EQUIVARIANT INTERSECTION THEORY AND BOTT'S RESIDUE FORMULA

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Introduction

The goal of these notes is to describe a version of Raoul Bott's residue formula and illustrate its applications to enumerative geometry.

The ideas for the formula go back at least to H. Hopf's well known result expressing the Euler number of a manifold in terms of local indices around the zeroes of a vector field. Bott's formula relate characteristic numbers (i.e., top intersection of Chern classes) to local invariants of a suitable vector field near its zeroes.

The coming into play of such a powerful tool in enumerative geometry is mainly due to S.A. Strømme & G. Ellingsrud [14], culminating with the landmark results of Kontsevich [30] about enumeration of rational curves.

We follow closely the articles of D. Edidin and W. Graham [11] and of M. Brion [7] for the basic notions of equivariant Chow rings. The proof of Bott's formula presented here is essentially copied from [7]. It rests on a theorem of localization that describes the equivariant Chow ring in terms of the usual Chow ring of the fixed points locus.

Our main interest goes towards applications to enumerative geometry. We include several examples showing how the residue formula can be used to compute characteristic numbers and a few Gromov-Witten invariants. Some of the applications are classical, *e.g.*, the 27 lines on a cubic surface. Our calculation of the number of canonical curves in $\mathbb{P}^3$ incident to 24 general lines is based on

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recent work by J. Rojas and the 2nd author [35] and part of the 1st author's doctoral dissertation [31].

We assume the reader is familiar with basic notions of algebraic group actions, quotient spaces and intersection theory, cf. the books by A. Borel [5] and W. Fulton [17].

It is a pleasure to thank the organizers for the opportunity to give a series of lectures at the first Escola de Álgebra of the new millennium for which the original Portuguese version of these notes were prepared.

Blanket assumptions

We work over the field of complex numbers. Schemes are quasi-projective over $\mathbb{C}$. Variety means integral scheme. All maps are morphisms over $\mathbb{C}$. A point in a variety is always a closed point (i.e., a $\mathbb{C}$-point).

A G-space is a scheme X endowed with an algebraic action $G \times X \to X$ where G is a linear group (often $G = T$, a torus). We assume throughout these notes that $X \subset \mathbb{P}^N$ is a quasi-projective subscheme and the action is induced by a representation of G in $\mathbb{C}^{N+1}$.

1 The Chow group

We introduce in this chapter the equivariant Chow group. It is defined in terms of the "classical" Chow group, for which we recall a few basic facts. We also offer a light invariant intermezzo.

1.1 The usual Chow group

The canonical reference for the material of this section is Fulton [17] or [18].

1.1.1 Group of Cycles

Let X be a scheme and set $n = \dim(X)$. The *group of cycles of dimension* k, or k*-cycles* in X is the free abelian group generated by the set of closed irreducible subvarieties of dimension k in X. It is denoted by $Z_k(X)$. The *group of cycles* of X is the graded group

$$Z_*(X) \ := \ \bigoplus_{k=0}^{n} Z_k(X)$$

By definition, each k-cycle c in $Z_k(X)$ can be written uniquely as a linear combination with integer coefficients,

$$c \ = \ \sum_V n_V \cdot V$$

where V ranges in the collection of (closed and irreducible) subvarieties of X of dimension k, with $n_V \neq 0$ for at most finitely many V's.

The *support* of a cycle $c = \sum n_V V$ is defined by

$$|c| \ = \ \bigcup_{n_V \neq 0} V.$$

Let $X_1, \cdots, X_m$ be the irreducible components of a scheme X. The *fundamental cycle* of X is defined by

$$[X] \ = \ \sum_{i=1}^{m} m_i X_i$$

where $m_i = l(\mathcal{O}_{X,X_i})$ is the length of the local ring of X along X_i.

Since the local ring $\mathcal{O}_{X,X_i}$ is artinian, the length is a positive integer, called the geometric multiplicity of X at X_i.

1.1.2 Rational equivalence

Let V be a variety and let $R(V)$ be the field of rational functions of V. Let $r \in R(V)$ be a nonzero rational function. We define the *order* of r along a subvariety $W \subset V$ of codimension 1 by

$$\mathrm{ord}_W(r) \ := \ l(A/(a)) \ - \ l(A/(b)),$$

where $A = \mathcal{O}_{V,W}$ and $r = a/b$ with $a, b \in A$.

We note that ord_W is well defined cf. [17] and

$$\mathrm{ord}_W(r \cdot s) = \mathrm{ord}_W(r) \; + \; \mathrm{ord}_W(s), \; \forall r, s \in R(V).$$

We define the *the divisor of a rational function r on a variety V* as

$$\mathrm{div}(r) \; := \; \sum_W \mathrm{ord}_W(r) \cdot W$$

where W ranges in the collection of closed and irreducible subvarieties of V of codimension 1.

The above formal sum is in fact a cycle on V since $\mathrm{ord}_W(r) \neq 0$ only for finitely many subvarieties of V, cf. [17].

Let X be a scheme. The group of *k-cycles rationally equivalent to zero* on X is defined as the subgroup $R_k(X)$ of $Z_k(X)$ spanned by divisors of rational functions of subvarieties of X of dimension $k+1$. The *group of cycles rationally equivalent to zero* is the graded group

$$R_*(X) \; := \; \bigoplus_{k=0}^{n} \; R_k(X)$$

The graded quotient group

$$A_*(X) \; := \; Z_*(X)/R_*(X) \; = \; \bigoplus_{k=0}^{n} Z_k(X)/R_k(X)$$

is called *the Chow group of X*.

1.2 The G-Invariant Group of Chow and the Theorem of Hirschowitz

We explain the construction of the G-invariant Chow groups. On the one hand, these groups have motivated the study of the G-equivariant groups. On the other hand, it so happens that in many interesting cases the two groups practically coincide.

1.2.1 Cycles and G-invariant rational equivalence

Let X be a G-space. The *G-invariant Chow group* of X is the quotient group $A_k(X,G) = Z_k(X,G)/R_k(X,G)$, where $Z_k(X,G) \subset Z_k(X)$ is the subgroup generated by the closed irreducible subvarieties of X that are G-invariant. The subgroup $R_k(X,G) \subset R_k(X)$ is generated by all divisors of rational eigenfunctions on G-invariant subvarieties of X of dimension $k+1$.

We recall that a rational function f on a G-invariant subvariety, $W \subset X$, is said to be an eigenfunction if $g \circ f = \chi(g) \cdot f$ for all $g \in G$ and some character $\chi = \chi_f$ of G.

Note that the inclusion $Z_k(X,G) \subset Z_k(X)$ induces a natural homomorphism $A_k(X,G) \to A_k(X)$. In general, it is neither injective nor surjective.

1.2.2 Exercice.

Let X be an elliptic curve. Consider the $\mathbb{Z}_2$-action induced by $x \mapsto -x$. Show that the invariant proper subvarieties consist of the 4 points of order 2. Deduce that $A_0(X,\mathbb{Z}_2) \to A_0(X)$ is not surjective. Find an example where injectivity fails.

When the linear group G is connected and solvable (*e.g.*, $G = T$ a torus), we have the following.

1.3 Theorem of Hirschowitz

If a solvable and connected linear algebraic group G acts on a projective variety X, then the natural homomorphism $A_k(X,G) \longrightarrow A_k(X)$ is bijective.

The above theorem was originally proven by André Hirschowitz [23] in 1984 in the case when X is a projective variety. In 1995, W. Fulton, R. MacPherson, F. Sottile and B. Sturmfels [19] proved the general case. The rest of this section is devoted to a sketch of the proof given by Hirschowitz in [23]. The hypotheses are required in order to validate the use of the principal tool, namely,

1.3.1 Borel's fixed point Theorem.

Let G be a solvable and connected linear algebraic group acting on a nonempty projective variety V. Then G has a fixed point in V. ([5], pág. 242.)

Now the idea is to apply the previous theorem to the Chow variety of the projective variety X, denoted by $CH(X)$. The latter parametrizes the effective cycles (of subvarieties) of the projective variety X. It is known that the irreducible components of $CH(X)$ are projective varieties, cf. [20].

Denote by $Z_*^+(X) \subset Z_*(X)$ the subgroup of effective cycles (similarly, $Z_*^+(X,G) \subset Z_*(X,G)$). By definition there exists a natural bijection

$$Z_*^+(X) \longleftrightarrow CH(X).$$

We shall say that a 1-cycle is *rational* if its support is a connected union of some family of irreducible curves of geometric genus zero. Thus, saying that two points of $CH(X)$ are members of a rational 1-cycle is tantamount to saying that the corresponding cycles in X are rationally equivalent. Conversely, if U is a cycle on X rationally equivalent to zero, then there exist a rational 1-cycle C in $CH(X)$ and two points U_1 and U_2 on C such that $U = U_1 - U_2$. We say a variety Y is rationally connected if each pair of points (x,y) of Y is contained in a rational 1-cycle of Y.

1.3.2 Proposition

Let Y be a unirational projective variety. Then Y is rationally connected.

Proof. Recall that Y unirational means that there is a dominant, rational map $\mathbb{P}^n \cdots \to Y$ for a suitable n. By resolution of singularities, there exists a surjective map $\widehat{X} \to Y$ such that $\widehat{X}$ is obtained from $\mathbb{P}^n$ by finitely many blowups along smooth subvarieties. Let X be a rationally connected smooth projective variety. Let $Z \subset X$ be a smooth subvariety. It suffices to show that the blowup $\widetilde{X}$ of X along Z is again rationally connected. Let $\widetilde{Z}$ denote the exceptional divisor. It is enough to connect a point $\tilde{z}$ on $\widetilde{Z}$ to some point outside the exceptional divisor. Say $\tilde{z}$ lies over $z \in Z$. Join z to a point $x \in X \smallsetminus Z$.

Let C be a component of a rational chain joining z to x such that $z \in C$. If C is not contained in Z then C lifts to a rational curve $\widetilde{C} \subset \widetilde{X}, \widetilde{C} \not\subset \widetilde{Z}$. Let $\tilde{z}' \in \widetilde{C}$ map to z. Then $\tilde{z}, \tilde{z}'$ lie on a fiber of the exceptional divisor and and we are done since $\widetilde{Z} \to Z$ is a projective bundle. If $C \subset Z$, then normalizing C we see that the pullback of $\widetilde{Z}$ over C is rationally connected, so that $\tilde{z}$ can be joined to a point $\tilde{z}'$ lying over a point in the intersection of C with another component of the rational chain. This brings us closer to the point x and we are done by induction on the number of components of the chain.

□

As a matter of fact, the main application of the theorem of Hirschowitz needed in the sequel is to assert the map is surjective, i.e., the Chow ring $A_*(X)$ is generated by classes of G-invariant subvarieties of X.

Proof. (*of the Theorem of Hirschowitz*)

Let $A_*^+(X)$ be the image of $Z_*^+(X)$ in $A_*(X)$. We will show that $Z^+(X,G) \to A^+(X)$ is surjective and hence, so is $A_*(X,G) \to A_*(X)$.

Given $U \in Z_*^+(X)$, then we have $U \in CH(X)$ and the closure $V = \overline{G \cdot U}$ of the orbit $G \cdot U$ is a G-invariant projective variety.

Therefore, by Borel's Theorem, there exists a fixed point $U_G \in V$. Since G is rational[1] we have that V is unirational. Hence, by the above proposition U and U_G are rationally equivalent, thereby determining the same class in $A_*(X)$.

For the injectivity, we just give a rough idea. We must show that the kernel of the map $Z_*(X,G) \to A_*(X)$ is equal to $R_*(X,G)$. Let U be a cycle rationally equivalent to zero. Then we may write $U = U^1 - U^2$ with $U^1, U^2 \in CH(X)$ points representing two effective cycles lying in a rational 1-cycle $C \subset CH(X)$. Thus, (U^1, C, U^2) is a point in $CH(X) \times CH(CH(X)) \times CH(X)$. The closure W of the orbit of that point under the G-action is a projective variety. Again by Borel's Theorem, there exists a fixed point $(U_G^1, C_G, U_G^2) \in W$. Looking at the two projection maps onto $CH(X)$, one shows that C_G is a rational 1-cycle

[1]This is obvious if G is a torus; for the general case, cf. [5], 15.8.

containing U_G^1 and U_G^2 and that $U = U_G^1 - U_G^2$, and so we get $U \in R(X, G)$.

□

1.4 The G-equivariant Chow group

In this section we introduce the G-equivariant Chow group of a G-space X. The functorial properties of flat pullback and proper pushforward will be reviewed in the G-equivariant context. Details will be omitted; most of them are consequence of (more or less) well known results about quotients of varieties by algebraic group actions that go far beyond the objectives of these notes. The canonical references for these foundations are Borel [5] and Mumford [33].

We shall also define G-equivariant Chern classes for a G-equivariant vector bundle E over X. Last but not least, we discuss at some length some examples which, trivial as they may appear at first sight, yet they will be enough for the applications we present in the final chapters.

1.4.1 G-principal bundle

Let G be a linear group and let X be a G-space. Set

$$g = \dim G \quad \text{and} \quad n = \dim X.$$

We choose an l-dimensional representation V of G such that V contains an invariant open dense subset U where the action is free. Such $U \subset V$ will be explicitly described in the main examples.

Let $\pi : U \to \overline{U} := U/G$ be the quotient G-principal bundle. This means that there exists an open cover $\{\overline{U}_i\}$ of $\overline{U}$ such that $\pi^{-1}\overline{U}_i \simeq \overline{U}_i \times G$, with transition functions $\varphi_{ij} : \overline{U}_{ij} \to G$. Such a quotient always exists as an algebraic space, since G acts freely on U.

For the cases we shall have a closer look, the quotient U/G is in fact a product of projective spaces and the construction is elementary, cf. 1.5.1.

Note that the diagonal action $(\gamma, x, u) \mapsto (\gamma \cdot x, \gamma \cdot u)$ over $X \times U$ is also free. Hence there exists a quotient $X \times U \to (X \times U)/G$ in the category of algebraic spaces which is a G-principal bundle. We denote the quotient $(X \times U)/G$ by $X \times^G U$, or X_G for short. Again, in all cases we are particularly interested, the quotient X_G is a projective scheme.

Henceforth, the notation $U \subset V$ means an open dense subset U of a representation V of G on which G acts freely, and X_G denotes the base of the G-principal quotient bundle, which will also be written

$$X \times U \longrightarrow X_G = X \times^G U.$$

The main observation here is that the choice of $U \subset V$ can always be made in such a way that the map of restriction of cycles of $X \times V$ to the open subset $X \times U$ is bijective for any pre-assigned dimension.

1.4.2 A very instructive example

Let us consider $G = T = \mathbb{C}^*$, the unidimensional torus, acting on $X = \mathbb{P}^1$ via $t\circ[x_0, x_1] = [x_0, t \cdot x_1]$. Fix $l > 1$ and look at the diagonal representation of T in $V = \mathbb{C}^l$, $(v \to t \cdot v)$. Now set $U = V \smallsetminus \{0\}$. It is clear that T acts freely on U. Our T-principal bundle $U \to U/T$ is nothing but the familiar construction $\mathbb{C}^l \smallsetminus \{0\} \to \mathbb{P}^{l-1}$. Continuing, $X \times U \to X_T$ also is a T-principal bundle, whose base X_T we go on to describe. Examine the map

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{C}^l \smallsetminus \{0\} & \xrightarrow{\psi} & \mathbb{P}^{l-1} \times \mathbb{P}^l \\ (x, y) = \big([x_0, x_1], (y_1, \ldots, y_l)\big) & \longmapsto & ([y_1, \ldots, y_l], [x_1,\, x_0y_1, \ldots, x_0y_l]). \end{array}$$

The reader should have no difficulty to verify that we have

$$\psi\big(t\circ(x, y)\big) = \psi(x, y),\ \forall t \in T, x \in \mathbb{P}^1, y \in \mathbb{C}^l \smallsetminus \{0\}.$$

In fact, $\psi^{-1}(\psi(x, y)) = T\circ(x, y) \cong T$. Taking $z_0, \ldots, z_l$ as homogeneous coordinates for the second factor, we see that the image of ψ is the subvariety $W \subset \mathbb{P}^{l-1} \times \mathbb{P}^l$ given by $y_iz_j = y_jz_i$, $1 \leq i, j \leq l$. The projection $W \to \mathbb{P}^{l-1}$ in fact identifies that variety with the $\mathbb{P}^1$-bundle $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{l-1}} \oplus \mathcal{O}_{\mathbb{P}^{l-1}}(-1)\big)$. Summarizing, we have in fact $X_T \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{l-1}} \oplus \mathcal{O}_{\mathbb{P}^{l-1}}(-1)\big) \to \mathbb{P}^{l-1}$. See 1.9 for a generalization.

