DECAY ESTIMATES FOR SOLUTIONS OF VARIOUS PARABOLIC PROBLEMS

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Abstract

In this paper we establish a maximum principle for solutions of some 1-dimensional parabolic equations. This maximum principle is then applied to construct exponential decay bounds (in time) for solutions of two classes of related boundary value problems.

1. Introduction

In a previous paper [8] we have constructed exponential decay bounds for some quantity involving $u(x,t)$ and $u_x(x,t)$ where $u(x,t)$ is the classical solution of the following initial-boundary value problem

$$[G(u)]_{xx} + f(u) = u_t \ , \ |x| < L \ , \ t > 0, \tag{1.1}$$

$$u(\pm L, t) = 0 \ , \ t \geq 0 \tag{1.2}$$

$$u(x,0) = h(x) \ , \ |x| < L. \tag{1.3}$$

This paper may be considered as a continuation of our previous work when the parabolic equation (1.1) is replaced by

$$g(u, u_x^2)u_{xx} + f(u) = u_t \ , \ |x| < L \ , \ t > 0, \tag{1.4}$$

with

$$g(u, u_x^2) = r(u) \ q(u_x^2) \ , \tag{1.5}$$

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or with

$$g(u, u_x^2) = r(u) + q(u_x^2). \tag{1.6}$$

The first case (1.5) is investigated in Section 3, whereas the second case (1.6) is investigated in Section 4. Section 2 contains some preliminary computations pertinent to both cases (1.5) and (1.6).

2. Maximum principles

In this section we want to establish some maximum principles for auxiliary functions Φ appropriately chosen in terms of the solutions u of the parabolic equations under investigation, of their first space derivatives u_x and of time t. In order to unify the next calculations we start with the general parabolic equation

$$g(u, u_x^2)u_{xx} + f(u) = u_t \ , \ |x| < L \ , \ t > 0 \ , \tag{2.1}$$

where f and g are stictly positive differentiable functions and we consider some auxiliary functions Φ of the form

$$\Phi(x, t) = \Psi(u, u_x^2)\, e^{2\alpha\beta t}, \tag{2.2}$$

where Ψ is a positive function to be specified later and α and β are positive parameters to be selected appropriately. We compute

$$\Psi_x = (\Psi' u_x + 2\dot{\Psi} u_x u_{xx})e^{2\alpha\beta t} \tag{2.3}$$

with $\Psi' := \dfrac{\partial\Psi}{\partial u}$, $\dot{\Psi} := \dfrac{\partial\Psi}{\partial u_x^2}$,

$$\Phi_{xx} = \left\{\Psi'' u_x^2 + 2\dot{\Psi}' u_x^2 u_{xx} + \Psi' u_{xx} + \left(\frac{2\dot{\Psi}}{g} u_x\right)_x g u_{xx} + 2\frac{\dot{\Psi}}{g} u_x (g u_{xx})_x\right\} e^{2\alpha\beta t} \tag{2.4}$$

$$= \left\{\Psi'' u_x^2 + 2\dot{\Psi}' u_x^2 u_{xx} + \Psi' u_{xx} + 2g u_{xx} u_x^2 \left(\frac{\dot{\Psi}}{g}\right)' + 4g u_x^2 u_{xx}^2 \left(\frac{\dot{\Psi}}{g}\right)^{\cdot}\right.$$

$$\left. + 2\dot{\Psi} u_{xx}^2 + \frac{2\dot{\Psi}}{g} u_x u_{xt} - \frac{2\dot{\Psi}}{g} f' u_x^2 \right\} e^{2\alpha\beta t}.$$

From (2.3) we obtain

$$u_{xx} = -\frac{\Psi'}{2\dot{\Psi}} + \frac{e^{-2\alpha\beta t}}{2\dot{\Psi}u_x}\Phi_x = -\frac{\Psi'}{2\dot{\Psi}} + \cdots \tag{2.5}$$

In (2.5) and later , dots stand for terms of the form $w(x,t)\ \Phi_x$ where $w(x,t)$ is some function singular at critical points of $u(x,t)$. Making systematic use of (2.5) in (2.4), we arrive at

$$g\Phi_{xx} + \cdots = \left\{ \left[g\Psi'' - g\frac{\dot{\Psi}'\,\Psi'}{\dot{\Psi}} - g^2\frac{\Psi'}{\dot{\Psi}}\left(\frac{\dot{\Psi}}{g}\right)' \right.\right. \tag{2.6}$$

$$\left.\left. +g^2\left(\frac{\Psi'}{\dot{\Psi}}\right)^2\left(\frac{\dot{\Psi}}{g}\right)^{\cdot} - 2\dot{\Psi}f' \right] u_x^2 + 2\dot{\Psi}u_x u_{xt} \right\} e^{2\alpha\beta t}.$$

