t) = \int_0^t c(x,s)ds$ **and** $d = -c_x$ **we have** $e^{c_1}\mathcal{L}\mathcal{V} = [I\partial_t + bJ(\partial_x + d)]\mathcal{U}$ **where** $\mathcal{V} = e^{-c_1}\mathcal{U}$.

So we write

$$e^{kf}e^{c_1}\mathcal{L}\mathcal{V} = [M + bJN]\mathcal{W},$$

where $\mathcal{W} = e^{kf}\mathcal{U}$, $M = I\partial_t - bkf_xJ + bdJ$ and $N = I\partial_x + kf_1J$. Observe that $M^* = -M$, $N^* = -N$ and $[M,N] = k\{\partial_t f_1 + \partial_x(b\partial_x f)\}J - (bd)_xJ$.
From $2 < M\mathcal{W} + JbN\mathcal{W}, -JN\mathcal{W} >= 2 < M\mathcal{W}, -JN\mathcal{W} > - \int_Q b|N\mathcal{W}|^2 dxdt$ and

$$2 < M\mathcal{W}, -JN\mathcal{W} >=< \mathcal{W}, J[M,N]\mathcal{W} >$$

we get
$2 < M\mathcal{W} + JbN\mathcal{W}, -JN\mathcal{W} >= - < \mathcal{W}, (k\{\partial_t f_1 + \partial_x(b\partial_x f)\} - (bd)_x)I\mathcal{W} >$
$-2\int_Q b|N\mathcal{W}|^2 dxdt$.
Therefore, from (4.2) we conclude that, for large k,

$$2 < M\mathcal{W} + JbN\mathcal{W}, JN\mathcal{W} >\ge kC\int_Q |\mathcal{W}|^2 dxdt,$$

which implies (4.3). This concludes the proof of Proposition 4.1.

Using the above proposition, we can prove uniqueness for the special case considered. As in [4], we argue by contradiction. Let $\mathcal{J}$ be a interval of $\mathbb{R}$ and $0 < T$ such that $\mathcal{U}(x,t) \neq 0$, $\forall (x,t) \in \mathcal{J} \times (0,T)$. Then we choose $\theta \in C^2(\mathcal{J})$, with

i) Graph$(\theta) \cap$supp $\mathcal{U} \neq \emptyset$.

ii) $t \geq \theta(x)$ for every $(x,t) \in$ supp $\mathcal{U}$.

Take $(x_0, \theta(x_0)) \in$ Graph$(\theta) \cap$supp$(\mathcal{U})$. We can assume that $b(x_0, \theta(x_0)) = 0$, because otherwise the problem reduces to the elliptic case discused in the previous section.

Now consider

$$f_0(x) = -\epsilon (x - x_0)^2 \quad \text{and} \quad f_1(t) = -e^{-t},$$

where $\epsilon > 0$. For these choice of θ, f_0 and f_1, define f as in Proposition 4.1.

As in the elliptic case, we multiply $\mathcal{U}$ by a cut off function and then use the Carleman estimate.

In this case, we have to localize the support of the obtained function near the segment l_0 joining $(x_0, \theta(x_0))$ to $(x_0, \bar{t}(x_0))$, where $\bar{t}(x_0)$ =sup$\{t \geq \theta(x_0); b(x_0, s) = 0,\ \theta(x_0) \leq s \leq t\}$. (From the equation we have that $\bar{t}(x_0) < T$). Let U_1 be an open neighbourhood of l_0 such that (4.2) is valid in U_1 for ϵ small. Take U_2 be an open neighbourhood of l_0 such that its clousure is contained in U_1. Consider $\delta > 0$ small and define $U_1^\delta = \{(x,t) \in U_1; f(x,t) \geq -\delta\}$.

Let $g \in C^2(U_1)$ be compactly supported such that $g(x) = 1$ for $x \in U_2$. Clearly, there is a $\delta_0 > 0$ so that supp grad(g)$\cap$ supp $\mathcal{U} \subset (U_1^{\delta_0})^c$. Let $0 < \delta_1 < \delta_0$ such that $-f(x,t) < \delta_1$; then there is a neighbourhood U_3 of l_0 with $U_3 \subset U_1^{\delta_1} \cap U_2$.