1.4.3 Proposition-Definition

We define the ith G-equivariant Chow group of X by

$$A_i^G(X) = A_{i+l-g}(X_G),$$

where $l = \dim(V)$*,* $g = \dim(G)$ *and* A_* *denotes the usual Chow group. The group is independent from the chosen representation, provided* $V - U$ *is of codimension sufficiently big, i.e.,* $> \dim X - i$*.*

Proof. We use the so called *double fibration* trick of Bogomolov. Let V_1 and V_2 be representations of G with respective dimensions l_1 and l_2, satisfying the conditions above. That is, there exist open subsets $U_1 \subset V_1$ and $U_2 \subset V_2$ such that G acts freely on U_1 and U_2 and the complements $V_1 - U_1$ and $V_2 - U_2$ have codimension bigger than $n - i$.

Let G act diagonally on $V_1 \oplus V_2$. Then, $V_1 \oplus V_2$ contains an open subset W that contains both $V_1 \oplus U_2$ and $U_1 \oplus V_2$ on which G acts freely. Thus, there exists the G-principal bundle quotient W/G. Hence, we have that

$$A_{i+l_1+l_2-g}(X \times^G W) = A_{i+l_1+l_2-g}(X \times^G (U_1 \oplus V_2)).$$

This holds because the closed subset taken away, $(X \times^G W) \smallsetminus (X \times^G (U_1 \oplus V_2))$, is of dimension smaller than $i + l_1 + l_2 - g$.

On the other hand, the projection $V_1 \oplus V_2 \to V_1$ renders $X \times^G (U_1 \oplus V_2)$ a vector bundle over $X \times^G U_1$ with fiber V_2 and structural group G. Thus, $A_{i+l_1+l_2-g}(X \times^G (U_1 \oplus V_2)) = A_{i+l_1-g}(X \times^G U_1)$, thereby implying the equality $A_{i+l_1+l_2-g}(X \times^G W) = A_{i+l_1-g}(X \times^G U_1)$. Similarly, one shows that the groups $A_{i+l_1+l_2-g}(X \times^G W)$ and $A_{i+l_2-g}(X \times^G U_2)$ are one and the same. Hence, $A_i^G(X)$ is independent from the representation.

□

Whenever we write $A_i^G(X) = A_{i+l-g}(X_G) = A_{i+l-g}(X \times^G U)$ it is always assumed that a representation V and the open subset $U \subset V$ were chosen in such a way that $V - U$ is of codimension bigger than $n - i$ in V.

1.4.4 Invariant cycles

If $Y \subset X$ is a G-invariant subvariety of X of dimension m, then Y gives rise to a fundamental G-equivariant class, $[Y]_G = [Y \times^G U] \in A_m^G(X)$.

In general, if V is an l-dimensional representation of G and $S \subset X \times V$ is an invariant subvariety of dimension $m + l$, then S admits a *G-equivariant fundamental class* $[S]_G \in A_m^G(X)$ given by

$$[S]_G = [\big(S \cap (X \times U)\big)/G].$$

1.4.5 Lemma

Let $\pi : X \times U \to X \times^G U = X_G$ be the quotient map and let $Z \subset X_G$ be a closed irreducible subvariety. Then $\pi^{-1}Z \subset X \times U$ is G-invariant (and irreducible if G is connected).

1.4.6 Non-triviality of $A_i^G(X)$.

Notice it may well happen that the usual Chow group $A_i(X)$ be trivial, but $A_i^G(X)$ be nonzero for some $i \leq n$, including negative i. Take for example $X = pt$, yes, just a single point, and $G = T = \mathbb{C}^*$. Presently, X_T is just U/T. Choosing a representation as in (1.4.2), we see that $A_i^T(X) = A_{i+l-1}(\mathbb{P}^{l-1})$ is zero for $i > 0$ and isomorphic to $\mathbb{Z}$ for all $i \leq 0$.

1.4.7 The ring structure

Whenever X is a smooth variety, the G-equivariant Chow group

$$A_*^G(X) = \bigoplus A_i^G(X)$$

inherits an intersection product from the ordinary Chow groups. This endows $A_*^G(X)$ with the structure of a graded ring. In this case, it is more convenient to take the grading given by codimension, writing

$$A_G^j(X) = A_{\dim G - j}^G(X) \quad \text{and} \quad A_G^*(X) = \bigoplus A_G^i(X).$$

1.4.8 Proposition

Let $\alpha \in A_m^G(X)$. Then there exists an l-dimensional representation V of G such that $\alpha = \sum a_i[S_i]_G$, where each S_i is a G-invariant subvariety of $X \times V$ of dimension $l+m$.

Proof. Given that $A_m^G(X) = A_{m+l-g}(X_G)$, cycles of dimension $m+l-g$ in X_G correspond exactly to G-invariant cycles of dimension $m+l$ in $X \times U$. Since $V - U$ has sufficiently big codimension, therefore G-invariant $(m+l)$-cycles in $X \times U$ extend uniquely to G-invariant $(m+l)$-cycles in $X \times V$. Employing the double fibration argument it can be shown that any finite number of such cycles will appear all in a suitable representation.

□

The representation V is not necessarily unique. For instance, $[X]_G$ and $[X \times V]_G$ define the same G-equivariant class.

We shall see later (1.10.1) that, indeed, for many interesting cases it suffices to consider just the cycles of G-invariant subvarieties of X, avoiding the general search in $X \times U$.

1.5 The T-equivariant ring of a point

1.5.1 Tori.

If $T = (\mathbb{C}^*)^{\times g}$ is a torus of dimension g, then we may take $U = \prod_{i=1}^g (V_i - \{0\})$, with $V_i = \mathbb{C}^{l_i}$ denoting a representation of dimensional l_i. We find for the quotient a product of projective spaces,

$$U/T = \prod_{i=1}^{g} \mathbb{P}^{l_i-1}.$$

Suppose $g = 1$, *i.e.*, $T = \mathbb{C}^*$. For each $i \geq 0$ pick $l > i$ and set $V = \mathbb{C}^l$ and U as in the above instructive example. With these choices, the codimension of $V - U = \{0\}$ in V is equal to l and we have $U/T = \mathbb{P}^{l-1}$. We may write,

$$A_T^i(pt) = A^i(U/T) = A^i(\mathbb{P}^{l-1}) = \mathbb{Z} \cdot h^i,$$

where $h = c_1(\mathcal{O}(1))$ denotes the class of a hyperplane section of $\mathbb{P}^{l-1}$. Hence we get,

$$A^*_{\mathbb{C}^*}(pt) = \bigoplus \mathbb{Z} \cdot h^i = \mathbb{Z}[h].$$

The equivariant ring of a point acted on by a torus plays a central role. We denote it by R_T. If T is a g-dimensional torus, choosing V and U as just above, we see that

$$R_T := A^*_T(pt) = \mathbb{Z}[h_1, \ldots, h_g] \tag{1.5-1}$$

is a ring of polynomials with integer coefficients. Here each indeterminate h_1, $\ldots, h_g$ represents a hyperplane section in some $\mathbb{P}^{l-1}$.

1.5.2 GL_n.

For the general linear group GL_n of $n \times n$ nonsingular matrices, take V as the space of $n \times p$ matrices (with $p > n$), and the action given by multiplication of matrices. Now the open subset $U \subset V$ can be selected as the open subset consisting of matrices of maximal rank. We see that U/GL_n is the Grassmann variety $Gr(n, p)$ of all linear subspaces of $\mathbb{C}^p$ of dimension n.

1.6 Functorial properties

All morphisms are assumed G-equivariant throughout this section.

Given a morphism $f : X \to Y$ of G-schemes, $(f \times id) : X \times U \to Y \times U$ induces a morphism $f_G : X_G \to Y_G$ that renders the following diagram commutative:

$$\begin{array}{ccc} X \times U & \stackrel{f \times id}{\longrightarrow} & Y \times U \\ \downarrow & & \downarrow \\ X_G & \underset{f_G}{\longrightarrow} & Y_G. \end{array}$$

Note that in the above cartesian diagram the projections are surjective and flat. We may deduce that if $f : X \to Y$ is either smooth, or proper, or flat of relative codimension k or an embedding, then $f_G : X_G \to Y_G$ will have the same property.

The *proper pushforward* $f_* : A_i^G(X) \to A_i^G(Y)$ is given by

$$f_{G*} : A_{i+l-g}(X_G) \to A_{i+l-g}(Y_G).$$

If $f : X \to Y$ is a flat morphism of relative dimension k, the *flat pullback* $f^* : A_i^G(Y) \to A_{i+k}^G(X)$ is defined by

$$f_G^* : A_{i+l-g}(X_G) \to A_{i+k+l-g}(Y_G).$$

1.6.1 Proposition

The maps f_* and f^* are well defined.

Proof. We use once more the double fibration argument of Bogomolov.

Let V_1 and V_2 be representations. We have the cartesian diagram

$$\begin{array}{ccc} X \times^G (U_1 \oplus V_2) & \longrightarrow & Y \times^G (U_1 \oplus V_2) \\ \downarrow & & \downarrow \\ X \times^G U_1 & \longrightarrow & Y \times^G U_1 \end{array}$$

The projections are flat, and we already know that their pullback induce isomorphisms with the A_i^G defined with the bottom part of the diagram. This implies that f^* is well defined. Finally, use the fact that proper pushforward is compatible with flat pullback in order to conclude that the pushforward f_* is also well defined.

$\square$

1.6.2 G-equivariant Chern classes

Let X be a G-space. Let E be a G-equivariant vector bundle on X. For each pair i, j we define a map $c_j^G(E) : A_i^G(X) \to A_{i-j}^G(X)$ as follows.

Let V be an l-dimensional representation of G and choose an open subset $U \subset V$ such that $V - U$ has sufficiently big codimension and the G-principal bundle $X \times U \to X_G$ exists. Then, according to [GIT [33], Prop.7.1] there exists a quotient E_G of $E \times U$. It can be shown that $E_G \to X_G$ is in fact a vector bundle. We give details for this later on in the cases when X is a T-space and either E is a trivial bundle (1.7.1), or when T acts trivially on X (2.2.1).

1.6.3 Definition-Proposition.

The j-th G-equivariant Chern class

$$c_j^G(E) : A_i^G(X) \to A_{i-j}^G(X)$$

is defined as the operator

$$\alpha \in A_{i+l-g}(X_G) \longmapsto c_j^G(E) \cap \alpha = c_j(E_G) \cap \alpha \in A_{i-j+l-g}(X_G).$$

This definition is independent from the choice of representation.

Proof. Let V_1 and V_2 be representations of G. Consider the diagram

$$\begin{array}{ccc} E \times^G (U_1 \oplus V_2) & \longrightarrow & E \times^G U_1 \\ \downarrow & & \downarrow \\ X \times^G (U_1 \oplus V_2) & \longrightarrow & X \times^G U_1 \,. \end{array}$$

Since the projections are surjective and flat, we see that the pullback $E \times^G U_1$ to $X \times^G (U_1 \oplus V_2)$ is isomorphic to the quotient $E \times^G (U_1 \oplus V_2)$. By Bogomolov's double fibration trick, we see that the definition above does not depend on the chosen representation.

□

1.6.4 Equivariant self-intersection

Let $i : Y \hookrightarrow X$ be an equivariant regular embedding of G-spaces of codimension d. The usual self-intersection formula,

$$i^* i_* \alpha = c_d(\mathcal{N}_{Y/X}) \cap \alpha, \;\; \alpha \in A_*(Y)$$

induces a similar equivariant formula,

$$i_G^* i_{G*} \alpha = c_d^G(\mathcal{N}_{Y/X}) \cap \alpha, \;\; \alpha \in A_*^G(Y) \tag{1.6-1}$$

This follows from the fact that under the given hypotheses, the normal bundle of the quotient $Y_G \hookrightarrow X_G$ is the quotient $(\mathcal{N}_{Y/X})_G$ of the normal $\mathcal{N}_{Y \times U/X \times U}$. Details are left for the reader.

1.7 The line bundle of a character

We focus on the action of a torus. The lemma below is a simplified version of the general construction of a vector bundle associated to a T-principal bundle.

1.7.1 Lemma

Let $U \to U/T$ be as before and let χ be a character of T. Let $\varphi : U \times_\chi \mathbb{C} \to U$ be the trivial line bundle endowed with the T-action

$$t \cdot (u, v) \mapsto (t \cdot u, \chi(t) \cdot v).$$

Then φ is a T-equivariant bundle and induces, passing to the quotient, a line bundle $\mathcal{L}_\chi \to U/T$.

Proof. The trivial bundle in the statement is obviously T-equivariant. We now describe a local trivialization of the quotient $\mathcal{L}_\chi$ with transition functions that yield a line bundle on U/T. For this, let $\{(U_i, \varphi_{ij})\}$ be a trivialization of the T-principal bundle $U \to U/T$. That is, we have

$$U = \bigsqcup (U_i \times T)\big/ \sim$$

where $\quad (u,t) \sim (u',t') \;\Leftrightarrow\; u = u' \in U_{ij} = U_i \cap U_j$ and $t' = \varphi_{ij}(u) \cdot t.$

This allows us to write,

$$\mathcal{L}_{\chi \mid U_i} = (U_i \times T \times_\chi \mathbb{C})\big/T = U_i \times \mathbb{C}.$$

The gluing in U_{ij} is done by first identifying representatives $(u, t, v) \in U_i \times T \times \mathbb{C}$ with $(u', t', v') \in U_j \times T \times \mathbb{C}$, and then passing to the quotient. Thus, we have

$$u' = u \in U_{ij},\; t' = \varphi_{ij}(u) \cdot t \text{ and } v' = \chi(\varphi_{ij}(u)) \cdot v.$$

Hence, $\{(U_i, \chi \circ \varphi_{ij})\}$ gives a local trivialization of $\mathcal{L}_\chi$ as a line bundle on U/T.

$\square$

1.8 Character versus Chern class

1.8.1 The multiplication by a character

Notation as in the lemma above, let X be a T-space. Form the diagram of natural maps

$$\begin{array}{ccccc} X \times U \times_\chi \mathbb{C} & \longrightarrow & \mathcal{L}_{\chi|X_T} & \longrightarrow & \mathcal{L}_\chi \\ \downarrow & & \downarrow & & \downarrow \\ X \times U & \longrightarrow & X \times^T U & \longrightarrow & U/T \end{array}$$

where the horizontal arrows on the left are the quotient maps by the action of T.

For each cycle $\alpha \in A_*(X_T)$, we write

$$\chi \cdot \alpha := c_1(\mathcal{L}_{\chi|X_T}) \cap \alpha. \tag{1.8-1}$$

This operation of the group $\widehat{T}$ of characters of the torus T in $A_*(X_T)$ plays a very important role.

1.8.2 Structure of R_T-module

Continuing with the above setup, the structure morphism $X \to pt$ induces a morphism

$$X_T \to U/T$$

that turns $A_*^T(X)$ into an R_T-module.