Moreover we compute

$$\Phi_t = (\Psi' u_t + 2\dot{\Psi}u_x u_{xt} + 2\alpha\beta\Psi)e^{2\alpha\beta t}. \tag{2.7}$$

From (2.1) and (2.5) we obtain

$$u_t = -\frac{\Psi'}{2\dot{\Psi}}g + f + \cdots \tag{2.8}$$

From (2.7) and (2.8) we have

$$\Phi_t + \cdots = \left\{ -\frac{(\Psi')^2}{2\dot{\Psi}}g + \Psi' f + 2\dot{\Psi}u_x u_{xt} + 2\alpha\beta\Psi \right\} e^{2\alpha\beta t}. \tag{2.9}$$

Combining (2.6) and (2.9)we obtain after some reduction

$$L\Phi := g\Phi_{xx} - \Phi_t + \cdots \tag{2.10}$$

$$= \left\{ \left[(g\Psi')' - \left(\frac{g\Psi'^2}{\dot{\Psi}}\right)^{\cdot} - 2\dot{\Psi}f' \right] u_x^2 + \frac{1}{2}\frac{g\Psi'^2}{\dot{\Psi}} - 2\alpha\beta\Psi - \Psi' f \right\} e^{2\alpha\beta t}.$$

The first class of parabolic equations considered in this paper is obtained from (2.1) when we choose $g(u,u_x^2) = r(u)\ q(u_x^2)$ where r and q are two given positive functions :

$$r(u)q(u_x^2)u_{xx} + f(u) = u_t\ ,\ |x| < L\ ,\ t > 0\ . \tag{2.11}$$

In this case we establish the following maximum principle:

Lemma 1. *Let $u(x,t)$ be a classical solution of (2.11). Assume that $q(\sigma)$ is a nondecreasing function of $\sigma > 0$, and that the functions f and r satisfy the condition*

$$f\int_0^u \frac{d\sigma}{r(\sigma)} - 2\int_0^u \frac{f(\sigma)}{r(\sigma)}d\sigma \geq 0\ . \tag{2.12}$$

We then conclude that the auxiliary function $\Phi(x,t) := \Psi e^{2\alpha\beta t}$ defined on solutions of (2.11) where α is an arbitrary nonnegative parameter with

$$\Psi(u,u_x^2) := \int_0^{u_x^2} q(\sigma)d\sigma + \alpha\left(\int_0^u \frac{d\sigma}{r(\sigma)}\right)^2 + 2\int_0^u \frac{f(\sigma)}{r(\sigma)}d\sigma \tag{2.13}$$

and with

$$\beta \leq \frac{1}{r_{max}}\ , \tag{2.14}$$

takes its maximum value either at $x = \pm L$ for some $t > 0$, or initially at $t = 0$ for some $|x| < L$, or at some critical point $(\overline{x},\overline{t})$ of u , i.e. at $(\overline{x},\overline{t})$ such that $u_x(\overline{x},\overline{t}) = 0$.The above assertion may be formulated as follows

$$\Phi(x,t) \leq \max\begin{cases} \Phi(\pm L,t)\ , & (i) \\ \max\limits_{|x|<L}\Phi(x,0)\ , & (ii) \\ \Phi(\overline{x},\overline{t})\ \textit{with}\quad u_x(\overline{x},\overline{t}) = 0\ . & (iii) \end{cases} \tag{2.15}$$

For the proof of Lemma 1 we insert $\Psi(u,u_x^2)$ defined by (2.13) into (2.10) with $g = rq$. This leads to

$$L\Phi = \left\{\frac{2\alpha q}{r}u_x^2 + \frac{2}{r}\left[\alpha\int_0^u \frac{d\sigma}{r(\sigma)} + f\right]^2 - 2\alpha\beta\Psi\right. \tag{2.16}$$

$$\left. -\frac{2\alpha f}{r}\int_0^u \frac{d\sigma}{r(\sigma)} - \frac{2f^2}{r}\right\}e^{2\alpha\beta t}.$$

