BOUNDARY BEHAVIOUR AND INTEGRABILITY OF LARGE SOLUTIONS TO p-LAPLACE EQUATIONS

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Abstract

Consider the problem $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u)$ in D, $u \to \infty$ as $x \to \partial D$, where $D \subset \mathbb{R}^N$ is a bounded smooth domain, $p > 1$, and $f : [0, \infty) \to [0, \infty)$ is increasing and satisfies suitable growth conditions. First we prove a boundary behaviour result of the solution $u(x)$ in a general domains D. Next we discuss the integrability of $u(x)$ in D according to the growth of f.

1. Introduction

Let $D \subset \mathbb{R}^N$ be a bounded smooth domain. We investigate the following problem

$$\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = f(u) \quad \text{in} \quad D, \quad u \to \infty \quad \text{as} \quad x \to \partial D, \tag{1.1}$$

where $p > 1$, and $f : [0, \infty) \to [0, \infty)$ is increasing and satisfying $f(0) = 0$. It is known [8,13] that a necessary and sufficient condition for the existence of a solution to problem (1.1) is that $f(t)$ satisfies the generalized Osserman condition

$$\int_1^\infty \frac{ds}{F(s)^{1/p}} < \infty, \quad F(s) = \int_0^s f(t)dt. \tag{1.2}$$

Note that if condition (1.2) holds then the function

$$\psi(t) = \int_t^\infty \frac{ds}{(qF(s))^{1/p}}, \quad q = p/(p-1) \tag{1.3}$$

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is well defined, decreasing and convex.

For $p = 2$ problem (1.1) has been widely studied, see [3,4,9,10,12] and the references therein. For general p and D convex we refer to [8,13].

In section 2 of this paper we shall find a precise estimate near the boundary ∂D for a solution $u(x)$ to problem (1.1), extending a result from [8] where D was assumed to be convex. In section 3 we study the integrability of $u(x)$ according to the growth of f.

2. Boundary estimate

We shall make use of the following Lemma.

Lemma 2.1. *Let $g(s) \in L^1(R, \rho)$, $g(s) > 0$, $k(s) \to \infty$ as $s \to R^+$, $k'(s) \leq 0$. We have*

$$\lim_{r \to R^+} \frac{\int_r^\rho g(s)k(s)ds}{k(r)} = 0. \tag{2.1}$$

Proof. Arguing as in the proof of Lemma 2.1 of [10] we have, for $\epsilon > 0$,

$$\frac{\int_r^\rho g(s)k(s)ds}{k(r)} = \frac{\int_r^{r+\epsilon} g(s)k(s)ds}{k(r)} + \frac{\int_{r+\epsilon}^\rho g(s)k(s)ds}{k(r)}$$

$$\leq \int_r^{r+\epsilon} g(s)ds + \frac{\int_{R+\epsilon}^\rho g(s)k(s)ds}{k(r)}.$$

The result follows easily.

We also need the following result proved in [8].

Lemma 2.2. *If condition* (1.2) *holds for some $p > 1$ then we have*

$$\lim_{t \to \infty} \frac{F(t)^{\frac{p-1}{p}}}{f(t)} = 0, \tag{2.2}$$

$$\lim_{t \to \infty} \frac{t^p}{F(t)} = 0, \tag{2.3}$$

$$\lim_{t \to \infty} \frac{t^{p-1}}{f(t)} = 0. \tag{2.4}$$

Let D be a ball centered at the origin and of radius R and let condition (1.2) hold. A radially symmetric solution to problem (1.1) is a function $v(r)$ satisfying

$$\left(r^{N-1}|v'|^{p-2}v'\right)' = r^{N-1}f(v), \quad v'(0)=0, \; v(r)\to\infty \;\text{ as } r\to R^-. \tag{2.5}$$

One can prove that a solution to problem (2.5) is increasing and convex in $(0,R)$. See [8] and the references therein.

Let $q = p/(p-1)$. Define

$$\Gamma(t) := \frac{1}{F(t)}\int_0^t \left(qF(\tau)\right)^{\frac{1}{q}} d\tau, \tag{2.6}$$

where F is the integral function introdued in (1.2). By using (2.2) or (2.3) we find that $\Gamma(t)\to 0$ as $t\to\infty$.

Lemma 2.3. *Assume condition* (1.2) *and let* $F(t)t^{-p}$ *be increasing for large* t. *If* $v(r)$ *is a solution to problem* (2.5) *then there is a constant* $\beta>0$ *such that*

$$v(r) < \phi(R-r) + (R-r)\beta\phi(R-r), \tag{2.7}$$

where ϕ *is the inverse of the function* ψ *defined in* (1.3).

Proof: We refer to Lemma 2.2 and Lemma 2.3 of [8].