To have a closer look at this action of R_T in $A_*^T(X)$, suppose

$$T = \mathbb{C}^*, \quad V = \mathbb{C}^{l+1}, \quad U = V \setminus \{0\}.$$

The action is given by multiplication with all the weights equal to -1. As in example (1.5.1), we have $U/T = \mathbb{P}^l$ and $A_*(U/T) = A_*(\mathbb{P}^l)$. Taking the limit for $l \to \infty$, we see that $R_T = \mathbb{Z}[h]$, where h represents a hyperplane section of some $\mathbb{P}^l$. The map $R_T \to A_*^T(X)$ is simply pullback of cycles in R_T. Since in each dimension R_T is generated by $c_1(\mathcal{L}) \cap [U/T]$, where $\mathcal{L}$ is a line bundle on U/T, we see that the multiplication

$$R_T \times A_*^T(X) \to A_*^T(X)$$

is obtained from generators of R_T as multiplication by characters of T, in the form described in (1.8-1).

1.8.3 The bundle $\mathcal{O}_{\mathbb{P}^l}(a)$

Assume further $T = \mathbb{C}^*$, $V = \mathbb{C}^{l+1}$, $U = V - \{0\}$ with the action on V as just above. We get the familiar T-principal bundle ,

$$\begin{array}{ccc} U & \longrightarrow & U/T \\ || & & || \\ \mathbb{C}^{l+1} \setminus \{0\} & \longrightarrow & \mathbb{P}^l. \end{array}$$

The latter can be described by the usual local chart, $\{(U_i, \varphi_{ij})\}_{i=0\ldots l}$ where

$$U_i = \{[x_0, \ldots, x_l] \in \mathbb{P}^l \mid x_i \neq 0\}$$

and the transition functions are

$$\varphi_{ij} : U_{ij} \to \mathbb{C}^*$$

$$[x_0, \ldots, x_l] \mapsto x_i/x_j.$$

Presently, every character $\chi : T = \mathbb{C}^* \to \mathbb{C}^*$ is of the form $t \mapsto t^a$ for some $a \in \mathbb{Z}$. In other words, we have an isomorphism

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \widehat{T} \\ a & \mapsto & (\chi_a : T \to \mathbb{C}) \end{array}$$

where χ_a denotes the character of T, $t \to t^a$. Consider the line bundle induced by χ_a,

$$\mathcal{L}_a = \mathcal{L}_{\chi_a} \to U/T = \mathbb{P}^l.$$

Recalling the recipe in the proof of (1.7.1), $\mathcal{L}_a$ is given by transition functions

$$\begin{array}{ccc} \psi_{ij} : U_{ij} & \longrightarrow & GL_1 = \mathbb{C}^* \\ x = [x_0, \ldots, x_l] & \mapsto & \chi_a(\varphi_{ij}(x)) = (x_j/x_i)^{-a}. \end{array}$$

These are precisely the transition functions of the line bundle $\mathcal{O}_{\mathbb{P}^l}(a)$. It follows that

$$\mathcal{L}_a = \mathcal{O}_{\mathbb{P}^l}(a) \longrightarrow U/T = \mathbb{P}^l. \tag{1.8-2}$$

Note that the choice of the C^*-action with all weights equal to -1 was done precisely to ensure the formula above.

For a torus of arbitrary dimension g, let $h_1, \ldots, h_g$ be a base of the group of characters $\widehat{T} \cong \mathbb{Z}^g$. Take $U \subset V$ as in (1.5.1) so that $U/T = \prod_i \mathbb{P}^{l_i}$. Arguing as in the previous case, it can be shown that each line bundle $\mathcal{L}_{h_i}$ arises from $\mathcal{O}_{\mathbb{P}^{l_i}}(1)$. The action (1.8-1) of the character h_i comes from multiplication by a hyperplane class. In other words, in the identification $R_T = \mathbb{Z}[h_1, \ldots, h_g]$, this ring of polynomials can be thought of as the symmetric algebra of the group of characters $\widehat{T}$.

1.8.4 Divisors of eigenfunctions

Let X be a T-variety. Let $f \in \mathbb{C}(X)$ be a nonzero rational function. Assume f is an eigenvector of T with character χ. Then the support of f is a T-invariant divisor. Hence it defines a class $\operatorname{div}^T(f)$ in the equivariant Chow ring $A_*^T(X)$. Precisely, write the principal divisor

$$\operatorname{div}(f) = \Sigma m_Z Z \in Z^1(X),$$

where Z ranges through all components of the support and m_Z denotes the respective multiplicity. We define

$$\operatorname{div}^T(f) = \Sigma m_Z [Z]_T \in A_*^T(X), \tag{1.8-3}$$

where $[Z]_T$ denotes the class (1.4.4) of the invariant subvariety Z in the T-equivariant ring of X. Note that a rational function f induces a rational section $f_{|X \times U}$ of the equivariant trivial bundle $X \times U \times_\chi \mathbb{C}$. Equivariance follows from the definition of $U \times_\chi \mathbb{C}$:

$$\begin{aligned} f_{|X\times U}(t \cdot (x,u)) &= (t \cdot x, t \cdot u, f(t \cdot x)) \\ &= (t \cdot x, t \cdot u, \chi(t) \cdot f(x)) \\ &= t \cdot (x, u, f(x)). \end{aligned}$$

This T-equivariant rational section, $f_{|X \times U}$, passes to the quotient. More precisely, let $\mathcal{L}_{\chi|X_T}$ be the pullback of the line bundle $\mathcal{L}_\chi$ under the map $X \times^T U \to$

U/T. We have an induced rational section,

$$\begin{array}{rccc} s_f: & X \times^T U & \cdots\longrightarrow & \mathcal{L}_{\chi|X_T} \\ & [x,u] & \mapsto & [(x,u), f(x)]. \end{array}$$

1.8.5 Lemma

Notation as above, $\mathrm{div}^T(f)$ *represents in* $A_*^T(X)$ *the operator* $c_1(\mathcal{L}_\chi)$ (1.8-1) *evaluated on the fundamental class of* $X_T = X \times^T U$*,*

$$\mathrm{div}^T(f) = \chi \cdot [X_T].$$

Proof. In the diagram of natural maps

$$\begin{array}{ccc} X \times U \times_\chi \mathbb{C} & \longrightarrow & \mathcal{L}_{\chi|X_T} \\ \downarrow & & \downarrow \\ X \times U & \longrightarrow & X \times^T U \end{array}$$

the horizontal arrows are the quotient maps by the action of T. The pullback of the rational section s_f is the rational section $f_{|X\times U}$. The cycle $c_1(\mathcal{L}_\chi) \cap [X \times^T U]$ can be computed in terms of the pseudo-divisor determined by the rational section s_f (cf. [17]). The maps in sight are faithfully flat. Therefore, the multiplicities of the components of the cycles associated to s_f in $X \times^T U$ and to f in $X \times U$ coincide.

$\square$

Note that, even though the class of the divisor of the rational function f is zero in the usual Chow group, the equivariant class $\mathrm{div}^T(f)$ is not necessarily zero in $A_*^T(X)$, cf. 1.8.6.

The relation $\mathrm{div}^T(f) = \chi \cdot [Y]_T$ enables us to compare the equivariant class of Y with a class of a T-invariant divisor contained in Y. That is, given a T-invariant subvariety $Y \subset X$, suppose that there is a rational eigenfunction f with non trivial character χ and such that f does not vanish on all of Y. Then, inverting the character χ in $A_*^T(X)$ if needed, we may compare the class $[Y]_T$ with an invariant cycle with support of dimension less than $\dim Y$. This is the key point in the proof of the localization theorem.

1.8.6 T-invariant divisors in $\mathbb{P}^n$

Let $T = \mathbb{C}^*$ be a unidimensional torus acting diagonally on $\mathbb{P}^n$ with weights $a_0, \ldots, a_n$, i.e., $t \circ [x_0, \ldots, x_n] = [t^{a_0} x_0, \ldots, t^{a_n} x_n]$. Later on we shall describe $\mathbb{P}^n_T = \mathbb{P}^n \times^T U$ as a projective variety for a suitable choice of $U \subset V$ as well as the ring $A^T_*(\mathbb{P}^n)$. Right now we limit ourselves to explain the computation of

$$\operatorname{div}^T(f) \in A^T_*(\mathbb{P}^n)$$

for a rational function $f : \mathbb{P}^n \cdots \to \mathbb{C}$ that is an eigenfunction with weight $a \in \mathbb{Z}$, that is, $f(t \circ x) = t^a \cdot f(x)$ for each $x \in \mathbb{P}^n$ in the domain of f and $t \in T$. For instance, if $f = x_i/x_j$ then f has weight $a = a_i - a_j$. We may write

$$\operatorname{div}(f) = H_i - H_j \in Z^1(\mathbb{P}^n)$$

with $H_k \subset \mathbb{P}^n$ denoting the T-invariant hyperplane given by $x_k = 0$. Thus, we get by 1.8-3

$$\operatorname{div}^T(f) = [H_i]_T - [H_j]_T \quad \text{in} \quad A^T_*(\mathbb{P}^n).$$

The lemma 1.8.5 tells us that

$$\operatorname{div}^T(f) = c_1(\mathcal{L}_\chi) \cap [\mathbb{P}^n_T]$$

with $\chi = \chi_a$. In this manner, recalling (1.8-2), we see that

$$[H_i]_T - [H_j]_T = c_1(\mathcal{O}_{\mathbb{P}^l}(a)) \cap [\mathbb{P}^n_T] = a \cdot t \cap [\mathbb{P}^n_T]$$

where, by deliberate abuse, we also write $t = c_1(\mathcal{O}_{\mathbb{P}^l}(1))$ for the pullback of the hyperplane section of $\mathbb{P}^l$. In particular, we see that $\operatorname{div}^T(f)$ is not necessarily zero in $A^T_*(\mathbb{P}^n)$.

1.9 The $\mathbb{C}^*$-equivariant Chow ring of $\mathbb{P}^n$

Let $T = \mathbb{C}^*$ act on $\mathbb{P}^n$ as in the previous example. We have $\mathbb{P}^n = \mathbb{P}(V)$, with $V = \mathbb{C}^{n+1}$; the action on $\mathbb{P}^n$ is induced by the diagonal representation,

$$\begin{array}{rcl} \rho : \mathbb{C}^* & \longrightarrow & GL_{n+1} \\ t & \longmapsto & \operatorname{diag}(t^{a_0}, t^{a_1}, \ldots, t^{a_n}). \end{array}$$

Thus, we get a decomposition into eigenspaces, which is customarily written as

$$\mathbb{C}^{n+1} = \oplus_{i=0}^{n} t^{a_i}$$

with weights $a_0, \ldots, a_n$.

We have already seen that, taking $U = \mathbb{C}^{l+1} - \{0\}$, we get $U/T = \mathbb{P}^l$. The trivial bundle $U \times_\rho \mathbb{C}^{n+1} \to U$ associated to the representation ρ is T-equivariant and induces, passing to the quotient, the vector bundle

$$U \times^T \mathbb{C}^{n+1} = \mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_n) \longrightarrow \mathbb{P}^l$$

which we shall write for short as $\mathcal{O}(\underline{a})$. Each eigenvector with weight a, passes to the quotient as a section of $\mathcal{O}(a)$. As in example (1.4.2), it can be shown that the quotient variety $(\mathbb{P}^n)_T = \mathbb{P}^n \times^T U$ is the projective bundle / $\mathbb{P}^l$,

$$(\mathbb{P}^n)_T = \mathbb{P}(\mathcal{O}(\underline{a})) \longrightarrow \mathbb{P}^l$$

with fiber $\mathbb{P}^n$. The quotient map

$$U \times \mathbb{P}^n \longrightarrow U \times^T \mathbb{P}^n = \mathbb{P}(\mathcal{O}(\underline{a}))$$

may be explicitly written in the following fashion. Given $(u, [v]) \in U \times \mathbb{P}^n$, write the vector $v = v_0 + \cdots + v_n$ as a sum of eigenvectors. Each v_i yields a class $[u, v_i] \in \mathcal{O}(a_i)$. Next, project $\Sigma[u, v_i] \in \mathcal{O}(\underline{a})$ to $\mathbb{P}(\mathcal{O}(\underline{a}))$.

The Chow ring of this projective bundle is given by

$$A_*(\mathbb{P}^n_T) = A_*(\mathbb{P}^l)[h]/\langle p(h,t)\rangle$$

where t is a generator of $A^1(\mathbb{P}^l)$ (a hyperplane section of $\mathbb{P}^l$), $h = c_1(\mathcal{O}_{(\mathcal{O}(\underline{a}))}(1))$ and

$$p(h,t) = \prod_{i=0}^{n} (h + a_i t).$$

The relation $\prod_{i=0}^{n}(h + a_i t) = 0$ in $A^*(\mathbb{P}^n_T)$ can be interpreted as follows. Each homogeneous coordinate x_i is an eigenvector with weight a_i. Therefore the hyperplane $H_i \subset \mathbb{P}^n$ given by $x_i = 0$ is T-invariant. The inclusion $(H_i)_T \subset \mathbb{P}^n_T$ is the inclusion of the projective subbundle over $\mathbb{P}^l$,

$$\mathbb{P}(\oplus_{j \neq i} \mathcal{O}(a_j)) \subset \mathbb{P}(\mathcal{O}(\underline{a})).$$

This allows us to compute $[H_i]_T$ as the zeros of a section of the line bundle

$$\mathcal{Q} \otimes O_{\mathcal{O}(\underline{a})}(1)$$

where we have set

$$\mathcal{Q} = (\mathcal{O}(\underline{a}))/(\oplus_{j \neq i} \mathcal{O}(a_j)) = \mathcal{O}(a_i)$$

Hence, (cf. [17], 3.2.17)

$$[H_i]_T = a_i \cdot t + h.$$

Since $\bigcap_i H_i = \emptyset$, we have $\bigcap_i (H_i)_T = \emptyset$ thus implying

$$\prod_{i=0}^{n} (h + a_i t) = 0.$$

Letting the dimension of the representation go to infinity, we see that

$$A_*^T(\mathbb{P}^n) = \mathbb{Z}[h, t] \,/\, \Pi(h + a_i t).$$

Notice that $A_*^T(\mathbb{P}^n)$ is a free module of rank $n+1$ over $\mathbb{Z}[t]$, the T-equivariant Chow ring of a point.

More generally, let $X \subset \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial f of degree d which is an eigenfunction with weight a. Then X is T-invariant and its equivariant class $[X]_T \in A_*^T(\mathbb{P}^n)$ is equal to $dh + at$.

This follows from the fact that we may produce, using f, the rational function $r = f/x_0^d : \mathbb{P}^n \cdots \to \mathbb{C}$ which is an eigenfunction with weight $a - d \cdot a_0$. It is easy to see that $\mathrm{div}(r)$ is the cycle $X - d \cdot H_0$. We have therefore

$$\mathrm{div}^T(r) = [X]_T - d \cdot [H_0]_T \in A_*^T(\mathbb{P}^n).$$

Recalling (1.8-2), we may write

$$\mathrm{div}^T(r) = c_1(\mathcal{L}_{(a - d \cdot a_0)}) \cap [\mathbb{P}^n_T] = (a - d \cdot a_0) \cdot t \cdot [\mathbb{P}^n_T] \tag{1.9-1}$$

whence we get (suppressing the factor $\cap[\mathbb{P}^n_T]$),

$$[X]_T = (a - d \cdot a_0) \cdot t + d \cdot (h + a_0 \cdot t) = a \cdot t - a_0.d \cdot t + d \cdot h + a_0.d \cdot t = a \cdot t + d \cdot h.$$

We leave the following generalization as an exercise. Let X be a complete intersection in $\mathbb{P}^n$ defined by homogeneous polynomials f_i of degree d_i that are eigenfunctions with weights a_i. Then, $[X]_T = \prod(d_i h + a_i t) \in A_*^T(\mathbb{P}^n)$.