Since q is nondecreasing , we have

$$qu_x^2 \geq \int_0^{u_x^2} q(\sigma)d\sigma. \tag{2.17}$$

Combining (2.16) with (2.17) we obtain, after adding and subtracting the quantity $\left(\frac{4\alpha}{r}e^{-2\alpha\beta t}\int_0^u \frac{f(\sigma)}{\sigma}d\sigma\right)e^{2\alpha\beta t}$ at the right hand side of (2.16)

$$L\Phi \geq 2\alpha\Psi\left[\frac{1}{r} - \beta\right] + \left\{\frac{2\alpha}{r}\left[f\int_0^u \frac{d\sigma}{r(\sigma)} - 2\int_0^u \frac{f(\sigma)}{r(\sigma)}d\sigma\right]\right\}e^{2\alpha\beta t}, \tag{2.18}$$

so that $L\Phi \geq 0$ in view of (2.14), and (2.12). The conclusion of Lemma 1 then follows from an application of Nirenberg's maximum principle [4,9].

The second class of parabolic equations considered in this paper is obtained from (2.1) when we choose $g(u, u_x^2) = r(u) + q(u_x^2) > 0$:

$$[r(u) + q(u_x^2)]u_{xx} + f(u) = u_t, |x| < L, t > 0. \tag{2.19}$$

In this case we establish the following maximum principle:

Lemma 2. *Let $u(x,t)$ be a classical solution of (2.19). Assume that $q(\sigma)$ is a nondecreasing function of $\sigma > 0$, and that $\dfrac{f(\sigma)}{\sigma}$ is nonincreasing. Assume moreover that f and r satisfy the two conditions*

$$r\,\frac{f(u)}{u} - (rf)' \geq 0\ , \tag{2.20}$$

and

$$rr'' - \frac{1}{2}r'^2 \geq 0\ ,\ r'' \geq 0\ . \tag{2.21}$$

We then conclude that the auxiliary function $\widetilde{\Phi}(x,t) := \widetilde{\Psi}\ e^{2\alpha t}$ defined on solutions of (2.19) with

$$\widetilde{\Psi}(u, u_x^2) := r(u)u_x^2 + \int_0^{u_x^2} q(\sigma)d\sigma + \alpha u^2 + u\int_0^u \frac{f(\sigma)}{\sigma}d\sigma, \tag{2.22}$$

where α is an arbitrary nonnegative parameter , takes its maximum value either at $x = \pm L$ for some $t > 0$, or initially at $t = 0$ for some $|x| < L$, or at some critical point $(\overline{x}, \overline{t})$ of u , i.e. at $(\overline{x}, \overline{t})$ such that $u_x(\overline{x}, \overline{t}) = 0$.The above assertion may be formulated as follows

$$\widetilde{\Phi}(x,t) \leq \max \begin{cases} \widetilde{\Phi}(\pm L, t)\ , & (i) \\ \max\limits_{|x|<L} \widetilde{\Phi}(x,0)\ , & (ii) \\ \widetilde{\Phi}(\overline{x}, \overline{t}) \textit{ with } \quad u_x(\overline{x}, \overline{t}) = 0\ . & (iii) \end{cases} \tag{2.23}$$

For the proof of Lemma 2 we insert $\widetilde{\Psi}(u, u_x^2)$ defined by (2.22) into (2.10) with $g = r + q$. Using the inequality (2.17) we are led to

$$L\widetilde{\Phi} \geq \left\{ \left[(r+q)r'' - \frac{1}{2}r'^2\right] u_x^4 + [(r+q)(F' - 2f)' - fr']\, u_x^2 \right. \tag{2.24}$$

$$+2\alpha[(F'-f)u-F]+\frac{1}{2}F'(F'-2f)\Big\}\, e^{2\alpha t},$$

with

$$F(u):=u\int_0^u \frac{f(\sigma)}{\sigma}d\sigma\ . \tag{2.25}$$

Using the assumptions on f , we compute from (2.25)

$$F'=\int_0^u \frac{f(\sigma)}{\sigma}+f(u)\geq 0\ , \tag{2.26}$$

$$F'-2f=\int_0^u \frac{f(\sigma)}{\sigma}-f(u)\geq 0\ , \tag{2.27}$$

$$(F'-2f)'=\frac{f(u)}{u}-f'(u)=-u\left(\frac{f}{u}\right)'\geq 0\ . \tag{2.28}$$

Making use of the inequalities (2.20), (2.21), (2.27) and (2.28) in (2.24), we conclude then that

$$L\widetilde{\Phi}\geq 0\ , \tag{2.29}$$

so that the conclusion of Lemma 2 holds true, in view of Nirenberg's maximum principle [4,9].