As in the elliptic case, applying Proposition 4.1 to $\mathcal{V} = g\mathcal{U}$, we have

$$\int_{U_3} |\mathcal{U}|^2 dx dt \leq C e^{-k(\delta_0 - \delta_1)},$$

where C is a constant depending only on $\mathcal{U}$ and its derivatives on U_1. As before this estimative implies the uniqueness result.

Step 2

The general case will be a consequence of the following procedure :
i) Using the first degree Taylor's expansion the nonlinear term is transformed into a linear one in $\mathcal{U}$.
ii) By the usual convex change of variables we can assume the retriction on the support of $\mathcal{U}$.
iii) Now, integrating along the characteristic of $\partial_t + a\partial_x$ we reduce the general case to $A_1 = bJ$. Observe that the reductions made above remain unchanged.
iv) For the zero order term $C\mathcal{W}$, analogous to $cI\mathcal{W}$, we determine a similar c_1 by solving the following system of ordinary differential equations: $\partial_t C_1(x,t) = C_1(x,t)C(x,t)$ with $C(x,0) = I$. And then a direct inspection of the above proof shows that the final estimate holds.

5. The Finite Order Case.

As in the previous cases we consider a simple situation. Let

$$\mathcal{L} = I\partial_t + b(x,t)J\partial_x + c(x,t)I,$$

where $b, c \in C^\infty$, and suppose that $t \to b(0,t)$ has a zero of finite order at $t = 0$. As in [4], the proof of the uniqueness result will follow from the steps given below.

Step 1

We can reduce the study to a operator of the form:

$$\mathcal{L} = I\partial_t + (\alpha I + \beta J)t^l\partial_x + \gamma I,$$

where $\beta(0,0) \neq 0$ and $l \in \mathbb{N}$.

Step 2

Assume that $\mathcal{L}$ has a more general form than in Step 1, namely:

$$\mathcal{L} = I\partial_t + (\alpha I + \beta J)t^l\partial_x + \frac{\gamma}{t}I,$$

where $\alpha, \beta \in C^1$, $\gamma \in C^0$ and $\beta(0,0) \neq 0$. We perform the change of variables $X = x$ and $T = \frac{t}{\delta - x^2}$, where $\delta > 0$ is small. We may assume that $\mathcal{L}$ is, up to multiplication by an invertible matrix,

$$\mathcal{L} = I\partial_t + (\tilde{\alpha}I + \tilde{\beta}J)(\delta - x^2)^{l+1}t^l\partial_x + \frac{\tilde{\gamma}}{t},$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are C^1 functions defined in a neighbourhood of the origin such that $\tilde{\beta}(0,0) \neq 0$, and $\tilde{\gamma}$ is a continuous function. Here to simplify the notation we kept the same coordinates (x,t) to represent the new variables (X,T).

Step 3

Consider $\mathcal{L}$ as in Step 2. Then for some positive constants C, λ_0, T_0 and r if $\lambda > \lambda_0$, $\mathcal{V} \in C^\infty$ and supp $\mathcal{V} \subset \{(x,t); 0 \leq t \leq T_0, |x| \leq r\}$ we have the following estimate

$$\lambda ||t^{-\lambda-1}\mathcal{V}||^2 \leq C||t^{-\lambda}\mathcal{P}\mathcal{V}||^2.$$

As before, the uniqueness result is a consequence of this estimate.

Now we prove the statements contained in the above steps.

Proof of Step 1

By the Malgrange-Weierstrass Theorem, there are $r > 0$ and $T > 0$ such that if $|x| \leq r$ and $0 < t < T$:

$$b(x,t) = a(x,t)(t^k + a_1(x)t^{k-1} + ... + a_k(x)),$$

where $a, a_j \in C^\infty$, $a(0,0) \neq 0$ and $a_j(0) = 0$ for $1 \leq j \leq k$. Taking $f = b/a$ we have the following lemma, the proof of which can be found in [4].

Lemma 5.1.