1.10 Invariant cycles suffice

We end this chapter showing that the invariant cycles are enough to handle many interesting cases.

1.10.1 Proposition

Let T be a torus and let X be a T-space. Then the T-equivariant Chow group $A_T^*(X)$ is generated as R_T-module by the T-invariant subvarieties of X.

Proof. Note that a generator for the T-equivariant Chow ring of X is given by a T-invariant subvariety $Y \subset X \times U \subset X \times V$ for a suitable choice of $U \subset V$.

Employing the isomorphism between $A_*(X)$ and $A_*(X \times V)$, we have that Y is rationally equivalent to $\sum m_i(Y_i \times V)$. Using the generalization of the theorem of Hirschowitz 1.3 to the (non necessarily projective) variety X, we may assume that each subvariety $Y_i \subset X$ is T-invariant. Applying once more the theorem, this time to $X \times V$, we see that the rational equivalence between Y and $\sum m_i(Y_i \times V)$ can also be achieved in a T-invariant way, namely, there exist T-invariant subvarieties $W \subset X \times V$, such that $\dim(W) = \dim(Y) + 1$ and rational eigenfunctions $f_W : W \cdots \to \mathbb{C}$ with characters χ_W, so that

$$Y - \sum m_i(Y_i \times V) = \sum \mathrm{div}(f_W).$$

Thus, passing to the equivariant classes we have

$$[Y]_T - \sum m_i[Y_i]_T = \sum \mathrm{div}^T(f_W) = \sum c_1(\mathcal{L}_{\chi_W}) \cap [W]_T = \sum \chi_W \cdot [W]_T,$$

where the dimension of each W is bigger than $\dim(Y)$. It follows by induction that $A_*^T(X)$ is generated as R_T module by T-invariant subvarieties $Y \subset X$.

□

2 The theorem of localization

For the rest of these notes, we restrict ourselves to the case of an action of a torus T on a scheme X.

2.1 The case of a trivial action

Suppose $\dim T = g$. We have seen in 1.5.1 that the T-equivariant Chow ring of a point can be described as

$$R_T = \mathbb{Z}[t_1, \ldots, t_g].$$

We have on the right hand side the ring of polynomials with integer coefficients. This ring appears in fact as the group algebra of $\widehat{T} = \mathbb{Z}^g$, the group of characters of T. The explicit isomorphism comes from the following recipe.

For each character $\chi \in \widehat{T}$, consider the line bundle $\mathcal{L}_\chi$ on U/T constructed in 1.7.1. We identify χ with $c_1(\mathcal{L}_\chi)$.

2.1.1 Lemma

Notation as above, if T acts trivially on X, then

$$A_*^T(X) = A_*(X) \otimes R_T.$$

Proof. Due to the triviality of the action of T on X, it follows that $X \times^T U = X \times (U/T)$. Thus, choosing U so that the quotient U/T is a product of projective spaces (cf. 1.5.1), we have the isomorphism

$$A_*\big(X \times (U/T)\big) = A_*(X) \otimes A_*(U/T).$$

Increasing the dimension of the representation we get the assertion.

□

Note that in general, the map of Künneth decomposition

$$A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$$

used above, need not be an isomorphism: take for instance $X = Y$ to be a curve of genus > 0. If the diagonal $\Delta \subset X \times X$ had an expression in the form $[\Delta] = \Sigma m_i[p_i \times X] + n_i[X \times q_i]$, with $m_i, n_i \in \mathbb{Z}, p_i, q_i \in X$ then, for each $p \in X$ we could write $[p \times p] = [p \times X] \cap \Delta = \Sigma n_i[p \times q_i]$. Projecting to the second factor, we may write $[p] = \Sigma n_i[q_i]$. Hence, any two points would be rationally equivalent, a contradiction for positive genus.

2.2 Decomposition into eigensubbundles

We still assume that the T-action on X is trivial.

Given a T-equivariant vector bundle $E \to X$ we get a canonical decomposition

$$E = \bigoplus_{\chi \in \widehat{T}} E^\chi$$

into a direct sum of subbundles, where E^χ denotes the eigensubbundle consisting of vectors in E on which T acts with the character χ.

It follows that the T-equivariant Chern classes of E can be expressed in term of the classes of eigensubbundles E^χ. For the latter, we shall describe the vector bundle E_T on X_T induced by $E = E^\chi$, noting that now $X_T = X \times (U/T)$ by the triviality of the action on the factor X. Let us look at E_T in terms of local charts.

2.2.1 Lemma

Notation and hypotheses as above, the quotient bundle $(E^\chi)_T$ on $X \times (U/T)$ is isomorphic to the tensor product of the pullback of the bundle E^χ by the pullback of the line bundle $\mathcal{L}_\chi$.

Proof. Choose trivializations $\{(U_\alpha, \varphi_{\alpha\beta})\}$ for the T-principal bundle $U \to U/T$ and $\{(V_i, \psi_{ij})\}$ for the equivariant bundle $E^\chi \to X$. Then, we have

$$U = \bigsqcup (U_\alpha \times T)/\sim, \quad E^\chi = \bigsqcup (V_i \times_\chi \mathbb{C}^r)/\approx$$

where

$$(u,t) \sim (u',t') \Leftrightarrow u = u' \in U_{\alpha\beta} := U_\alpha \cap U_\beta,\ t' = \varphi_{\alpha\beta}(u) \cdot t$$

and

$$(x,v) \approx (x',v') \Leftrightarrow x = x' \in V_{ij} := V_i \cap V_j,\ v' = \psi_{ij}(x) \cdot v.$$

This implies

$$(E^\chi)_T \mid_{U_\alpha \times V_i} = (U_\alpha \times V_i \times_\chi \mathbb{C}^r \times T)/T = U_\alpha \times V_i \times \mathbb{C}^r,$$

which yields a local trivialization for $(E^\chi)_T$ as a vector bundle. The gluing over

$$(U_\alpha \times V_i) \cap (U_\beta \times V_j)$$

is done by identifying representatives

$$U_\alpha \times V_i \times_\chi \mathbb{C}^r \times T \ni (u, x, v, t) \text{ with } (u', x', v', t') \in U_\beta \times V_j \times_\chi \mathbb{C}^r \times T$$

and then passing to the quotient. Namely, we have

$$u' = u \in U_{\alpha\beta},\ x' = x \in V_{ij}$$
$$\text{and}$$
$$t' = \varphi_{\alpha\beta}(u) \cdot t, \quad v' = \chi(\varphi_{\alpha\beta}(u)) \cdot \psi_{ij}(x) \cdot v.$$

Hence, $\{(U_\alpha \times V_i, (\chi \circ \varphi_{\alpha\beta}) \cdot \psi_{ij})\}$ is a local trivialization of $(E^\chi)_T$ as a vector bundle over $X_T = X \times U/T$. Furthermore,

$$(E^\chi)_T \simeq E^\chi \otimes \mathcal{L}_\chi,$$

where E^χ and $\mathcal{L}_\chi$ denote the pullbacks to $X \times (U/T)$ of the "same" bundles over X and U/T.

□

2.2.2 Corollary

Let X be a T-space with trivial action. Let $E = E^\chi \to X$ be a T-equivariant vector bundle of rank r on X such that the action of T on each fiber is given by a character χ. Then, for all i, we have

$$c_i^T(E^\chi) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E^\chi)\chi^{i-j}.$$

Proof. The assertion follows from example (3.2.2) of [17], where the (usual) Chern class of a tensor product like $(E^\chi)_T \simeq E^\chi \otimes \mathcal{L}_\chi$ is computed.

□

2.3 Fixed points locus

The following result ensures nontriviality of the weights of the normal bundle of the locus of fixed points.

2.3.1 Lemma

If X is a smooth T-variety then the locus X^T of fixed points is also smooth. If F is a component of X^T, then the normal bundle $\mathcal{N}_{F/X}$ is T-equivariant over F. Furthermore, we have $(\mathcal{T}_x X)^T = \mathcal{T}_x F$ for all $x \in F$, and therefore the T-action on $(\mathcal{N}_{F/X})_x$ is non trivial.

Proof. See B. Iversen, [25].

□

2.3.2 Localize to invert

Let X be a T-variety and let $F \subseteq X^T$ be a component of the fixed points locus. Let E be an equivariant bundle on X. Taking into account that T acts trivially on F, we have by 2.1.1 that $A_T^*(F) = R_T \otimes A^*(F)$. The restriction $E_{|F}$ decomposes as a sum of eigensubbundles $E^\chi_{|F}$. The lemma 2.2.2 tells us that the component of $c_i^T(E^\chi_{|F})$ in R_T^i is given (setting $j = 0$) by $\binom{r}{i}\chi^i$.

Since $A^N(F) = 0$ for $N > \dim(F)$, we have that, for $j > 0$, the elements of $A^j(F)$ are nilpotent in the ring $A_T^*(F)$. It follows that $c_i^T(E^\chi_{|F}) \in (R_T \otimes A^*(F))^i$ is invertible if and only if its component in $R_T^i \otimes A^0(F) \cong R_T^i$ is invertible. Hence, $c_i^T(E^\chi_{|F})$ is invertible in the localization $R_T \otimes A^*(F)[\chi^{-1}]$.

2.3.3 Lemma

Let X be a smooth T-variety and let F be a component of codimension d of the fixed points locus X^T. Then there exist finitely many nontrivial characters $\lambda_1, \ldots, \lambda_r$ such that $c_d^T(\mathcal{N}_{F/X})$ becomes invertible in the ring of fractions

$$A_T^*(F)[1/\lambda_1, \ldots, 1/\lambda_r].$$

Proof. By the previous lemma, we know that T acts with nontrivial weights on the normal space $(\mathcal{N}_{F/X})_x = \mathcal{T}_x X/\mathcal{T}_x F$. Hence, the characters λ_i that occur in the decomposition of the normal space $(\mathcal{N}_{F/X})$ into eigenbundles are all nontrivial. By the previous remark, we see that the component of $c_d^T(\mathcal{N}_{F/X})$ in R_T^d is nonzero. Hence, the class $c_d^T(\mathcal{N}_{F/X})$ becomes invertible in the ring $A_T^*(F)[1/\lambda_1, \ldots, 1/\lambda_r]$, as claimed.

□

2.4 The theorem of localization

We present in this section the version of M. Brion [7] for the theorem of localization. The principal point in favor of his approach is to avoid the need of the higher Chow groups as required in [12].

2.4.1 Lemma

Let X be an affine T-scheme. Let Y be a T-invariant subvariety. If Y is not fixed pointwise, then there exists a regular eigenfunction f on X with nontrivial weight whose restriction $f_{|Y} \neq 0$.

Proof. Pick $y \in Y$ not in the fixed points locus X^T. Hence, there exists $t \in T$ such that $t \circ y \neq y$. Since T is a torus, we know that the coordinates ring of X is generated by eigenfunctions. Hence there exists an eigenfunction f, say associated to the character χ, which separates those points: $f(t \circ y) \neq f(y)$. Hence, $f(t \circ y) = \big(\chi(t) \cdot f\big)(y) \neq f(y)$. This implies at once that $f(y) \neq 0$ and $\chi(t) \neq 1$.

□

2.4.2 Lemma

Let X an affine T-scheme. The fixed points locus $X^T \subset X$ is an intersection of schemes of zeros of the regular eigenfunctions on X with nontrivial weights.

Proof. Let $x \in X^T$, and let f be a regular eigenfunction with weight $\chi \neq 1$. We have $f(x) = f(t \circ x) = \chi(t) \cdot f(x)$, $\forall t \in T$. Hence, if $\chi(t) \neq 1$ then $f(x) = 0$. Conversely, if $x \notin X^T$, we apply the previous lemma with $Y = X$ to conclude that x is not a common zero of all eigenfunctions referred to in the statement.

□

Denote by

$$i_T : X^T \hookrightarrow X$$

the map of inclusion of the fixed points locus. We know that i_T induces a homomorphism of R_T-modules

$$i_{T*} : A_*^T(X^T) \longrightarrow A_*^T(X)$$

for the T-equivariant Chow groups (cf. 1.6.1).

Recall that we have a natural isomorphism

$$A_*^T(X^T) \simeq R_T \otimes A_*(X^T),$$

since the action of T in X^T is trivial.

We may at last handle the important

2.4.3 Theorem of localization

Let X be a T-space. Then the R_T-linear map

$$i_{T*} : A_*^T(X^T) = A_*(X^T) \otimes R_T \longrightarrow A_*^T(X)$$

becomes an isomorphism after inverting finitely many nontrivial characters.

Proof. By our blanket assumption, X can be covered by finitely many T-invariant affine open subsets X_i.

By the previous lemma, each fixed points locus $X_i^T \subset X_i$ is an intersection of zeros of the regular functions on X_i which are eigenvectors of the action of T on X_i with nontrivial weights. By quasi-compactness, we may extract a finite intersection. That is, there exists a finite set of eigenfunctions $\{f_{ij}\}$ with respective weights $\{\chi_{ij}\}$, *all nontrivial*, such that $x \in X_i$ is in X_i^T if and only if $f_{ij}(x) = 0,\ \forall j$.

In order to show that i_{T*} is surjective, we invoke the theorem 1.3: the usual Chow ring of X is generated by the cycles of T-invariant subvarieties. An immediate consequence is the fact that $A_*^T(X)$ is generated, as R_T-module, by the cycles of the form $[Y \times^T U]$ with $Y \subset X$ a T-invariant closed subvariety.

Let now $Y \subset X$ be a T-invariant subvariety. Suppose that Y is not pointwise fixed by T. Then one of the f_{ij} defines a nonzero rational function on Y. This rational function defines in turn a rational section of the pullback of the line bundle $\mathcal{L}_{\chi_{ij}}$ (1.7.1) by the map $Y \times^T U \to U/T$, still denoted by $\mathcal{L}_{\chi_{ij}}$.

Therefore, $\chi_{ij} \cdot [Y]_T = c_1(\mathcal{L}_{\chi_{ij}}) \cdot [Y]_T = \mathrm{div}^T(f_{ij}) \in A_*^T(X)$, which implies

$$[Y]_T = \chi_{ij}^{-1}\mathrm{div}^T(f_{ij}) \in A_*^T(X) \otimes R_T[1/\chi_{ij}].$$

Well, the support of $\mathrm{div}^T(f_{ij})$ is of dimension smaller than $\dim(Y)$ and is made up of invariant subvarieties. By noetherian induction it follows that, upon inverting finitely many χ_{ij}'s, the induced map i_{T*} becomes surjective.

For injectivity, notice that if $X^T = X$ then i_{T*} is the identity map.

Hence, we may assume that X is not fixed pointwise by T. Let Y be an irreducible component of X that is not contained in X^T. Choose f_{ij} as before, namely, $f_{ij\,|Y} \neq 0$.

Denote by $|D|$ the union of the support of the divisor of f_{ij} in Y and the irreducible components of X distinct from Y. Then, by construction, $|D|$ contains all the fixed points of X. Let $\iota : |D| \to X$ be the map of inclusion.

Consider the T-principal bundle $U \to U/T$ given as in 1.5.1, and let $\mathcal{L}_{\chi_{ij}}$ be the line bundle on U/T associated to the weight χ_{ij}. Let $p : X \times^T U \to U/T$ be the projection. We have a pseudo-divisor on $X \times^T U$ (cf. [17], 2.2),

$$(\ p^*\mathcal{L}_{\chi_{ij}}, |D| \times^T U, f_{ij}\ )$$

which defines a homogeneous map of degree -1,

$$\iota^* : A_*^T(X) \longrightarrow A_*^T(|D|)$$

such that the composition $\iota^*\circ\iota_*$ is multiplication by χ_{ij}. Examining the diagram,

$$\begin{array}{ccc} A_*^T(X^T) = A_*(X^T)\otimes R_T & \xrightarrow{i_{T*}} & A_*^T(X) \\ || & & \uparrow \iota_* \\ A_*^T(X^T = |D|^T) & \longrightarrow & A_*^T(|D|) \end{array}$$

we see that the map $\iota_* : A_*^T(|D|) \to A_*^T(X)$ becomes injective after inverting χ_{ij}. Therefore injectivity follows using again noetherian induction.