To prove the next Lemma we make the following further assumption (see [4]). There exists a constant K such that for all $\delta>0$ we have

$$\frac{\phi'(\delta)}{\phi'(2\delta)} \le K. \tag{2.8}$$

Now let D be an annulus centered at the origin with radii R and R', $R<R'$. A radially symmetric solution to problem (1.1) is a function $z(r)$ satisfying

$$\left(r^{N-1}|z'|^{p-2}z'\right)' = r^{N-1}f(z), \quad z'(\rho)=0, \; z(r)\to\infty \;\text{ as } r\to R^+, \tag{2.9}$$

for some ρ with $R<\rho<R'$.

Lemma 2.4. *Assume conditions* (1.2) *and* (2.8) *and let* $F(t)t^{-p}$ *be increasing for large* t. *If* $z(r)$ *is a solution to problem* (2.9) *then there is a constant* $\beta>0$ *such that*

$$z(r) > \phi(r-R) - (r-R)\beta\phi(r-R). \tag{2.10}$$

Proof: By (2.9) one finds easily that $z'(r) < 0$ for $R < r < \rho$. Hence, in this interval we have

$$-\Big(r^{N-1}(-z')^{p-1}\Big)' = r^{N-1} f(z),$$

or

$$(p-1)(-z')^{p-2} z'' - \frac{N-1}{r}(-z')^{p-1} = f(z).$$

Note that the last equation implies $z''(r) > 0$. Multiplying by $-z'$ and integrating over (r, ρ) we find

$$\frac{p-1}{p}(-z')^p - (N-1)\int_r^\rho \frac{1}{s}(-z'(s))^p ds = F(z) - F(\lambda), \quad \lambda = z(\rho). \tag{2.11}$$

By Lemma 2.1 with $k(s) = (-z'(s))^p$, we get

$$\lim_{r \to R^+} \frac{\int_r^\rho \frac{1}{s}(-z'(s))^p ds}{(-z'(r))^p} = 0.$$

Hence, by (2.11) we have

$$(-z')^p \leq c_1 q F(z), \quad q = p/(p-1).$$

Here and in the sequel we denote by c_i suitable positive constants. We may rewrite the latter inequality as

$$(-z')^{p-1} \leq c_2 \Big(qF(z)\Big)^{1/q}.$$

Using (2.11) together with the last estimate we find

$$\frac{(-z')^p}{qF(z)} \leq 1 + \frac{N-1}{R} \frac{\int_\lambda^{z(r)} (-z'(\tau))^{p-1} d\tau}{F(z)}$$

$$\leq 1 + c_3 \frac{\int_0^{z(r)} \Big(qF(\tau)\Big)^{1/q} d\tau}{F(z)} = 1 + c_3 \Gamma(z(r)).$$

As a consequence,

$$\frac{-z'}{\Big(qF(z)\Big)^{1/p}} \leq 1 + c_3 \Gamma(z(r)).$$

Integration over (R,r) leads to

$$\int_{z(r)}^{\infty} \frac{ds}{\big(qF(s)\big)^{1/p}} \le r - R + c_3 \int_R^r \Gamma(z(s))ds.$$

Recalling the definition of ψ given in (1.3) we have

$$\psi(z(r)) \le r - R + c_3 \int_R^r \Gamma(z(s))ds. \tag{2.12}$$

Putting

$$\omega := c_3 \int_R^r \Gamma(z(s))ds,$$

inequality (2.12) reads as

$$\psi(z(r)) \le r - R + \omega,$$

whence

$$z(r) \ge \phi(r - R + \omega) \ge \phi(r-R) + \phi'(r-R)\omega.$$

Using condition (2.8) and recalling that $\phi' < 0$, we have

$$z(r) \ge \phi(r-R) + K\phi'(2(r-R))c_3 \int_R^r \Gamma(z(s))ds. \tag{2.13}$$

On the other hand, since $F(t)$ is increasing we find

$$\Gamma(z(r)) < q^{1/q} \frac{z(r)}{F(z(r))^{1/p}}.$$

Using the latter estimate and the monotonicity of $z(r)/F(z(r))^{1/p}$, (2.13) yields

$$z(r) \ge \phi(r-R) - c_4 F\Big(\phi(2(r-R))\Big)^{1/p}(r-R)\frac{z(r)}{F(z(r))^{1/p}}. \tag{2.14}$$

Finally, by (2.12) we find, for r near R,

$$\psi(z(r)) \le 2(r-R), \quad \text{whence} \quad z(r) \ge \phi(2(r-R)).$$

Hence, from (2.14) we get

$$z(r) \ge \phi(r-R) - c_4(r-R)z(r).$$

The lemma follows easily.