In the next two sections , we apply Lemmas 1 and 2 in order to derive exponential decay in time of the quantities $\Psi(u,u_x^2)$ and $\widetilde{\Psi}(u,u_x^2)$ (for some values of α) associated to positive solutions of some initial-boundary value problems involving the parabolic equations (2.11) and (2.19).

3. $\mathbf{r(u)\,q(u_x^2)\,u_{xx}+f(u)=u_t}$

In this section we consider the following initial- boundary value problem

$$r(u)q(u_x^2)u_{xx}+f(u)=u_t\ ,\ |x|<L\ ,\ t>0\ , \tag{3.1}$$

$$u(\pm L,t)=0\ ,\ t>0\ , \tag{3.2}$$

$$u(x,0)=h(x)\geq 0\ ,\ |x|<L\ . \tag{3.3}$$

We assume that $0<r_0\leq r(\sigma),\sigma\geq 0$, $0<q_0\leq q(\sigma)$, $\sigma\geq 0$, and that $f(s)\geq 0$, $s>0$, $f(0)=0$. Moreover we make the assumptions of Lemma 1, so that the conclusion (2.15) of Lemma 1 holds . However the first possibility (i) in (2.15) can be eliminated since we have $\Psi' = 0$ and $u_{xx} = 0$ at $x = \pm L$, implying that

$$\Phi_x(\pm L, t) = (\Psi' u_x + 2\dot{\Psi} u_x u_{xx}) e^{2\alpha\beta t} = 0, \tag{3.4}$$

so that Φ cannot take its maximum at $x = \pm L$ in view of Friedman's maximum principle [3,9]. Moreover depending of the behaviour of the ratio $\dfrac{f(s)}{r(s)}$ we may select α small enough in order to eliminate (iii) in (2.15).

We first consider the following particular case:

$$\frac{f(s)}{r(s)} = \mu s \, , \; \mu = \text{const.} \tag{3.5}$$

Suppose now that (iii) holds in (2.15) , i.e. suppose $\Phi(x,t) \le \Phi(\overline{x}, \overline{t})$ with $u_x(\overline{x}, \overline{t}) = 0$. Evaluated at $t = \overline{t}$, we obtain

$$\Psi(u, u_x^2) \le \Psi(u_M, 0), \tag{3.6}$$

with $u_M := \max\limits_{|x|<L} u(x, \overline{t})$ and with

$$\Psi(u, u_x^2) = \int_0^{u_x^2} q(\sigma) d\sigma + \alpha \left(\int_0^u \frac{d\sigma}{r(\sigma)} \right)^2 + \mu u^2, \tag{3.7}$$

i.e.

$$\int_0^{u_x^2} q(\sigma) d\sigma \le \mu(u_M^2 - u^2) + \alpha \left\{ \left(\int_0^{u_M} \frac{d\sigma}{r(\sigma)} \right)^2 - \left(\int_0^u \frac{d\sigma}{r(\sigma)} \right)^2 \right\} . \tag{3.8}$$

From the monotonicity of $q(\sigma)$ we have the inequality

$$q_0 u_x^2 \le \int_0^{u_x^2} q(\sigma) d\sigma \, . \tag{3.9}$$

Using the mean value theorem , we have for some intermediate values ξ_1, ξ_2, ξ_3

$$\begin{aligned} \left(\int_0^{u_M} \frac{d\sigma}{r(\sigma)} \right)^2 - \left(\int_0^u \frac{d\sigma}{r(\sigma)} \right)^2 &= \left[\int_0^{u_M} \frac{d\sigma}{r(\sigma)} + \int_0^u \frac{d\sigma}{r(\sigma)} \right] \int_u^{u_M} \frac{d\sigma}{r(\sigma)} \\ &= \frac{1}{r(\xi_1)} (u_M - u) \left[\int_0^{u_M} \frac{d\sigma}{r(\sigma)} + \int_0^u \frac{d\sigma}{r(\sigma)} \right] \end{aligned} \tag{3.10}$$

$$= \frac{1}{r(\xi_1)}(u_M^2 - u^2)\frac{\int_0^{u_M} \frac{d\sigma}{r(\sigma)} + \int_0^{u} \frac{d\sigma}{r(\sigma)}}{u_M + u}$$

$$\leq \frac{1}{r(\xi_1)}(u_M^2 - u^2) \max \left(\frac{1}{u_M}\int_0^{u_M} \frac{d\sigma}{r(\sigma)} , \frac{1}{u}\int_0^{u} \frac{d\sigma}{r(\sigma)}\right)$$

$$= \frac{1}{r(\xi_1)} \max \left(\frac{1}{r(\xi_2)} , \frac{1}{r(\xi_3)}\right) (u_M^2 - u^2) \leq \frac{1}{r_0^2}(u_M^2 - u^2) .$$