□

Recall that the equivariant ring of a point, $R_T = \mathbb{Z}[t_1,\ldots,t_g]$, is a ring of polynomials. Let R_T^+ denote the multiplicative system of homogeneous elements of positive degree. We define the ring of fractions

$$\mathcal{R}_T = (R_T^+)^{-1}\cdot R_T.$$

Thus, in $\mathcal{R}_T$ (the image of) all nontrivial characters are invertible elements.

We get the following consequence.

2.4.4 Corollary

The map $i_* : A_*(X^T)\otimes\mathcal{R}_T \to A_*^T(X)\otimes\mathcal{R}_T$ is an isomorphism.

2.4.5 Theorem (explicit localization).

Let X be a smooth T-variety. Let $\alpha\in A_T^*(X)\otimes\mathcal{R}_T$. Then

$$\alpha = \sum_F i_{F*}\left(\frac{i_F^*\alpha}{c_{d_F}^T(\mathcal{N}_{F/X})}\right),$$

where the sum is taken over all components of X^T and d_F denotes the codimension of F in X.

Proof. From the surjectivity ensured by the Localization Theorem, we may write $\alpha = \sum_F i_{F*}(\beta_F)$. Since the irreducible components F of X^T are disjoint, it follows that $i_F^*\alpha = i_F^* i_{F*}(\beta_F)$ because the remaining components of X^T do not contribute for cycles in F. The formula of self intersection 1.6-1 yields

$$i_F^* i_{F*}(\beta_F) = c_{d_F}^T(\mathcal{N}_{F/X})\cdot\beta_F,$$

and so, by 2.3.3, we get $\beta_F = i_F^*\alpha / c_{d_F}^T(\mathcal{N}_{F/X})$ as desired.

$\square$

2.4.6 Homomorphism of integration

When X is a complete variety, the projection $\pi_X : X \to pt$ induces a pushforward $\pi_{X*} : A_*^T(X) \to R_T$ which is zero in A_i^T for $i > 0$ and is given by the degree of zero cycles for $i = 0$. Tensorizing by $\mathcal{R}_T$, we get the homomorphism of integration

$$\begin{array}{rccc} \pi_{X*}: & A_*^T(X) \otimes \mathcal{R}_T & \longrightarrow & \mathcal{R}_T \\ & \alpha & \longmapsto & \int_X \alpha. \end{array}$$

Replacing X by F, a component of X^T, we have a similar map π_{F*}.

Let us apply π_{X*} to both sides in the theorem of explicit localization.

Using the fact of that $\pi_{F*} = \pi_{X*} \circ i_{F*}$, we get the following.

2.4.7 Corollary. (Formula of integration)

Let X be a smooth and complete T-variety and let $\alpha \in A_T^*(X) \otimes \mathcal{Q}$. Then

$$\int_X \alpha = \sum_{F \subset X^T} \pi_{F*} \left(\frac{i_F^* \alpha}{c_{d_F}^T(\mathcal{N}_{F/X})} \right),$$

as an element of $\mathcal{R}_T$. $\square$

The previous corollary yields a formula of integration which is particularly useful for elements of the usual Chow group $A_0(X)$ of the form pullback of an element of $A_0^T(X)$. More precisely, consider the commutative diagram

$$\begin{array}{ccccccc} X & \overset{i}{\hookrightarrow} & X_T & \leftarrow & X \times U \\ \pi_X \downarrow & \square & \downarrow \pi_X^T & & \downarrow \\ pt & \overset{j}{\hookrightarrow} & U/T & \leftarrow & U \end{array} \qquad (2.4\text{-}1)$$

where the horizontal maps on the right are the quotient maps. Note that, by construction of the T-principal bundle, the inverse image of the point $pt \in U/T$ in U is the orbit $T \cdot u \cong T$ for some $u \in U$. On the other hand, the inverse image in $X \times U$ is $X \times (T \cdot u)$. The image of the latter subvariety in $X_T = (X \times U)/T$ is isomorphic to X. Since X is smooth, we have that i is a regular embedding of codimension $d = \dim(U/T)$. Recalling the definition, we see that i induces the homomorphism,

$$i^* : A_0^T(X) = A_d(X_T) \longrightarrow A_0(X).$$

2.4.8 Proposition

With the hipotheses as in the previous corollary, put

$$a = i^*\alpha, \quad \textit{with} \quad \alpha \in A_0^T(X).$$

Then

$$\int_X a = \sum_{F \subset X^T} \pi_{F*} \left(\frac{i_F^* \alpha}{c_{d_F}^T(\mathcal{N}_{F/X})} \right).$$

Proof. We have $\pi_{X*}(a) = \pi_{X*} i^*(\alpha) = j^* \pi_{X*}^T(\alpha)$. Applying the integration formula, the assertion follows.

□

2.4.9 Remark.

Let $E \to X$ be a T-equivariant vector bundle. Consider in the diagram (2.4-1), the vector bundle E_T induced over X_T. The identification of the fiber $(\pi_X^T)^{-1} pt$ with X induces the identification $i^*(E_T) = E$.

2.5 Bott's residues formula

We describe in this section an equivariant version of Bott's formula.

Let $E_1, \ldots, E_s$ be T-equivariant vector bundles on a smooth and complete variety X of dimension n. Let $p(x_1^1, \ldots, x_s^1, \ldots, x_1^n, \ldots, x_s^n)$ be a weighted homogeneous polynomial of degree n in the variables x_j^i, where x_j^i has degree i. Denote by $p(E_1, \ldots, E_s)$ the polynomial in the Chern classes of $E_1, \ldots, E_s$, obtained by substituting $x_j^i = c_i(E_j)$. The integration formula computes the degree of the zero cycle $p(E_1, \ldots, E_s) \cap [X]$ in terms of the restrictions of the bundles E_i's to the locus $X^T \subset X$ of fixed points.

Set for short

$$p(E.) = p(E_1, \ldots, E_s) \text{ and } p^T(E.) = p(E_{1\,T}, \ldots, E_{s\,T})$$

the corresponding polynomial for the T-equivariant Chern classes of the bundles $E_1, \ldots, E_s$. Note that

$$p(E) \cap [X] = i^*(p^T(E) \cap [X]_T).$$

Employing the proposition 2.4.8, we get the following.

2.6 (Bott's residues formula)

Let X be a smooth, complete variety and let $E_1, \ldots, E_s$ be T-equivariant vector bundles over X. Then

$$\int_X (p(E) \cap [X]) = \sum_{F \subset X^T} \pi_{F*} \left(\frac{p^T(E_{|F}) \cap [F]_T}{c_{d_F}(\mathcal{N}_{F/X})} \right), \tag{2.6-1}$$

where d_F denotes the codimension of the component F in X.

In spite of the possibly awe-inspiring appearance of the formula at first (and perhaps even a few subsequent) sight, we hope to convince the reader that it is in fact very efficient and rather simple to apply in practice.

2.7 Contribution of fixed points

Let X be a nonsingular T-variety and $F \subseteq X^T$ a connected (=irreducible) component of the fixed points locus. Write $d_F = \mathrm{codim}(F)$.

The T-equivariant Chern classes $c_k^T(E_{\,|\,F})$ and $c_{d_F}^T(\mathcal{N}_{F/X})$ can be computed in the equivariant Chow ring $A_*^T(F)$ in terms of the characters that occur in the decomposition of $E_{\,|\,F}$ and $\mathcal{N}_{F/X}$ into eigensubbundles and of the Chern classes of the latter.

2.7.1 Isolated fixed points

When X^T is a finite set of points, the classes

$$c_k^T(E_{|F}) \quad \text{and} \quad c_{d_F}^T(\mathcal{N}_{F/X}) = c_{\dim(X)}^T(\mathcal{T}X)$$

can be described purely in terms of characters associated to eigenbundles. More precisely, once we have the decomposition

$$E_{|F} = \bigoplus\nolimits_\chi E^\chi_{|F}$$

into eigenspaces, the equivariant Chern classes of $E_{|F}$ can be determined. Indeed 2.2.2 yields the classes of each summand,

$$c_k^T(E^\chi_{|F}) = \binom{r}{k}\chi^k, \quad r = \text{rank of } E^\chi_{|F}. \tag{2.7-1}$$

Note that, in the above expression, χ^k means the kth.-iteration of the operator first Chern class introduced in (1.8-1). In particular, we deduce that $c_{\max}^T(E_{|F})$ is represented in the equivariant Chow ring of the fixed point F by the product of all characters that occur in the decomposition of the fiber $E_{|F}$ into eigenspaces, with their respective multiplicities. Here each character is already being considered as acting on the equivariant ring, according to (1.8-1). See the example 2.7.4.

For the case $T = \mathbb{C}^*$, a unidimensional torus, let us have a closer look at the replacement of $R_T = \mathbb{Z}[t]$ by

$$\mathcal{R}_T = (R_T^+)^{-1}(R_T) = \mathbb{Q}[t, t^{-1}].$$

On the right hand side of Bott's formula (2.6-1), the numerator $p^T(E_{|F})$ is a homogeneous polynomial of degree $n = \dim X$ in as many variables as there are characters appearing in the decomposition into eigensubbundles. Typically, suppose that the original polynomial contains a term equal to $c_1^{n-2} \cdot c_2$, while say, $E_{|F} = 2\chi_1 + \chi_2$. Then we have $c_1^T(E_{|F}) = 2\chi_1 + \chi_2$, the right side now with the meaning of (1.8-1). Similarly, $c_2^T(E_{|F}) = \chi_1^2 + 2\chi_1 \cdot \chi_2$. That term yields at last the operator of degree n, to wit, $(2\chi_1 + \chi_2)^{n-2} \cdot (\chi_1^2 + 2\chi_1 \cdot \chi_2)$. Since

$T = \mathbb{C}^*$, each character is of the form $\chi_i = t^{a_i}$, $a_i \in \mathbb{Z}$. The operator induced on R_T is $a_i \cdot t$, sorry for the abuse, cf. (1.9-1) where this time t means a hyperplane class! That term gets the final form $(2a_1 + a_2)^{n-2} \cdot (a_1^2 + 2a_1 \cdot a_2) \cdot t^n \in R_T$.

That is, the numerator and the denominator on the right hand side of (2.6-1) are integer multiples of t^n. Cancelling t^n, we get in this way a rational number. So, the right hand side of (2.6-1) is a finite sum of rational numbers obtained from the weights as described in (2.7-1).

More precisely, denote by $\tau_1(E,F), \ldots, \tau_r(E,F)$ the weights that occur in the decomposition of $E_{|F}$ into eigensubbundles, and for each integer $k \geq 0$, let $\sigma_k(E,F)$ denote the k-th elementary symmetric function on these weights. We have therefore the following corollaries.

2.7.2 Corollary

With notation as just above, we have that each equivariant Chern class $c_k^T(E_{|F})$ is represented in the equivariant Chow ring of the fixed point F by $\sigma_k(E,F)$.

2.7.3 Corollary

The equivariant top Chern class of the tangent bundle of X is given in the equivariant Chow ring of a fixed point F by the product of the weights that occur in the decomposition of the respective tangent space.

In the next chapter we explain systematically how to apply Bott's formula to a few situations of interest in enumerative geometry. We can't resist however the compulsion to exhibit right away how the above result can be used to retrieve the number of zeros of a vector field in $\mathbb{P}^n$.

2.7.4 Zeros of vector fields in $\mathbb{P}^n$

Write $\mathcal{F} = \langle x_0, \ldots, x_n \rangle$ for the vector space of linear forms on these variables, a choice of homogeneous coordinates for $\mathbb{P}^n$. Consider the action of $T = \mathbb{C}^*$ given by $t \circ x_i = t^i x_i$. One sees at once that the set of fixed points in $\mathbb{P}^n$ is formed by the $n+1$ unitary points

$$P_0 = [1, 0, \ldots, 0], \ldots, P_n = [0, \ldots, 0, 1].$$

Let $\mathcal{A}$ be the subbundle of the trivial vector bundle $\mathcal{F}$ with fiber over each $P \in \mathbb{P}^n$ given by the space of linear forms that vanish at the point P. The tangent bundle admits the expression (cf. [21], p.200)

$$\mathcal{T}\mathbb{P}^n = \mathcal{A}^\vee \otimes (\mathcal{F}/\mathcal{A}). \tag{2.7-2}$$

The fiber over, say P_0, is given, with evident notation, by

$$\langle x_1, \ldots, x_n \rangle^\vee \otimes \langle \overline{x_0} \rangle.$$

Hence, the decomposition of the tangent space into eigenspaces can be described in symbols by

$$\mathcal{T}_{P_0}\mathbb{P}^n = (t^{-1} + \cdots + t^{-n}) \cdot t^0 = t^{-1} + \cdots + t^{-n}.$$

(Here we have used the property that the weight of the dual (resp. of a tensor product) is...) The product of characters that occur in the decomposition gives

$$c_n^T(\mathcal{T}_{P_0}\mathbb{P}^n) = (-1)^n n!\, t^n,$$

where the term t^n now means n-iteration of hyperplane classes.

One knows that the class in the usual Chow group of the cycle of zeros of a section of a vector bundle represents, under suitable conditions of regularity, the top Chern class of the bundle. In particular, if the zeros of a vector field in $\mathbb{P}^n$ are isolated, then $\int c_n(\mathcal{T}\mathbb{P}^n) \cap [\mathbb{P}^n]$ yields the number of zeros (counted with multiplicities).

Now Bott's formula (2.6-1) applied to the present situation displays, on its right hand side, $n+1$ terms, each equal to 1!

Indeed, each component F is just a point P_i, so that the homomorphism of integration $\pi_{F*} : \mathcal{R}_T \to \mathcal{R}_T$ is the identity. The numerator and denominator of the fractions that occur in the formula are both equal to $c_n^T(\mathcal{T}\mathbb{P}^n_{|P_i})$. We retrieve in this manner the well known fact that the number of zeros of a vector field in $\mathbb{P}^n$ is $n+1$.

Of course, the same argument shows that, in general, if X is a smooth complete variety endowed with a $\mathbb{C}^*$-action with exactly N isolated fixed points, then the Euler characteristic of X is equal to $\int_X c_n(\mathcal{T}X) = N$. The 1-parameter subgroup induces, by differentiation, a vector field $X \to \mathcal{T}X$, $x \mapsto \frac{d}{dt}(t \circ x)_{|t=1}$, whose set of zeros is precisely X^T.

3 Applications to enumerative geometry, I

Our goal in the final two chapters is to give an idea of the usefulness of Bott's formula for the computation of some characteristic numbers.

We begin with rather simple examples which certainly could be handled with more elementary, cheaper tools than explained here. The first two will hopefully serve the purpose of acquainting the reader with the computational gadgetry involved.

3.1 Two lines in $\mathbb{P}^2$

Probably one of the simplest and yet instructive problems is the counting of the number of points of intersection of two general lines in the projective plane $\mathbb{P}^2$. The reader will quickly realize that the answer to our question is *one*...

Now that we know from the start the size of the answer, we may try and mess up the discussion a little bit and go on to perform the calculation using the usual Chow ring,

$$A^*(\mathbb{P}^2) = \mathbb{Z}[h]/\langle h^3 \rangle$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ represents the class of a line in $\mathbb{P}^2$. Similarly, h^2 is the class of a point. Recalling that the product is induced by intersection, one sees at once that we want to compute the degree

$$\int c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2.$$

Our familiarity with the Chow ring of $\mathbb{P}^2$ is certainly enough to proclaim that this degree is 1.