Theorem 2.5. *Let $D \subset \mathbb{R}^N$ be a bounded smooth domain satisfying a uniform interior and exterior sphere condition. Assume conditions* (1.2), (2.8) *and suppose $F(t)t^{-p}$ is increasing for large t. If $u(x)$ is a positive solution to problem* (1.1) *then there exists a constant $\beta > 0$ such that*

$$|u(x) - \phi(\delta(x))| < \beta\delta(x)\phi(\delta(x)).$$

Here, $\delta(x) = dist(x, \partial D)$.

Proof: Take a point $P \in \partial D$. We may assume that $P = (R, 0, \dots, 0)$, that D is contained in the annulus $\mathbb{A}(R, R')$ with center at $(2R, 0, \dots, 0)$, and that D contains the ball $B(R)$ with center at $(0, \dots, 0)$. Note that $\mathbb{A}(R, R')$ and $B(R)$ are tangent to ∂D at P. If u, v and z are solution of (1.1) in D, $B(R)$, and $\mathbb{A}(R, R')$ respectively then, by the comparison principle, we have

$$z(x) \le u(x) \le v(x) \quad \forall x \in B(R).$$

The result follows now by Lemmas 2.3 and 2.4 using a comparison principle.

Remark. For a convex domain D, Theorem 2.5 has been proved in [8] without using condition (2.8).

3. Integrability

By Theorem 2.5 one gets

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\phi(\delta(x))} = 1. \tag{3.1}$$

As an application of this fact we prove the following result of integrability.

Theorem 3.1. *Let $D \subset \mathbb{R}^N$ be a bounded smooth domain satisfying a uniform interior and exterior sphere condition. Assume* (2.8) *and suppose $F(t)t^{-p}$ is increasing for large t. If the condition*

$$\int_1^\infty \frac{dt}{F(t)^{\frac{1}{\alpha}}} < \infty \tag{3.2}$$

holds for some $\alpha > p$ and if $u(x)$ is a positive solution to problem (1.1) *then $u \in L^{\frac{\alpha-p}{p}}(D)$.*

Proof: Note that condition (3.2) implies (1.2). By (3.1), the solution $u(x)$ near the boundary ∂D behaves like the corresponding solution in one dimension. Therefore, it suffices to prove that the solution $v(r)$ to the problem

$$\Big((v')^{p-1}\Big)' = f(v), \quad v'(0) = 0, \quad v(R) = \infty \tag{3.3}$$

belongs to $L^{\frac{\alpha-p}{p}}(0, R)$.

By (3.3) we find, for a positive constant c_1 and r near R

$$\frac{p-1}{p}(v'(r))^p = \int_{v(0)}^{v(r)} f(t)dt \geq c_1 F(v),$$

whence,

$$F(v(r))^{\frac{1}{p}} \leq c_2 v'(r). \tag{3.4}$$

By (2.3) of Lemma 2.2 (with p replaced by α) we have, for all r greater than some r_1

$$v(r) \leq F(v(r))^{\frac{1}{\alpha}}.$$

Using the latter inequality and (3.4) we find

$$\int_{r_1}^{R} v(r)^{\frac{\alpha-p}{p}} dr \leq \int_{r_1}^{R} F(v(r))^{\frac{1}{p}-\frac{1}{\alpha}} dr$$

$$\leq c_2 \int_{r_1}^{R} \frac{v'(r)dr}{F(v(r))^{\frac{1}{\alpha}}} = c_2 \int_{v(r_1)}^{\infty} \frac{dt}{F(t)^{\frac{1}{\alpha}}} < \infty.$$

The theorem is proved.

Remarks. 1) If D is convex then Theorem 3.1 holds without condition (2.8).
2) Osserman condition (1.2) does not imply that the solution $v(r)$ to the problem

$$\Big((v')^{p-1}\Big)' = f(v), \qquad v(R) = \infty \tag{3.5}$$

belongs to $L^{\epsilon}(0, R)$ for any $\epsilon > 0$. Indeed, using the function

$$v(r) = e^{(R-r)^{-1}} \tag{3.6}$$

we find

$$v'(r) = v\Big(\log v\Big)^2$$

and

$$\Big((v'(r))^{p-1}\Big)' = (p-1)v^{p-1}\Big[\Big(\log v\Big)^{2p} + 2\Big(\log v\Big)^{2p-1}\Big].$$

Hence, the function $v(r)$ is a solution to problem (3.5) with

$$f(v) = (p-1)v^{p-1}\Big[\Big(\log v\Big)^{2p} + 2\Big(\log v\Big)^{2p-1}\Big].$$

A primitive of such a function is

$$F(v) = \frac{p-1}{p}v^p\Big(\log v\Big)^{2p},$$

which clearly satisfies condition (1.2). Note that the function $v(r)$ defined in (3.6) does not belong to $L^\epsilon(0,R)$ for any $\epsilon > 0$.

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