From (3.8) , (3.9) , (3,10) we obtain

$$q_0 u_x^2(x,\overline{t}) \leq \left(\mu + \frac{\alpha}{r_0^2}\right) [u_M^2 - u^2(x,\overline{t})] . \tag{3.11}$$

Rewriting (3.11) as

$$\frac{du(x,\overline{t})}{\sqrt{u_M^2 - u^2(x,\overline{t})}} \leq \sqrt{\frac{1}{q_0}\left(\mu + \frac{\alpha}{r_0^2}\right)}\, dx , \tag{3.12}$$

and integrating from the critical point $\overline{x}$ to the nearest endpoint of the interval $[-L, L]$ we obtain the inequality

$$\mu + \frac{\alpha}{r_0^2} \geq \frac{\pi^2 q_0}{4L^2} =: \alpha_0 . \tag{3.13}$$

The above inequality is a necessary condition to make (iii) possible in (2.15). If (3.13) is violated , (iii) cannot hold , i.e. Φ must take its maximum value at $t = 0$. Suppose that $\mu \leq \alpha_0$.Then for each $\alpha < (\alpha_0 - \mu) r_0^2 =: \alpha_1$ we have $\Phi(x,t) \leq \max_{|x|<L} \Phi(x,0)$. Increasing α to α_1 we obtain the desired inequality

$$\int_0^{u_x^2} q(\sigma)d\sigma + \alpha_1 \left(\int_0^u \frac{d\sigma}{r(\sigma)}\right)^2 + \mu u^2 \leq H^2 e^{-2\alpha_1 \beta t} , \tag{3.14}$$

with

$$H^2 := \max_{|x|<L} \left\{\int_0^{h'^2} q(\sigma)d\sigma + \alpha_1 \left(\int_0^h \frac{d\sigma}{r(\sigma)}\right)^2 + \mu h^2\right\}, \tag{3.15}$$

and

$$\alpha_1 := (\alpha_0 - \mu) r_0^2 = \left(\frac{\pi^2 q_0}{4L^2} - \mu\right) r_0^2 . \tag{3.16}$$

It is worthwhile to mention that the decay bounds (3.14) hold even in the more general case corresponding to

$$\frac{f(s)}{sr(s)} \leq \mu \leq \alpha_0 := \left(\frac{\pi^2 q_0}{4L^2}\right) , \forall s \geq 0. \tag{3.17}$$

Clearly if (3.17) holds , we have

$$r(u)q(u_x^2)u_{xx} + \mu\, u\, r(u) - u_t = u\, r(u)\left[\mu - \frac{f(u)}{ur(u)}\right] \geq 0\,, \tag{3.18}$$

so that $u(x,t)$ is not greater than the solution of (3.1)-(3.3) corresponding to the particular case (3.5). Moreover we can repeat the preceeding argument with only minor modifications. In fact with (2.13) , inequality (3.8) has to be replaced by

$$\int_0^{u_x^2} q(\sigma)d\sigma \leq \alpha\left\{\left(\int_0^{u_M}\frac{d\sigma}{r(\sigma)}\right)^2 - \left(\int_0^{u}\frac{d\sigma}{r(\sigma)}\right)^2\right\} + 2\int_u^{u_M}\frac{f(\sigma)}{r(\sigma)}d\sigma. \tag{3.19}$$

Using (3.17), the last term in (3.19) may now be estimated as follows

$$2\int_u^{u_M}\frac{f(\sigma)}{r(\sigma)}d\sigma = 2\int_u^{u_M}\frac{f(\sigma)}{\sigma r(\sigma)}\sigma d\sigma \leq 2\mu\int_u^{u_M}\sigma d\sigma = \mu(u_M^2 - u^2). \tag{3.20}$$

Combining (3.19) and (3.20) we obtain (3.8), and the subsequent computations remain valid without any change.