(a) There are k open sets $\mathcal{O}_1, ..., \mathcal{O}_k$ contained on $|x| < r$ such that the closure of $\cup_{j=1}^k \mathcal{O}_j = \{x; |x| < r\}$ and for any $x \in \mathcal{O}_j$, $1 \leq j \leq k$, the function

$$t \to f(x,t) \quad \textit{has exactly } j \textit{ distinct roots.}$$

(b) In each $\mathcal{O}_j$, the distinct roots of f can be represented by j functions $\rho_i^j \in C^\infty$, $1 \leq i \leq j$.

(c) If $\rho_i^j(x)$ is real at x_0, then it is real at the connected component of x_0 in $\mathcal{O}_j$.

To reduce to the canonical form take $\mathcal{U} \in C^1$, as in the Theorem. Assume that F=supp $\mathcal{U} \neq \emptyset$; then there exist j and a non-empty connected component $\tilde{\mathcal{O}}_j$ of $\mathcal{O}_j$ such that $F \cap (\tilde{\mathcal{O}}_j \times (0,T) \neq \emptyset$. In fact $F \cap (\cup_{j=1}^k (\mathcal{O}_j \times (0,T)) \neq \emptyset$ because $\mathcal{U}$ is continuous and $\cup_{j=1}^k \mathcal{O}_j$ is dense, so there exists j such that $F \cap (\mathcal{O}_j \times (0,T)) \neq \emptyset$ which implies the existence of such a component $\tilde{\mathcal{O}}_j$.
Now, let $\rho_1^j < ... < \rho_l^j$, $l \leq j$, the real roots of f on $\tilde{\mathcal{O}}_j$ and take $l = 0$ if there is no real root. Define $\tilde{\rho}_0^j(x) = 0$, $\tilde{\rho}_i^j(x)$ =sup$\{0, \inf\{\rho_i^j(x), T\}\}$ for $1 \leq i \leq l$ and $\tilde{\rho}_{l+1}^j(x) = T$.
For $0 \leq i \leq l$, define $\mathcal{A}_i = \{(x,t) \in \tilde{\mathcal{O}}_i \times (0,T); \tilde{\rho}_i(x) \leq t \leq \tilde{\rho}_{i+1}(x)\}$ and i_0 =inf$\{i; F \cap \mathcal{A}_i \neq \emptyset\}$. If $l = 0$, then $\mathcal{A}_0 = \tilde{\mathcal{O}}_i \times (0,T)$ and $i_0 = 0$.
Suppose that there is $(x_0, t_0) \in F \cap \mathcal{A}_{i_0}$ with $t_0 = \rho_{i_0}^j(x_0)$. We perform the change of variables $x' = x - x_0$ and $t' = t - \rho_{i_0}^j(x)$. In these coordinates, near (x_0, t_0), the operator $\mathcal{L}$ is transformed into a multiple of the operator

$$\mathcal{L} = I\partial_t + (\alpha I + \beta J)t^l \partial_x + \gamma I,$$

where we keep the same notation (x,t). This is done as follows:
i)By using this change of variable we get

$$\mathcal{L}\mathcal{U} = ((I - (\rho_{i_0}^j)_x bJ)\partial_{t'} + bJ\partial_{x'} + cI)\mathcal{U}'.$$

ii) Since the matrix $(I - b(\rho_{i_0}^j)_x J)$ is invertible with inverse given by

$$(I - (\rho_{i_0}^j)_{x'} bJ)^{-1} = \frac{1}{1 + ((\rho_{i_0}^j)_{x'} b)^2}(I + (\rho_{i_0}^j)_{x'} bJ),$$

we have

$$\mathcal{L}\mathcal{U} = (I - (\rho_{i_0}^j)_{x'} bJ)[\partial_{t'} + \tfrac{1}{1+((\rho_{i_0}^j)_{x'} b)^2}(I + (\rho_{i_0}^j)_{x'} bJ)bJ\partial_{x'} + \tfrac{1}{1+((\rho_{i_0}^j)_{x'} b)^2}(I + (\rho_{i_0}^j)_{x'} bJ)c]\mathcal{U}'.$$