However, the point here is to illustrate the use of Bott's formula. For this end, what really matters is realizing that the cycle which solves the present geometric question can be expressed as a polynomial function on the Chern classes of equivariant vector bundles for a suitable action of a torus.

3.1.1 Choice of the torus

In practice, it suffices to consider actions of $\mathbb{C}^*$, *i.e.*, 1-parameter subgroups judiciously selected in $T \subset GL_3$, a maximal torus acting diagonally on $\mathbb{P}^2$.

Choosing a 1-parameter subgroup $\mathbb{C}^* \subset T$ is equivalent to picking a point (w_0, w_1, w_2) in the free group of weights $\mathrm{Hom}(\mathbb{C}^*, T) = \mathbb{Z}^3$. The characters associated to the diagonal action of $\mathbb{C}^*$ are given by $\lambda_i = t^{w_i}$.

Henceforth we shall write $T = \mathbb{C}^*$, unidimensional torus acting on $\mathbb{P}^2$ so that the homogeneous coordinates x_0, x_1, x_2 are eigenvectors with weights w_0, w_1, w_2. That is,

$$t \circ x_i = t^{w_i} \cdot x_i \quad \text{for all} \quad t \in \mathbb{C}^*.$$

The fixed points locus of this $\mathbb{C}^*$-action is given by the system of equations

$$x_0 = t^{w_0} \cdot x_0 \;\; x_1 = t^{w_1} \cdot x_1 \;\; x_2 = t^{w_2} \cdot x_2 \quad \forall t \in \mathbb{C}^*.$$

The set of solutions of this system is simply

$$F = \{[1,0,0],\ [0,1,0],\ [0,0,1]\} \subset \mathbb{P}^2,$$

provided that the w_i's be chosen *all distinct* from each other. This will be assumed throughout the rest of this discussion.

We shall apply at first the version of Bott's formula for the case where the fixed points locus is a finite set F and the localization of the equivariant Chow ring of a point (1.5-1) is $\mathcal{R}_T = \mathbb{Q}[t, t^{-1}]$.

3.1.2 Decomposition into eigensubbundles

Recall that each T-vector bundle E restricted to the locus of fixed points decomposes canonically as a direct sum of subbundles $\bigoplus_\lambda E^\lambda$, where each E^λ

denotes the eigensubbundle of E where the a action is given by the character λ.

Presently, we must examine the decompositions of the tangent bundle $\mathcal{T}\mathbb{P}^2$ and the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$, both after restriction to a fixed point. The latter is the line bundle obtained by the quotient of the trivial bundle $\mathcal{F} = \langle x_0, x_1, x_2 \rangle$ of linear forms of $\mathbb{P}^2$, by the subbundle, $\mathcal{A}$, with fiber over a point $P \in \mathbb{P}^2$ given by the space of linear forms that vanish at P. Recalling (2.7-2), we have

$$\mathcal{T}\mathbb{P}^2 = \mathrm{Hom}(\mathcal{A}, \mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{A}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1).$$

3.1.3 The Chern classes

Continuing, we must study the weights of the representations induced on the fibers E_P for $E = \mathcal{O}_{\mathbb{P}^2}(1)$ and $E = \mathcal{T}\mathbb{P}^2$, at each fixed point P. Once this is done, each Chern class $c_k^T(E_P^\lambda)$ will be represented in the $\mathbb{C}^*$-equivariant Chow ring of the point P, by the k-th. power of the character λ, for $k \leq rk(E^\lambda)$, cf. (2.7.2).

At the fixed point $P = [1, 0, 0]$, we have $\mathcal{A}_P = \langle x_1, x_2 \rangle$. Thus, with obvious notation, we have $\mathcal{O}_{\mathbb{P}^2}(1)_P = \langle x_0, x_1, x_2 \rangle / \langle x_1, x_2 \rangle = \langle \overline{x_0} \rangle$. Here the weight is w_0.

Meanwhile,

$$\mathcal{T}_P\mathbb{P}^2 = \mathcal{A}_P^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1)_P = \langle x_1, x_2 \rangle^\vee \otimes \langle \overline{x_0} \rangle$$

decomposes as a direct sum of eigenspaces of dimension 1 with the $\mathbb{C}^*$-action given by the characters $t^{w_0 - w_1}$ and $t^{w_0 - w_2}$. The respective weights are $w_0 - w_1$ and $w_0 - w_2$. Hence, the class $c_2^T(\mathcal{T}\mathbb{P}_P^2)$ is represented by the weight $(w_0 - w_1) \cdot (w_0 - w_2)$ in the Chow ring $A_*^T(P)$ of the fixed point $P = [1, 0, 0]$. Similarly, we see that $c_2^T(\mathcal{T}_P\mathbb{P}^2)$ is represented by the weights

$$(w_1 - w_0) \cdot (w_1 - w_2) \ \text{ at } [0,1,0] \ \text{ and } \ (w_2 - w_0) \cdot (w_2 - w_1) \quad \text{at} \quad [0,0,1].$$

The weights of $\mathcal{O}_{\mathbb{P}^2}(1)$ at the fixed points $[0,1,0]$ and $[0,0,1]$ are w_1 and w_2.

Finally, we apply Bott's formula and to get the incredible identity

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = \sum_{P \in F} \int_{[P]} \frac{c_1^T(\mathcal{O}_{\mathbb{P}^2}(1)_P)^2 \cap [P]_T}{c_2^T(\mathcal{T}\mathbb{P}_P^2)}$$

$$= \frac{w_0^2}{(w_0-w_1).(w_0-w_2)} + \frac{w_1^2}{(w_1-w_0).(w_1-w_2)} + \frac{w_2^2}{(w_2-w_0).(w_2-w_1)} \equiv 1!!!$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ [1,0,0] & [0,1,0] & [0,0,1] \end{array}$$

3.2 Two lines in $\mathbb{P}^2$, bis

We now give an example of a simple application of the formula in the case where the set of fixed points is infinite. This actually occurs in our treatment of canonical curves in $\mathbb{P}^3$, as well as in the celebrated work of M. Kontsevich [30]. This is why we think it may be helpful to see how it works in a geometrically easier case, for which the answer is God-given.

Once again we let $T = \mathbb{C}^*$ act diagonally on $\mathbb{P}^2$, with weights

$$w_0 = w_1 = a,\ w_2 = b \neq a.$$

That is, $t \circ x_0 = t^a \cdot x_0$, $t \circ x_1 = t^a \cdot x_1$, $t \circ x_2 = t^b \cdot x_2$, for all $t \in \mathbb{C}^*$. Hence, the fixed points locus $X^T \subset X$ consists of two components:

- the line ℓ given by $x_2 = 0$, and
- the point $P = [0,0,1]$.

In the previous example, we have computed $\int_{\mathbb{P}^2} c_1(\mathcal{O}(1))^2$ in the case where X^T is a finite set. Now, Bott's formula's (2.6-1) yields

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}(1))^2 = \int_\ell \frac{c_1^T(\mathcal{O}(1)_\ell)^2 \cap [\ell]_T}{c_1^T(\mathcal{N}_{\ell/\mathbb{P}^2})} + \int_P \frac{c_1^T(\mathcal{O}(1)_P)^2 \cap [P]_T}{c_2^T(\mathcal{T}_P(\mathbb{P}^2))}$$

where for the second term in the sum we already know what comes out:

$$w_2^2 / \big((w_2 - w_0)\cdot(w_2 - w_1)\big) = b^2/(b-a)^2.$$

To find the contribution of the positive dimensional component $\ell \cong \mathbb{P}^1$, we need the values of $c_1^T(\mathcal{O}(1)_{|\ell})$ and $c_1^T(\mathcal{N}_{\ell/\mathbb{P}^2})$ in the equivariant Chow ring of ℓ. We have $\mathcal{O}_{\mathbb{P}^2}(1)_{|\ell} = \mathcal{O}_\ell(1)$, eigenbundle on ℓ of rank 1 with weight a. In view of the lemma 2.2.2, it follows that

$$c_1^T(\mathcal{O}_\ell(1)) = h + at \ \in A_*^T(\ell),$$

where $A^T_*(\ell) = A_*(\ell) \otimes R_T$ with $R_T = \mathbb{Z}[t]$ and $A_*(\ell) = A_*(\mathbb{P}^1) = \mathbb{Z}[h]/\langle h^2\rangle$.

Warning! As to the normal bundle, even though $\mathcal{N}_{\ell/\mathbb{P}^2}$ and $\mathcal{O}_\ell(1)$ *are* isomorphic, they *are not so as $T-$bundles*: the weight this time is $a-b$, not a, as we had in the previous case! In fact, let us study the natural sequence,

$$\mathcal{T}\mathbb{P}^1 \hookrightarrow \mathcal{T}\mathbb{P}^2_{|\ell} \twoheadrightarrow \mathcal{N}_{\ell/\mathbb{P}^2}.$$

We know that $\mathcal{T}\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$. But here the weight is trivial, given that the action on ℓ is trivial. To determine the weights of the central term at each point $Q = [\alpha, \beta, 0] \in \ell$, look at $\mathcal{A}_Q = \langle x_2, \beta x_0 - \alpha x_1\rangle$, the corresponding space of linear forms. Its decomposition is $t^b + t^a$. The fiber $\mathcal{T}_Q\mathbb{P}^2$ is given by $\mathcal{A}_Q^\vee \otimes \mathcal{F}/\mathcal{A}_Q$. The decomposition into eigenespaces can now be written as $(t^{-b} + t^{-a}) \cdot (2t^a + t^b - t^b - t^a) = t^{a-b} + 1$. Discounting the trivial character, 1, which comes from $\mathcal{T}\mathbb{P}^1$, we may conclude that the weight in $\mathcal{N}_{\ell/\mathbb{P}^2}$ is $a-b$. Hence, again by (2.2.2) we have

$$c_1^T(\mathcal{N}_{\ell/\mathbb{P}^2}) = h + (a-b)t.$$

It remains to compute

$$\int_\ell \frac{(h+at)^2}{h+(a-b)t}.$$

Recalling that $(a-b)t$ is invertible in $A^T_*(\ell) \otimes \mathcal{R}_T$ and using the fact that $h^2 = 0$, we may write

$$\big(h + (a-b)t\big)^{-1} = \frac{\big(h - (a-b)t\big)}{\big(-(a-b)^2t^2\big)}.$$

This implies

$$\frac{(h+at)^2}{(h+(a-b)t)} = -\frac{(h+at)^2(h-(a-b)t)}{(a-b)^2t^2}.$$

Collecting the coefficient of h we get

$$\int_\ell \frac{(h+at)^2}{h+(a-b)t} = -\frac{2a(b-a)+a^2}{(b-a)^2} = \frac{a^2-2ab}{(b-a)^2}.$$

Finally, enjoy another bizarre manner to find the number 1:

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}(1))^2 = \frac{a^2-2ab}{(b-a)^2} + \frac{b^2}{(b-a)^2} = \frac{(b-a)^2}{(b-a)^2} = 1.$$

3.3 The 27 lines on a cubic surface

The next educational example will be the calculation of the number of lines contained in a general cubic surface $S \subset \mathbb{P}^3$.

Let x_0, x_1, x_2, x_3 be homogeneous coordinates in $\mathbb{P}^3$. Once again $T = \mathbb{C}^*$ acts with weights (w_0, w_1, w_2, w_3), $t \circ x_i = t^{w_i} \cdot x_i$, for all $t \in \mathbb{C}^*$.

Let $\mathcal{F} = \langle x_0, x_1, x_2, x_3 \rangle$ be the trivial vector bundle of linear forms on $\mathbb{P}^3$.

Denote by $Gr(2,4)$ the Grassmann variety that parametrizes the lines in $\mathbb{P}^3$. It carries the tautological sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where the fiber of $\mathcal{A}$ over $\ell \in Gr(2,4)$ is the bidimensional subspace of linear forms that vanish on the line ℓ.

The action of the torus in $\mathbb{P}^3$ induces a natural action on $Gr(2,4)$. The maps $\mathcal{A} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{Q}$ are equivariant.

Each cubic surface $S \subset \mathbb{P}^3$ is given as zeros of a section of the trivial symmetric power bundle, $\mathcal{S}_3\mathcal{F}$. Composing with the quotient map $\mathcal{S}_3\mathcal{F} \twoheadrightarrow \mathcal{S}_3\mathcal{Q}$, we get a section $s : \mathcal{O} \to \mathcal{S}_3\mathcal{Q}$ over $Gr(2,\mathcal{F})$. It can be easily checked that, for each line $\ell \in Gr(2,\mathcal{F})$, the section s vanishes in the fiber $\mathcal{S}_3\mathcal{Q}_\ell$ if and only if the surface S contains the line ℓ. We see that the cycle of the sought for locus in $Gr(2,4)$ is given by the top Chern class of the bundle $\mathcal{S}_3\mathcal{Q}$. That is, the number we are after is the degree

$$\int_{Gr(2,4)} c_4(\mathcal{S}_3\mathcal{Q}).$$

We proceed to compute it using Bott's formula.

Choosing the weights w_i's distinct, the set of fixed points $F \subset Gr(2,4)$ is

$$F = \{\langle x_0, x_1\rangle, \langle x_0, x_2\rangle, \langle x_0, x_3\rangle, \langle x_1, x_2\rangle, \langle x_1, x_3\rangle, \langle x_2, x_3\rangle\}$$

where $\langle x_i, x_j \rangle$ represents the line given by $x_i = x_j = 0$.

Let us study the weights of the induced representations on the fibers $\mathcal{S}_3\mathcal{Q}_\ell$ and $\mathcal{T}_\ell Gr(2,4)$ for each fixed point $\ell \in F$.

We recall the identification of the tangent space cf. [21],

$$\mathcal{T}Gr(2,4) = \mathcal{A}^\vee \otimes \mathcal{Q}.$$

On the fiber over the fixed point $\ell = \langle x_0, x_1 \rangle$, we have

$$\mathcal{T}_\ell Gr(2,4) = \langle x_0, x_1 \rangle^\vee \otimes (\mathcal{F}/\langle x_0, x_1 \rangle).$$

This space is generated by the eigenvectors $x_2 \otimes x_0^\vee, x_2 \otimes x_1^\vee, x_3 \otimes x_0^\vee, x_3 \otimes x_1^\vee$ whose weights are equal to $w_2 - w_0, w_2 - w_1, w_3 - w_0, w_3 - w_1$ respectively. Thus, the equivariant Chern class $c_4^T(\mathcal{T}_\ell Gr(2,4))$ is represented by the weight

$$(w_0 - w_2) \cdot (w_0 - w_3) \cdot (w_1 - w_2) \cdot (w_1 - w_3)$$

in the $\mathbb{C}^*$-equivariant Chow ring of the fixed point $\ell = \langle x_0, x_1 \rangle \in Gr(2,4)$.