Finally we want to investigate the problem (3.1)-(3.3) under the assumption that $\dfrac{f(\sigma)}{\sigma\, r(\sigma)}$ is nondecreasing for $\sigma > 0$, i.e. under the assumption

$$\left(\frac{f(\sigma)}{\sigma r(\sigma)}\right)' \geq 0\,, \sigma > 0\,. \tag{3.21}$$

In this case we want to show that $u(x,t)$ cannot blow up if the initial data $h(x)$ are small enough . Our first analysis will be confined on any time interval $(0,T)$ with T prior an (hypothetic) blow up time $\hat{t}$. In a first step we establish the following comparison result:

Lemma 3. *Under the assumption of Lemma 1 and assumption (3.21) , the solution $u(x,t)$ of problem (3.1)-(3.3) may be estimated as follows*

$$0 \leq u(x,t) \leq Ue^{-(\alpha_0-\mu)r_0^2\beta t}\,,\ |x| < L\,, 0 < t < T\,, \tag{3.22}$$

with

$$U := \max_{|x|<L}\sqrt{\frac{1}{\mu}\int_0^{h'^2} q(\sigma)d\sigma + \frac{\alpha_1}{\mu}\left(\int_0^h \frac{d\sigma}{r(\sigma)}\right)^2 + h^2}\,, \tag{3.23}$$

$$\mu := \frac{f(u_M)}{u_M r(u_M)} \ , \ \ u_M := \max_{(-L,L)\times(0,T)} u(x,t), \tag{3.24}$$

$$\alpha_0 := \frac{\pi^2 q_0}{4L^2} \ , \ \alpha_1 := (\alpha_0 - \mu) r_0^2 \ , \tag{3.25}$$

$$\beta \ \leq \ \frac{1}{r_{max}} \ . \tag{3.26}$$

For the proof of Lemma 3, we note that (3.18) is valid with μ defined in (3.24). It then follows that $u(x,t)$ is not greater than the solution of (3.1)-(3.3) with $f(u)$ in (3.1) replaced by $\mu\, u\, r(u)$, for which we have established the estimate (3.14) with μ defined by (3.24). This establishes Lemma 3.

In a second step , we establish the next result:

Lemma 4. *Assuming the hypotheses of Lemma 3 and that $h(x) \geq 0$ is small enough in the following sense*

$$\frac{f(U)}{Ur(U)} < \alpha_0 := \frac{\pi^2 q_0}{4L^2} \ , \tag{3.27}$$

where U is defined by (3.23),we then conclude that the solution $u(x,t)$ of (3.1)-(3.3) exists for all time (i.e. $T = \infty$). Moreover we have

$$\max_{|x|<L} \ \frac{f(u(x,t))}{u(x,t) r(u(x,t))} < \alpha_0 \ , \ \forall t > 0 \ . \tag{3.28}$$

For the proof of Lemma 4, we observe that (3.21), (3.23) and (3.27) imply the inequality

$$\frac{f(h)}{hr(h)} \leq \frac{f(U)}{Ur(U)} < \alpha_0 \ . \tag{3.29}$$

Suppose now that (3.28) does not hold for all time. In view of (3.29) there must be a first time T at which we have

$$\mu := \frac{f(u_M))}{u_M r(u_M)} \leq \ \max_{|x|<L} \ \frac{f(u(x,T))}{u(x,T) r(u(x,T))} = \alpha_0. \tag{3.30}$$

It then follows from Lemma 3 that

$$u(x,t) \leq U e^{-(\alpha_1 - \mu) r_0^2 \beta t} \leq U \ , \ |x| < L \ , 0 \leq t \leq T \ . \tag{3.31}$$

From (3.31) and (3.27) we obtain

$$\max_{|x|<L} \frac{f(u(x,T))}{u(x,T)r(u(x,T))} \leq \frac{f(U)}{Ur(U)} < \alpha_0, \tag{3.32}$$

from which we conclude indeed that (3.28) cannot be violated for any finite value of T. This achieves the proof of Lemma 4.

In a third and last step we establish the desired decay bound for $\Psi(u, u_x^2)$ formulated in the next theorem:

Theorem 5. *Assuming the hypotheses of Lemma 3 and that the data $h(x)$ are small enough in the sense that there exists a constant α_2 such that*

$$\frac{f(U)}{Ur(U)} < \alpha_0 - \frac{\alpha_2}{r_0^2} = \frac{\pi^2 q_0}{4L^2} - \frac{\alpha_2}{r_0^2}\,, \tag{3.33}$$

where U is defined by (3.23),we have the following decay estimate:

$$\int_0^{u_x^2} q(\sigma)d\sigma + \alpha_2 \left(\int_0^u \frac{d\sigma}{r(\sigma)} \right)^2 + 2\int_0^u \frac{f(\sigma)}{r(\sigma)} d\sigma \leq \mathcal{H}^2 e^{-2\alpha_2 \beta t}\,, \tag{3.34}$$

with β defined in (3.26) and with

$$\mathcal{H}^2 := \max_{|x|<L} \left\{ \int_0^{h'^2} q(\sigma)d\sigma + \alpha_2 \left(\int_0^h \frac{d\sigma}{r(\sigma)} \right)^2 + 2\int_0^h \frac{f(\sigma)}{r(\sigma)} d\sigma \right\}. \tag{3.35}$$