On the other hand, $b(x', t') = \tilde{a}(x', t')t'^{m_{i_0}}$, where $\tilde{a}(x'_0, t'_0) \neq 0$. Therefore

$$(I + (\rho_{i_0}^j)_{x'} bJ)bJ = \alpha I + \beta J t^l,$$

where $\beta(x_0', t_0') \neq 0$, proving the canonical form of Step 1, for this case.
Now, suppose that $F \cap A_{i_0} \cap \{t = \rho_{i_0}(x)\} = \emptyset$; taking x_0 to be the middle point of $\tilde{\mathcal{O}}_j$ we consider the same change of variables as before. Under this change of variables the set $\{(x,t) \in \tilde{\mathcal{O}}_j \times (0,T);\ t = \rho_{i_0}(x)\}$ is mapped onto the set $\{(x',t'); t' = 0,\ |x'| < r'\}$, for some $r' > 0$.
Let $\epsilon_0 = \mathrm{Min}\{\epsilon > 0; (\text{graph of } t' - \epsilon = -\epsilon(x'/r')^2) \cap A_{i_0} \cap \mathrm{supp}\ \mathcal{U}' \neq \emptyset\}$; by hypothesis $\epsilon_0 > 0$. Taking $(x_0', t_0') \in \tilde{A}_{i_0} \cap \mathrm{supp}\, \mathcal{U}'$ such that $t_0' = -\epsilon_0(x_0'/r')^2 + \epsilon_0$ we have $(x_0', t_0') \in A_{i_0}$. Then near this point the operator is elliptic therefore the uniqueness follows.

Proof of Step 2

Consider the change of variables given by $X = x$ and $T = t/(\delta - x^2)$, with $\delta > 0$ small. Then $\mathcal{L}$ becomes

$$\bar{\mathcal{L}} = \frac{1}{\delta - X^2}\{[I + 2XT(\bar{\alpha}I + \bar{\beta}J)]\partial_T + (\bar{\alpha}I + \bar{\beta}J)(\delta - X^2)^{l+1}T^l\partial_X + \frac{\bar{\gamma}}{T}I\},$$

where $\bar{\alpha}$ and $\bar{\beta}$ are, respectively, the transformed functions of α and β under the change of variables.
Multiplying $\bar{\mathcal{L}}$ by $\delta - X^2$ and then by the inverse of $I + 2XT(\bar{\alpha}I + \bar{\beta}J)$, the resulting operator is

$$\mathcal{P} = I\partial_t + (\tilde{\alpha}I + \tilde{\beta}J)(\delta - x^2)^{l+1}t^l\partial_x + \frac{\tilde{\gamma}}{t}.$$

Here $\tilde{\beta}(0,0) \neq 0$ and we have returned to the notation (x,t) instead of (X,T).

Proof of Step 3

With $\mathcal{V} \in C^1$ we have supp $\mathcal{V} \subset \{(x,t);\ t \geq 0,\ (\delta - x^2) \geq 0\}$. Consider $\mathcal{V} = t^\lambda \mathcal{W}$. Let $\mathcal{P}_0$ be the principal part of the operator $\mathcal{P}$. It follows that

$$t^{-\lambda}\mathcal{P}_0\mathcal{V} = X\mathcal{W} + Y\mathcal{W},$$

where $X = I\partial_t + \tilde{\alpha}(\delta - x^2)^{l+1}t^l I\partial_x$ and $Y = \lambda t^{-1}I + \tilde{\beta}(\delta - x^2)^{l+1}t^l J\partial_x$.
Then we have

$$||t^{-\lambda}\mathcal{P}_0\mathcal{V}||^2 = ||X\mathcal{W}||^2 + ||Y\mathcal{W}||^2 + 2 < X\mathcal{W}, Y\mathcal{W} > .$$

But

$2 < X\,\mathcal{W}, Y\,\mathcal{W} >= 2 < I\partial_t\mathcal{W}, \lambda\, t^{-1}\, I\mathcal{W} > +2 < \tilde{\alpha}(\delta - x^2)^{l+1}t^l\, I\partial_x\mathcal{W}, \lambda t^{-1}\, I\mathcal{W} >$
$+2 < \tilde{\alpha}(\delta - x^2)^{l+1}t^l\, I\partial_x\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l\, J\partial_x\mathcal{W} > +2 < I\partial_t\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l\, J\partial_x\mathcal{W} >$
$= (I) + (II) + (III) + (IV)$, respectively.