As to $\mathcal{S}_3\mathcal{Q}$, its fiber over the the same fixed point ℓ is the quotient space generated by the classes

$$\bar{x}_2^3,\ \bar{x}_2^2 \cdot \bar{x}_3,\ \bar{x}_2 \cdot \bar{x}_3^2,\ \bar{x}_3^3$$

with respective weights

$$3w_2,\ 2w_2 + w_3,\ w_2 + 2w_3,\ 3w_3.$$

Hence, the Chern class $c_4^T(\mathcal{S}_3\mathcal{Q}_\ell)$ is represented by the weight

$$3w_2 \cdot (2w_2 + w_3) \cdot (w_2 + 2w_3) \cdot 3w_3$$

in the $\mathbb{C}^*$-equivariant Chow ring of the fixed point ℓ. Similarly, we can find the weights at the other five fixed points and compute, using 2.6-1,

$$\begin{aligned}
\int_{Gr(2,4)} c_4(\mathcal{S}_3\mathcal{Q}) &= \sum_{\ell\in F}\int_{[l]} \frac{c_4^T(\mathcal{S}_3\mathcal{Q}_{|\ell})}{c_4^T(\mathcal{T}_\ell Gr(2,4))} \\
&= 9\frac{w_0\cdot(2w_0+w_1)\cdot(w_0+2w_1)\cdot w_1}{(w_0-w_2)\cdot(w_0-w_3)\cdot(w_1-w_2)\cdot(w_1-w_3)} \\
&+9\frac{w_0\cdot(2w_0+w_2)\cdot(w_0+2w_2)\cdot w_2}{(w_0-w_1)\cdot(w_0-w_3)\cdot(w_2-w_1)\cdot(w_2-w_3)} \\
&+9\frac{w_0\cdot(2w_0+w_3)\cdot(w_0+2w_3)\cdot w_3}{(w_0-w_1)\cdot(w_0-w_2)\cdot(w_3-w_0)\cdot(w_3-w_2)} \\
&+9\frac{w_1\cdot(2w_1+w_2)\cdot(w_1+2w_2)\cdot w_2}{(w_1-w_0)\cdot(w_1-w_3)\cdot(w_2-w_0)\cdot(w_2-w_3)} \\
&+9\frac{w_1\cdot(2w_1+w_3)\cdot(w_1+2w_3)\cdot w_3}{(w_1-w_0)\cdot(w_1-w_2)\cdot(w_3-w_0)\cdot(w_3-w_2)} \\
&+9\frac{w_2\cdot(2w_2+w_3)\cdot(w_2+2w_3)\cdot w_3}{(w_2-w_0)\cdot(w_2-w_1)\cdot(w_3-w_0)\cdot(w_3-w_1)}.
\end{aligned}$$

After simplifications, or substitution of explicit distinct values for the w_i's, (for instance $w_0=-1, w_1=0, w_2=1, w_3=2$) we get

$$= 9(0-1/3+0+0+0+40/12) = 27.$$

Hence, a general cubic surface of $\mathbb{P}^3$ contains exactly 27 lines. The role these lines have played in the literature, both recent and classical, can hardly be overestimated, cf. [28], [9].

3.4 The 81 conics

Continuing the previous example, let us describe now the computation of the number of conics in $\mathbb{P}^3$ contained in a general cubic surface $S\subset\mathbb{P}^3$ and incident to a line in general position.

In the preceding section, the Grassmann variety $Gr(2,4)$ served as a space of parameters for the family of lines of $\mathbb{P}^3$. Thus, to start with, we need to construct a space of parameters for the family of conics in $\mathbb{P}^3$. For this, let $\check{\mathbb{P}}^3$ be the dual projective space of planes of $\mathbb{P}^3$, with tautological sequence

$$0\to\mathcal{O}_{\check{\mathbb{P}}^3}(-1)\longrightarrow\mathcal{F}\longrightarrow\mathcal{H}\to 0$$

where $\mathcal{F}$ is the vector space of linear forms in $\mathbb{P}^3$. The fiber $\mathcal{O}_{\check{\mathbb{P}}^3}(-1)(-1)_h \subset \mathcal{F}$ is the subspace of multiples of an equation of the plane $h \in \check{\mathbb{P}}^3$.

The vector bundle $\mathcal{E} = \mathcal{S}_2\mathcal{H}$ is of rank 6. Its fiber over $h \in \check{\mathbb{P}}^3$ is the space of quadratic forms in the plane h. Each point of the projective bundle $\mathbb{P}(\mathcal{E})$ corresponds to a pair (h, κ) where $h \in \check{\mathbb{P}}^3$ represents the supporting plane of a conicκ of h. In other words, the projective bundle $\mathbb{P}(\mathcal{E})$ is the space of parameters of the family of conics of $\mathbb{P}^3$. We have, by construction, a diagram of vector bundles $/\mathbb{P}(\mathcal{E})$,

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathcal{S}_2\mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{E}}(-1) & \hookrightarrow & \mathcal{E}. \end{array}$$

The fiber of $\mathcal{A}$ over a point of $\mathbb{P}(\mathcal{E})$ that represents a conic κ in the plane h is the 5-dimensional space of quadrics which contain κ.

3.4.1 Lemma

Let $S \subset \mathbb{P}^3$ be a general cubic surface. Fix a line $\ell \subset \mathbb{P}^3$ and a point $p \in \mathbb{P}^3$. Let $L \subset \mathbb{P}(\mathcal{E})$ denote the subvariety of conics incident to ℓ. Write P for the variety of conics with supporting plane h passing through p. Finally, denote by C the variety of conics contained in S. Then their classes in $A^*(\mathbb{P}(\mathcal{E}))$ are given by

$$[P] = c_1(\mathcal{O}_{\check{\mathbb{P}}^3}(1))$$

$$[L] = 2c_1(\mathcal{O}_{\check{\mathbb{P}}^3}(1)) + c_1(\mathcal{O}_{\mathcal{E}}(1))$$

$$[C] = c_7(\mathcal{B})$$

where $\mathcal{B}$ is the cokernel of the map induced by multiplication,

$$\mathcal{A} \otimes \mathcal{F} \longrightarrow \mathcal{S}_3\mathcal{F}.$$

Proof. Let $\mathcal{O}^{\oplus 3} \hookrightarrow \mathcal{F}$ be a choice of three independent linear forms that vanish at the point p. We have the diagram of bundles over $\check{\mathbb{P}}^3$,

$$\begin{array}{ccccc} & & \mathcal{O}(-1) & & \\ & & \downarrow & \searrow & \\ \mathcal{O}^{\oplus 3} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O} \end{array}$$

The section $s : \mathcal{O} \to \mathcal{O}_{\check{\mathbb{P}}^3}(1)$ induced by the slant arrow vanishes on the fiber over $h \in \check{\mathbb{P}}^3$ if and only if an equation of h, which is the image of the vertical arrow, belongs to the subspace generated by equations of p, *i.e.*, if and only if $p \in h$. That is, the class we are looking for is given by

$$[P] = c_1(\mathcal{O}_{\check{\mathbb{P}}^3}(1)) \cap [\mathbb{P}(\mathcal{E})].$$

To determine the class $[C]$ in $A^*(\mathbb{P}(\mathcal{E}))$, let $\sigma \in \mathcal{S}_3\mathcal{F}$ be an equation of the cubic surface $S \subset \mathbb{P}^3$. We may construct the following diagram of bundles over $\mathbb{P}(\mathcal{E})$,

$$\begin{tikzcd} & \mathcal{O} \arrow[d, "\sigma"'] \arrow[dr, "\bar{\sigma}"] & \\ \mathcal{A} \otimes \mathcal{F} \arrow[r] & \mathcal{S}_3\mathcal{F} \arrow[r] & \mathcal{B}. \end{tikzcd}$$

The section $\bar{\sigma} : \mathcal{O} \to \mathcal{B}$ has a locus of zeros consisting of all conics contained in the surface S. Therefore, the desired class is given by

$$[C] = c_7(\mathcal{B}) \cap [\mathbb{P}(\mathcal{E})].$$

A similar reasoning shows that $[L] = \big(2c_1(\mathcal{O}_{\check{\mathbb{P}}^3}(1)) + c_1(\mathcal{O}_{\mathcal{E}}(1))\big) \cap [\mathbb{P}(\mathcal{E})]$.

□

Now for the explicit calculation employing Bott's formula, we choose again a T-action with distinct weights w_i. The set $F \subset \mathbb{P}(\mathcal{E})$ of fixed points of the induced action is finite ($\#F = 4.6 = 24$). Although this number is small enough to be handled with bare hands, we introduce here the computational method which will rescue us in the cases where such approach would be very unpleasant, as in the examples of the final chapter. There is a script in MAPLE available for download at [37] for the calculation of the degree of these zero-cycles. It was adapted by the first author, profiting from P. Meurer [32]. We obtain in particular the following table:

0-cycle Σ	$\int_{\mathbb{P}(\mathcal{E})} \Sigma$
$[C] \cdot [L]$	81
$[C] \cdot [P]$	27
$[L]^8$	92
$[L]^7 \cdot [P]$	34
$[L]^6 \cdot [P]^2$	8
$[L]^5 \cdot [P]^3$	1

The table above tells us, for example, that a general cubic $S \subset \mathbb{P}^3$ contains $\int [C] \cdot [L] = 81$ conics incident to a line ℓ in general position.

Of course one may also retrieve these numbers recalling the classical fact that the cubic surface of $\mathbb{P}^3$ is a Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in 6 points $p_1, \ldots, p_6$ in general position. It is embedded in $\mathbb{P}^3$ by the linear system of plane cubics that pass through the six points. If you are familiar with this description (cf. [4]), you must have seen the counting of the 27 lines on the surface. They correspond precisely to the following plane configurations:

lines passing through 2 of the six points	$\Longrightarrow$	$\binom{6}{2} = 15$
conics passing through 5 of the six points	$\Longrightarrow$	$\binom{6}{5} = 6$
exceptional divisors	$\Longrightarrow$	6
TOTAL	$=$	27

A similar analysis shows that the 81 conics are gotten as follow. First, note that $\ell \cap S$ is formed by 3 points q_1, q_2, q_3 whose images in $\mathbb{P}^2$ will be denoted by the same symbols.

lines passing through some p_i and some q_j	$\Longrightarrow$	$6 \cdot 3 = 18$
conics passing through 4 of p_i's and some q_j	$\Longrightarrow$	$\binom{6}{4} \cdot 3 = 45$
cubics passing through 5 of p_i's, singular at the sixth point and passing through some q_j	$\Longrightarrow$	$6.3 = 18$
TOTAL	$=$	....

4 Enumerative applications, II

The following sections publicize applications of Bott's formula in recent works on smooth compactifications of parameter spaces for some families of projective varieties (cf. the articles [1],[2],[3], [35], [36]), [39] and [38]). The point to stress here is that Bott's formula essentially trivializes the difficulties for describing the Chow ring of the spaces of parameters.

4.1 Twisted cubics in the quintic

We explain the computation of the number of twisted cubics contained in a general quintic hypersurface $S \subset \mathbb{P}^4$. This question has had great historical importance since the pioneering paper of Clemens [8] (cf. also [27] for an update on related topics). The answer was first found by physicists in the context of string theory. The first mathematical confirmation was due to Ellingsrud and Strømme [15]. In their initial approach, the norwegian team aimed at finding that number by means of intensive use of presentation and relations for the Chow ring of the component $\mathbb{H}$ of the Hilbert scheme of twisted cubics. Quite surprisingly, the first result that they announced diverged from the one published by the physicists. It has turned out that an error was found in the computational script employed by the two mathematicians! It seems fair to say that, at least in part thanks to the mistake, a vigorous activity has started towards a better understanding of mathematical aspects of the physics involved. See the book of D. A. Cox & S. Katz [10].

We shall describe the computation, using Bott's formula on the variety introduced in the docotoral dissertation of *Fernando Xavier* [39], [38]. He constructs a smooth compactification for the space of twisted cubics that avoids the use of Geometric Invariant Theory, a central tool in [14].

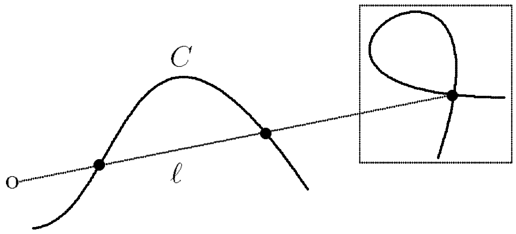
4.1.1 Idea of the compactification

We start by recalling that a *twisted cubic* is the image of the map

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ [u,v] & \mapsto & [u^3, u^2v, uv^2, v^3] \end{array}$$

for a suitable choice of homogeneous coordinates.

The principal idea of the construction rests on the following elementary fact: for each point $o \in \mathbb{P}^3$ not on a twisted cubic C there is a unique line ℓ passing through o that is bisecant (possibly tangent) to C.



The configuration $\ell \cup C$ is a complete intersection of a pencil of quadrics. A compactification of the space of twisted cubics that miss a point $o \in \mathbb{P}^3$ is constructed by simply *reverting* the process: for each line ℓ, one takes a pencil $\pi = \langle q_1, q_2 \rangle$ formed by the quadrics that contain the line. One obtains, at least generically, a twisted cubic residual to the line. The point now is to solve the indeterminacy of the cubic for the cases of special pencils. That is, we wish to ensure that our family of twisted cubics will be *complete*.

This will be done step by step. First of all, one associates to each pair (ℓ, π) as above, the *net* of quadrics $\nu(\ell, \pi) = \langle q_1, q_2, q_3 \rangle$ which, at least generically, give equations for the residual twisted cubic. Next, we must resolve the indeterminacies of the rational map ν. However, the family of nets of quadrics thus constructed is not enough to complete the desired family of twisted cubics. Indeed, it is well known that there are degenerations which require cubic equations (cf. [22], p.259). The final steps consist in resolving the locus of indeterminacy of the rational map that associates to each net of the above type, a 10-dimensional system of cubics.

4.1.2 First step

We need the following notation:

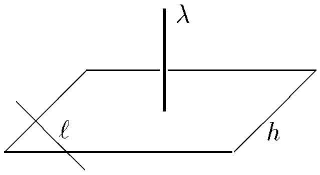
$$X = \left\{ (\ell, \pi) \;\middle|\; \begin{array}{l} \ell \subset \mathbb{P}^3 \text{ denotes a line and} \\ \pi = \langle q_1, q_2 \rangle \text{ is a pencil of} \\ \text{quadrics that contain } \ell \end{array} \right\}.$$

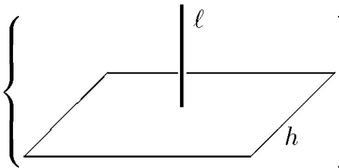
The projection $(\ell, \pi) \mapsto \ell$ exhibits X as a fibration over the grassmannian $Gr(2, 4)$ of lines in $\mathbb{P}^3$, with fiber the Grassmann variety $Gr(2, 7)$ of pencils of quadrics that contain ℓ.

We clearly have $\dim X = 4 + 10$, while the family of twisted cubics is of dimension 12. The excess of 2 is due to the ∞^2 bisecant lines that accompany each twisted cubic. We shall cut to size the excess on due time, restricting the family to the Schubert subvariety in $Gr(2,4)$ of lines passing through a point $o \in \mathbb{P}^3$, fixed once and for all.

Given a pair $(\ell, \pi) \in X$ as above, let us start by making explicit the net $\langle q_1, q_2, q_3 \rangle$ of quadrics whose base locus is equal to the residual twisted cubic, at least generically. In concrete terms, if a line ℓ is given by linear forms l_1, l_2, we have $q_i = \alpha_{i1} l_1 + \alpha_{i2} l_2$, $i = 1, 2$, where the α_{ij} denote linear forms. The third quadric must vanish where $q_1 = q_2 = 0$, outside of ℓ. This suggests taking $q_3 = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$. We obtain a rational map $X \cdots \to Gr(3,10)$ of the variety X to the grassmannian of nets of quadrics.

The locus of indeterminacy if this rational map is the nonsingular variety $Y_9 \subset X$, formed by pairs (ℓ, π) where π is a pencil of quadrics with a fixed component $h \supset \ell$; the moving part of π defines a line, denoted by λ in the picture below. We pinpoint another nonsingular subvariety $Y_7 \subset X$, again formed by pairs (ℓ, π) where π is a pencil of quadrics with a fixed component h except that now the moving part of π contains the distinguished line ℓ. Look at the pictures.

$Y_9 := \Bigg\{$ 

$\Bigg\}$, $Y_7 := \Bigg\{$ 

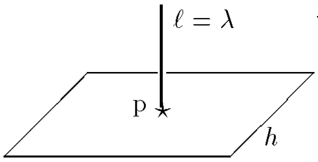
$\Bigg\}$.