For the proof of Theorem 5, we first observe that (3.33) implies (3.27), so that the solution $u(x,t)$ of (3.1)-(3.3) does not blow up in any finite time. We want now to eliminate the third possibility (iii) in (2.15) for the auxiliary function $\Phi(x,t) = \Psi(u, u_x^2)e^{2\alpha_2\beta t}$, with Ψ defined in (2.13). Suppose on the contrary that we have $\Phi(x,t) \leq \Phi(\overline{x}, \overline{t})$ with $u_x(\overline{x}, \overline{t}) = 0$. With $t = \overline{t}$ we obtain, using again (3.9) and (3.10)

$$q_0 u_x^2(x, \overline{t}) \leq \frac{\alpha_2}{r_0^2}(u_M^2 - u^2) + 2\int_u^{u_M} \frac{f(\sigma)}{r(\sigma)} d\sigma \leq \left(\frac{\alpha_2}{r_0^2} + \mu \right)(u_M^2 - u^2), \tag{3.36}$$

from which we obtain as usual

$$\frac{\alpha_2}{r_0^2} + \mu \;\geq\; \frac{\pi^2 q_0}{4L^2}\,. \tag{3.37}$$

It remains to show that (3.37) cannot hold under assumption (3.33). Indeed from (3.28) we have the strict inequality

$$\mu := \frac{f(u_M)}{u_M r(u_M)} < \alpha_0 \ , \tag{3.38}$$

so that $u_M \leq U$ by (3.22). It then follows from(3.21) , (3.38) , and (3.33) that

$$\mu \leq \frac{f(U)}{U r(U)} < \alpha_0 - \frac{\alpha_2}{r_0^2} \ , \tag{3.39}$$

in contradiction to (3.37).This achieves the proof of Theorem 5.

4. $[\mathbf{r(u) + q(u_x^2)}]\mathbf{u_{xx} + f(u) = u_t}$

In this section we consider the following initial-boundary value problem

$$[r(u) + q(u_x^2)] \ u_{xx} + f(u) = u_t \ , \ |x| < L \ , \ t > 0 \ , \tag{4.1}$$

$$u(\pm L, t) = 0 \ , \ t > 0 \ , \tag{4.2}$$

$$u(x, 0) = h(x) \geq 0 \ , \ |x| < L \ , \tag{4.3}$$

with

$$r(\sigma) + q(\tau) \geq \epsilon \ \text{ in } R^+ \times R^+, \epsilon := \text{const.} > 0 \, , \tag{4.4}$$

and with $f(\sigma) \geq 0, \sigma > 0$ such that

$$f(0) = 0 \ , \ \lim_{\sigma \nearrow 0} \frac{f(\sigma)}{\sigma} =: \gamma \leq \alpha_0 := \frac{\pi^2 \epsilon}{4L^2} \ . \tag{4.5}$$

Moreover we assume that the conditions of Lemma 2 are satisfied so that the conclusion (2.23) holds. Assuming in addition that $r'(0) = 0$, we have from (2.22),(4.2) and (4.3)

$$\widetilde{\Phi}_x(\pm L, t) = \widetilde{\Psi}_x(\pm L, t) e^{2\alpha t} = 0 \ , \tag{4.6}$$

so that the first possibility (i) in (2.23) cannot hold as a consequence of Friedman's maximum principle [3,9]. We now investigate two cases for which the nonnegative parameter α can be selected small enough in order to make (iii) impossible in (2.23). We first consider the following case

$$\frac{f(\sigma)}{\sigma} = \gamma \leq \alpha_0 \ , \ \forall \sigma > 0 \ . \tag{4.7}$$

In this case we have

$$\widetilde{\Phi}(x,t) := \left\{ r(u)u_x^2 + \int_0^{u_x^2} q(\sigma)d\sigma + (\alpha+\gamma)u^2 \right\} e^{2\alpha t}. \tag{4.8}$$