Now, we analyze each of the terms above:

$$(I) = 2\lambda \int_Q (\partial_t w_1 t^{-1} w_1 + \partial_t w_2 t^{-1} w_2)\,dxdt = \lambda ||t^{-1}\mathcal{W}||^2.$$

We also have

$$|(II)| = |\lambda \int \tilde{\alpha}(\delta - x^2)^{l+1} t^{l-1} \partial_x |\mathcal{W}|^2 dxdt| \leq \lambda C ||t^{\frac{l-1}{2}}\mathcal{W}||^2,$$

where C is a constant depending on l, $||\tilde{\alpha}||_\infty$, $||\partial_x\tilde{\alpha}||_\infty$, T and δ.

From the fact that $< \theta\mathcal{Z}, J\mathcal{Z} >= 0$ it follows that

$$(III) = 0.$$

Finally, we consider (IV). Note that

$$< I\partial_t\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l J\partial_x\mathcal{W} >= - < I\mathcal{W}, \tilde{\beta}_t(\delta - x^2)^{l+1}t^l J\partial_x\mathcal{W} >$$

$$- < I\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1} l t^{l-1} J\partial_x\mathcal{W} > + < I\partial_x\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l J\partial_t\mathcal{W} >$$

$$+ < I\mathcal{W}, \tilde{\beta}_x(\delta - x^2)^{l+1}t^l J\partial_t\mathcal{W} > -2 < I\mathcal{W}, \tilde{\beta}(l+1)(\delta - x^2)^l x t^l J\partial_t\mathcal{W} > .$$

But

$$< I\partial_x\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l J\partial_t\mathcal{W} >= - < I\partial_t\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l J\partial_x\mathcal{W} > .$$

Therefore

$$(IV) = - < I\mathcal{W}, \tilde{\beta}_t(\delta - x^2)^{l+1}t^l J\partial_x\mathcal{W} > - < I\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1} l t^{l-1} J\partial_x\mathcal{W} >$$

$$+ < I\mathcal{W}, \tilde{\beta}_x(\delta - x^2)^{l+1}t^l J\partial_t\mathcal{W} > -2 < I\mathcal{W}, \tilde{\beta}(l+1)(\delta - x^2)^l x t^l J\partial_t\mathcal{W} >=$$

$$= (IV)_a + (IV)_b + (IV)_c + (IV)_d, \text{ respectively.}$$

Now we study each of the terms above. We have

$$(IV)_a = - < I\mathcal{W}, \tilde{\beta}_t(\delta - x^2)^{l+1}t^l J\partial_x \mathcal{W} >= - < I\mathcal{W}, \tilde{\beta}_t\tilde{\beta}^{-1}(Y - \lambda t^{-1}I)\mathcal{W} >$$

$$= - < \tilde{\beta}_t\tilde{\beta}^{-1}I\mathcal{W}, Y\mathcal{W} > +\lambda < \tilde{\beta}_t\tilde{\beta}^{-1}I\mathcal{W}, t^{-1}I\mathcal{W} >,$$

hence

$$|(IV)_a| \leq C\lambda||t^{-1/2}\mathcal{W}||^2 + \epsilon||Y\mathcal{W}||^2 + C_\epsilon||\mathcal{W}||^2,$$

where C, C_ϵ are positive constants depending on $||\tilde{\beta}_t\tilde{\beta}^{-1}||_\infty$, with C_ϵ depending also on ϵ.

For $(IV)_b$ we observe that

$$(IV)_b = -l < t^{-1}I\mathcal{W}, (Y - \lambda t^{-1}I)\mathcal{W} >= l\lambda||t^{-1}I\mathcal{W}||^2 - l < t^{-1}\mathcal{W}, Y\mathcal{W} >;$$

thus

$$|(IV)_b - l\lambda||t^{-1}\mathcal{W}||^2| \leq \epsilon||Y\mathcal{W}||^2 + C_\epsilon||t^{-1}\mathcal{W}||^2,$$

where $\epsilon > 0$ and C_ϵ is a positive constant depending of ϵ.

Now

$$(IV)_c =< I\mathcal{W}, \tilde{\beta}_x(\delta - x^2)^{l+1}t^l J\partial_t \mathcal{W} >=< \tilde{\beta}_x(\delta - x^2)^{l+1}t^l I\mathcal{W}, J\partial_t\mathcal{W} > .$$

With $f = (\delta - x^2)^{l+1}t^l\tilde{\beta}_x$, we can write

$$(IV)_c =< fI\mathcal{W}, J\partial_t\mathcal{W} >=< fI\mathcal{W}, [X - \tilde{\alpha}(\delta - x^2)^{l+1}t^l I\partial_x]J\mathcal{W} >$$

$$=< fI\mathcal{W}, XJ\mathcal{W} > - < \tilde{\beta}^{-1}\tilde{\alpha}fI\mathcal{W}, \tilde{\beta}(\delta - x^2)^{l+1}t^l J\partial_x\mathcal{W} >$$

$$=< fI\mathcal{W}, XJ\mathcal{W} > - < \tilde{\beta}^{-1}\tilde{\alpha}fI\mathcal{W}, (Y - \lambda t^{-1}I)\mathcal{W} > .$$

From this we have

$$|(IV)_c| \leq \epsilon(||X\mathcal{W}||^2 + ||Y\mathcal{W}||^2) + C_\epsilon||\mathcal{W}||^2 + C\lambda||t^{-1/2}\mathcal{W}||^2.$$

And

$$(IV)_d = - < I\mathcal{W}, 2(l+1)x\tilde{\beta}(\delta - x^2)^l t^l J\partial_t\mathcal{W} >= - < I\mathcal{W}, 2(l+1)x\tilde{\beta}(\delta - x^2)^l t^l JX\mathcal{W} >$$

$$+ < I\mathcal{W}, 2(l+1)x\tilde{\beta}\tilde{\alpha}(\delta - x^2)^{2l+1}t^{2l}J\partial_x\mathcal{W} >;$$

writing the last term of the above equality in terms of Y we get

$$|(IV)_d| \leq \epsilon(||X\mathcal{W}||^2 + ||Y\mathcal{W}||^2) + C_\epsilon||\mathcal{W}||^2 + C\lambda||t^{(l-1)/2}\mathcal{W}||^2,$$

with C and C_ϵ chosen as before.
Then
$||X\mathcal{W}||^2+||Y\mathcal{W}||^2+(l+1)\lambda||t^{-1}\mathcal{W}||^2 \leq ||X\mathcal{W}||^2+||Y\mathcal{W}||^2+(I)+(l\lambda||t^{-1}\mathcal{W}||^2-(IV)_b)+(IV)_b$.
Introducing $||t^{-\lambda}\mathcal{P}_0\mathcal{V}||^2$ on the right hand side and considering the inequalities proved before for (I) through (IV), we have, for all T_0 with $0 < T_0 < 1$,

$$||X\mathcal{W}||^2+||Y\mathcal{W}||^2+(l+1)\lambda||t^{-1}\mathcal{W}||^2 \leq ||t^{-\lambda}\mathcal{P}_0\mathcal{V}||^2+\epsilon\, C(||X\mathcal{W}||^2+||Y\mathcal{W}||^2)+$$

$$+C_\epsilon||t^{-1}\mathcal{W}||^2 + C\lambda||t^{-1/2}\mathcal{W}||^2,$$

with possible different ϵ, C and C_ϵ. Take $\epsilon > 0$ small such that $\epsilon C < 1/2$, λ_0 and T_0^{-1} sufficiently large such that $C\lambda||t^{-1/2}\mathcal{W}||^2 \leq CT_0\lambda||t^{-1}\mathcal{W}||^2$ and $(l+1) > C_\epsilon/\lambda_0 + CT_0$, for $\lambda > \lambda_0$ and $T < T_0$. Then using $\mathcal{W} = t^{-\lambda}\mathcal{V}$ we obtain

$$\lambda||t^{-\lambda-1}\mathcal{V}||^2 \leq \tilde{C}||t^{-\lambda}\mathcal{P}_0\mathcal{V}||^2,$$

for some positive constant $\tilde{C}$.
Since $\mathcal{P} = \mathcal{P}_0 + \tilde{\gamma}/t$ and $\tilde{\gamma} \in L^\infty$ the result holds. Following the same arguments given in the condition P case, we reduce the general case to the special one treated above.

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