Suppose now that (iii) holds in (2.23) i.e. suppose $\widetilde{\Phi}(x,t) \leq \widetilde{\Phi}(\overline{x},\overline{t})$ with $u_x(\overline{x},\overline{t}) = 0$. Evaluated at $t = \overline{t}$ we obtain using (4.4)

$$\epsilon u_x^2(x,\overline{t}) \leq r(u)u_x^2 + \int_0^{u_x^2} q(\sigma)d\sigma \leq (\alpha+\gamma)[u_M^2 - u^2(x,\overline{t})], \tag{4.9}$$

with $u_M := \max\limits_{|x|<L} u(x,\overline{t})$. From (4.9) we obtain as usual the inequality

$$\alpha + \gamma \geq \frac{\pi^2 \epsilon}{4L^2} =: \alpha_0. \tag{4.10}$$

We then conclude that if $\alpha < \alpha_0 - \gamma$, (iii) in (2.23) cannot hold so that we must have (ii) in (2.23). With $\alpha \nearrow \alpha_0 - \gamma$ we are led to the following exponential decay estimate

$$r(u)u_x^2 + \int_0^{u_x^2} q(\sigma)d\sigma + (\alpha_0 - \gamma)u^2 \leq \widetilde{H}^2 e^{-2(\alpha_0-\gamma)t}, \tag{4.11}$$

with

$$\widetilde{H}^2 := \max_{|x|<L} \left\{ r(h)h'^2 + \int_0^{h'^2} q(\sigma)d\sigma + (\alpha_0-\gamma)h^2 \right\} . \tag{4.12}$$

Finally let us consider the case where the ratio $\dfrac{f(\sigma)}{\sigma}$ is nonincreasing with respect to σ . In this case we have

$$[r(u) + q(u_x^2)]\ u_{xx} + \gamma u - u_t = u \left[\gamma - \frac{f(u)}{u} \right] \geq 0 \ , \ |x| < L \ , \ t > 0 \ , \tag{4.13}$$

so that the solution of (4.1)-(4.3) is dominated by the solution associated to the previous case and exists therefore for all time. Suppose now that (iii) holds in

(2.23) with $\widetilde{\Phi}(x,t) := \widetilde{\Psi}(u, u_x^2)e^{2\alpha t}$, where $\widetilde{\Psi}$ is defined by (2.22).We then obtain using (4.4)

$$\epsilon u_x^2(x,\overline{t}) \le \alpha[u_M^2 - u^2(x,\overline{t})] + u_M \int_0^{u_M} \frac{f(\sigma)}{\sigma} d\sigma - u\int_0^u \frac{f(\sigma)}{\sigma} d\sigma. \qquad (4.14)$$

Using the generalized mean value theorem and the monotonicity of $\dfrac{f(\sigma)}{\sigma}$ we have

$$u_M \int_0^{u_M} \frac{f(\sigma)}{\sigma} d\sigma - u\int_0^u \frac{f(\sigma)}{\sigma} d\sigma \qquad (4.15)$$

$$= \frac{u_M \int_0^{u_M} \frac{f(\sigma)}{\sigma} d\sigma - u\int_0^u \frac{f(\sigma)}{\sigma} d\sigma}{u_M^2 - u^2}(u_M^2 - u^2) = \frac{\int_0^\xi \frac{f(\sigma)}{\sigma} d\sigma + f(\xi)}{2\xi}[u_M^2 - u^2] \le \gamma[u_M^2 - u^2].$$

Combining (4.14) and (4.15) , we obtain again (4.9) , so that if $\alpha < \alpha_0 - \gamma$, (iii) cannot hold in (2.23), implying that we must have (ii) in (2.23). With $\alpha \nearrow \alpha_0 - \gamma$ we obtain the following exponential decay estimate

$$r(u)u_x^2 + \int_0^{u_x^2} q(\sigma)d\sigma + (\alpha_0 - \gamma)u^2 + u\int_0^u \frac{f(\sigma)}{\sigma} d\sigma \le \widetilde{\mathcal{H}}^2 \, e^{-2(\alpha_0-\gamma)t}, \qquad (4.16)$$

with

$$\widetilde{\mathcal{H}}^2 := \max_{|x|<L} \left\{ r(h)h'^2 + \int_0^{h'^2} q(\sigma)d\sigma + (\alpha_0 - \gamma)h^2 + h\int_0^h \frac{f(\sigma)}{\sigma} d\sigma \right\}. \qquad (4.17)$$

It is worthwhile to mention that if one of the two functions r and q is constant, we may apply the results of Section 3.

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