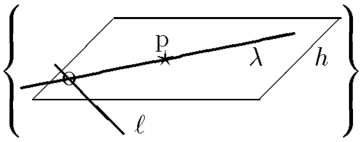
Blowing up $Y_9 \subset X$, we get a nonsingular variety X' together with a subbundle $\mathcal{A} \subset \mathcal{S}_2\mathcal{F}$ of rank 3, and a map

$$\nu : X' \to Gr(3,10) = Gr(3, \mathcal{S}_2\mathcal{F}).$$

4.1.3 Production of cubics

Let $\mathcal{F}$ be the vector space of linear forms in the variables x_1, x_2, x_3, x_4. The map of multiplication $\mathcal{A} \otimes \mathcal{F} \to \mathcal{S}_3\mathcal{F}$ is of generic rank 10. This enables us to define the rational map $\rho : X' \cdots \to Gr(3, \mathcal{S}_3\mathcal{F})$. The locus of indeterminacy is the subscheme of Fitting $Y' \subset X'$ defined locally by 10×10 minors of a local representation of our multiplication map. One checks that Y' is equal to the union of two smooth components Y'_7 and Y'_8, whose elements correspond to the configurations depicted below.

$Y'_7 := \Bigg\{$ 

$\Bigg\}$, $Y'_8 := \Bigg\{$ 

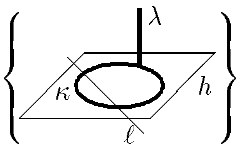
$\Bigg\}$.

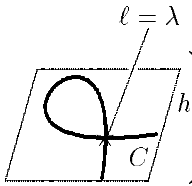
The subvariety Y'_7 is the strict transform of Y_7. It is isomorphic to the blowup of $Gr(2,4) \times \check{\mathbb{P}}^3$ along the incidence subvariety formed by pairs (ℓ, h) with the line ℓ contained in the plane h. Thus, the point $\mathrm{p} \in h \cap \ell$ is always well defined. The subvariety Y'_8 parametrizes the configurations $(\mathrm{p} \in \lambda \subset h \supset \ell)$. The intersection $Y'_7 \cap Y'_8$ is isomorphic to the flag variety $(\mathrm{p} \in \ell \subset h)$.

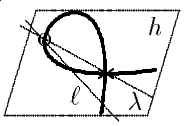
4.1.4 Final blowups

Blowup first Y'_7 in X', thereby producing $X'' \to X'$. the strict transform $Y''_8 \subset X''$ is now enriched by the point $\mathrm{o} \in \ell \cap \lambda$ everywhere well defined. At last, the desired compactification is X''', the blowup of X'' along Y''_8.

The subvariety of X''' that maps to Y consists of 3 nonsingular hypersurfaces E'''_1, E'''_2, E'''_3 which meet transversally. Their generic members are given by following configurations,

$E'''_1 := \Bigg\{$ 

$\Bigg\}$, $E'''_2 := \Bigg\{$ 

$\Bigg\}$, $E'''_3 := \Bigg\{$ 

$\Bigg\}$.

On the first picture, we have a union of a conic and an incident line. In the last two, we have a singular plane cubic C, with an embedded point $\star \in \lambda \cap h$.

4.1.5 Passing to $\mathbb{P}^4$

We will apply Bott's formula in a space of parameters $\mathbb{X}'''$ for the space of twisted cubics of $\mathbb{P}^4$. It is obtained repeating the previous construction, but letting now $\mathbb{P}^3$ vary in the family of hyperplanes $h \cong \mathbb{P}^3 \subset \mathbb{P}^4$.

We have to begin with a fibration $\mathbb{X} \to \check{\mathbb{P}}^4$ with fiber over each $h \in \check{\mathbb{P}}^4$ the variety X described above. This fibration factors as $\mathbb{X} \to \mathbb{G} \to \check{\mathbb{P}}^4$ into a fibration in grassmannians $\mathbb{G} \to \check{\mathbb{P}}^4$, with fiber $\mathbb{G}_h \cong Gr(2,4)$, the variety of lines contained in $h = \mathbb{P}^3$. The tautological vector bundles over $\check{\mathbb{P}}^4$ and $\mathbb{G}$ needed in the sequel will be denoted by

$$\mathcal{H} \hookrightarrow \mathbb{C}^5, \qquad \mathcal{H} \twoheadrightarrow \mathcal{Q}. \tag{4.1-1}$$

Here the fiber $\mathcal{H}_h$ is the subspace of $\mathbb{C}^5$ corresponding to the hyperplane h and the fiber $\mathcal{Q}_{(\ell,h)}$ is the quotient space of $\mathcal{H}_h$ by the subspace (of dimension 2) corresponding to the line $\ell \subset h$.

We perform the sequence of blowing ups

$$\mathbb{X}''' \to \mathbb{X}'' \to \mathbb{X}' \to \mathbb{X}$$

with centers varieties which are fibered over $\check{\mathbb{P}}^4$. Fiberwise, we have exactly the situation considered before. Note in particular the subvarieties $\mathbb{Y}_{9+4}, \mathbb{Y}_{11} \subset \mathbb{X}$ with respective fibers Y_9, Y_7, as well as $\mathbb{Y}' = \mathbb{Y}'_{11} \cup \mathbb{Y}'_{12} \subset \mathbb{X}'$ and $\mathbb{Y}''_{12} \subset \mathbb{X}''$, in addition to the exceptional divisors $\mathbb{E}'''_i$.

We have that $\mathbb{X}'''$ is a smooth projective variety of dimension 18, whose fiber over each $h \in \check{\mathbb{P}}^4$ is isomorphic to the variety X''' of the preceding subsection. A general point of $\mathbb{X}'''$, off the exceptional divisors, can be thought of as a triple (h, ℓ, π) where h denotes a hyperplane of $\mathbb{P}^4$, the line ℓ is contained in h and π represents a pencil of quadrics in h that contain ℓ.

Set now $\mathcal{F} = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, the space of linear forms in the homogeneous coordinates of $\mathbb{P}^4$. By construction, $\mathbb{X}'''$ is endowed with a vector subbundle $\mathcal{C} \subset \mathcal{S}_3\mathcal{F}$. Each fiber of $\mathcal{C}$ is a linear system of cubics in $\mathbb{P}^4$, with base locus a twisted cubic curve of $\mathbb{P}^4$. We have similarly a vector subbundle of $\mathcal{S}_5\mathcal{F}$, of rank $\binom{4+5}{4} - (5 \times 3 + 1) = 110$, image of the natural map $\mathcal{C} \otimes \mathcal{S}_2\mathcal{F} \to \mathcal{S}_5\mathcal{F}$. Define the bundle $\mathcal{E}$ as the cokernel of the latter map. We have therefore

$$\mathcal{S}_5\mathcal{F} \twoheadrightarrow \mathcal{E}. \tag{4.1-2}$$

Note that rank $\mathcal{E}$=16. As in the case of the 27 lines (3.3), we deduce that, for each quintic hypersurface of $\mathbb{P}^4$, given by a section of $\mathcal{S}_5\mathcal{F}$, the induced section of $\mathcal{E}$ vanishes exactly in the locus in $\mathbb{X}'''$ formed by points whose associated twisted cubic is contained in the quintic. We may state the following

4.2 Theorem

The number of twisted cubics contained in a general quintic hypersurface of $\mathbb{P}^4$ *is given by the degree*

$$\int_{\mathbb{X}'''} c_{16}(\mathcal{E}) \cdot c_2(\mathcal{Q})$$

where $\mathcal{Q}$ *denotes the quotient bundle* (4.1-1).

Proof. We restrict ourseleves to the justification of the factor $c_2(\mathcal{Q})$. It is in charge of cutting the excess of two dimensions of $\mathbb{X}'''$ with respect to the effective dimension (sixteen) of the family of twisted cubics in $\mathbb{P}^4$. Let $\mathbb{W}$ be the locus of zeros of the section of the vector bundle $\mathcal{E}$ above described. We have that $\mathbb{W}$ consists of a certain number, N, of disjoint subvarieties, of dimension 2, all situated in the complement of the exceptional divisors and each one contained in fiber of $\mathbb{X}''' \smallsetminus \bigcup \mathbb{E}_i''' = \mathbb{X} \smallsetminus \bigcup \mathbb{Y}_i$ over $\check{\mathbb{P}}^4$ (cf. remark below). Let $Z \subset \mathbb{W}$ be one of those N components and let $h \cong \mathbb{P}^3$ be the corresponding hyperplane. This variety $Z \subset \mathbb{X}_h$ maps isomorphically onto the surface of $Gr(2,4)$, still denoted by Z, which parametrizes the chords (*i.e.*, bisecant lines) of a twisted cubic. Let $\iota : Gr(2,4) \hookrightarrow \mathbb{G}$ be the inclusion in the fiber of $\mathbb{G}$ over h. The natural action of $\mathrm{Aut}(\mathbb{P}^4)$ permutes these varieties of chords, thus implying the equality of cycles, $[\mathbb{W}] = N \cdot \iota_*[Z]$ in $A_2(\mathbb{G})$. In order to determine N, it suffices to intersect with a suitable cycle of codimension 2. Here enters $c_2(\mathcal{Q})$. We may write $c_2(\mathcal{Q}) \cap \iota_*[Z] = \iota_*(\iota^* c_2(\mathcal{Q}) \cap [Z])$ and perform this last calculation in the fiber $Gr(2,4)$. Now it should be easy to convince ourselves that $\int_{Gr(2,4)} \iota^* c_2(\mathcal{Q}) \cap [Z]$ is the number of chords passing through a general point in $\mathbb{P}^3$.

$\square$

4.2.1 Remark.

The locus of zeros, $\mathbb{W}$, described in the above proof is pullback of the locus of zeros of a section of a similar bundle built over the component $\mathbb{H}$ of the Hilbert scheme. One knows that, for a sufficiently general choice of the quintic, this locus of zeros is formed by a finite number of points in $\mathbb{H}$, corresponding to non degenerated twisted cubics and not contained in a same $\mathbb{P}^3 \subset \mathbb{P}^4$ (cf. [29]). Hence, the inverse image of such point up on $\mathbb{X}'''$ sits in fact in $\mathbb{X}$ (outside the blowup center) and matches that surface of chords.

4.2.2 Employing Bott's formula

We stress once again that, feasible as it might be, the computation of the degree in 4.2 by means of the explicit structure of the Chow ring of $\mathbb{X}'''$ is strongly not recommended! The use of Bott's formula allows one to perform that task without pain.

As in the preceding cases, we start with a diagonal action of $T = \mathbb{C}^*$, with distinct weights $(w_1, w_2, w_3, w_4, w_5)$. We now examine the fixed points of the induced actions on each of the varieties to be considered, from $\check{\mathbb{P}}^4$ till $\mathbb{X}'''$.

The set of fixed points in $\check{\mathbb{P}}^4$ is $F = \{x_1, x_2, x_3, x_4, x_5\}$. Let us study the fibers over each of these 3-planes.

For $x_5 = 0$, we begin with the tangent space $\mathcal{T}_{x_5}\check{\mathbb{P}}^4 = (\mathcal{F}/\langle x_5\rangle) \otimes \langle x_5\rangle^\vee$.

From a computational perspective, given a T-vector bundle E, it will be very handy to write its expression as a sum of eigensubbundles,

$$E = \bigoplus\nolimits_\chi E^\chi, \qquad \text{in the “simplified” form:} \qquad E = \Sigma_\chi \chi.$$

In the present case, we have, $\mathcal{T}_{x_5}\check{\mathbb{P}}^4 = \chi_1/\chi_5 + \chi_2/\chi_5 + \chi_3/\chi_5 + \chi_4/\chi_5$.

We further simplify the notation just writing, henceforth, $\mathcal{F} = x_1 + x_2 + x_3 + x_4 + x_5$, where x_i stands for both the functional that is an eigenvector and the character χ_i it corresponds to.

Similarly, we may rewrite

$$\begin{gathered}\mathcal{T}_{x_5}\check{\mathbb{P}}^4 = (\mathcal{F}/\langle x_5\rangle) \otimes \langle x_5\rangle^\vee = \\ ((x_1+x_2+x_3+x_4+x_5) \;-\; x_5)\cdot(x_5)^{-1} = \\ x_1/x_5 + x_2/x_5 + x_3/x_5 + x_4/x_5.\end{gathered}$$

Look at the fiber of $\mathbb{X}'''$ over the 3-plane $x_5 = 0$, taking into account the construction

$$\mathbb{X}''' \to \mathbb{X}'' \to \mathbb{X}' \to \mathbb{X} \to \mathbb{G} \to \check{\mathbb{P}}^4.$$

Start with the fiber $\mathbb{G}_{x_5} = Gr(2,4)$ of the grassmannian of lines in this 3-plane. The action on $\mathbb{G}_{x_5}$ has the fixed points

$$F_2 = \{\langle x_1,x_2\rangle, \langle x_1,x_3\rangle, \langle x_1,x_4\rangle, \langle x_2,x_3\rangle, \langle x_2,x_4\rangle, \langle x_3,x_4\rangle\}$$

where $\langle x_i,x_j\rangle$ represents the line $x_i = x_j = 0$ in the 3-plane $x_5 = 0$. For instance, at the fixed point $\langle x_1,x_2\rangle$ we have

$$\begin{gathered}\mathcal{T}_{\langle x_1,x_2\rangle}\mathbb{G}_{x_5} = \big((x_1+x_2+x_3+x_4)/(x_1+x_2)\big) \otimes \langle x_1,x_2\rangle^\vee = \\ \big((x_1+x_2+x_3+x_4) - (x_1+x_2)\big)\cdot((x_1)^{-1} + (x_2)^{-1}) = \\ x_3/x_1 + x_4/x_1 + x_3/x_2 + x_4/x_2.\end{gathered}$$

The fiber of $\mathbb{X}$ over the point that represents the line $\langle x_1,x_2\rangle$ is the grassmannian $Gr(2,Q_{(\ell)})$ of subspaces of dimension 2 of the space $Q_{(\ell)}$ of quadratic forms that contain the distinguished line $\ell = \langle x_1,x_2\rangle$. We may write the decomposition into eigenspaces

$$Q_{(\ell)} = x_1^2 + x_1\cdot x_2 + x_1\cdot x_3 + x_1\cdot x_4 + x_2^2 + x_2\cdot x_3 + x_2\cdot x_4.$$

One checks that $Gr(2,Q_{(\ell)})$ has the following set of fixed points:

$$F_3 = \{\langle x_i\cdot x_j, x_r\cdot x_s\rangle \mid 1\le i\le r\le 2,\, j,s = 1..4,\, i\le j,\, r\le s, \quad x_i\cdot x_j \ne x_r\cdot x_s\}.$$

At this point we recall that the first blowup center, $\mathbb{Y}_{13}$, has as fiber over $\langle x_1,x_2\rangle$ the variety $\mathbb{P}(\langle x_1\rangle + \langle x_2\rangle)\times\mathbb{G}_{x_5}$. One sees that

$$\mathbb{Y}_{13}\cap F_3 = \{\langle x_i\cdot x_j, x_i\cdot x_k\rangle \mid i = 1,2,\; 1\le j<k\le 4\}.$$

On the other hand, the fiber of $\mathbb{Y}_{11}$ over $\langle x_1, x_2\rangle$ is the variety $\mathbb{P}(\mathcal{F}/\langle x_5\rangle)$. We have therefore

$$\mathbb{Y}_{11} \cap F_3 = \{\langle x_1^2, x_1 \cdot x_2\rangle, \langle x_1 \cdot x_2, x_2^2\rangle, \langle x_1 \cdot x_3, x_2 \cdot x_3\rangle, \langle x_1 \cdot x_4, x_2 \cdot x_4\rangle\}$$

and

$$\mathbb{Y}_{13} \cap \mathbb{Y}_{11} \cap F_3 = \{\langle x_1^2, x_1 \cdot x_2\rangle, \langle x_1 \cdot x_2, x_2^2\rangle\}.$$

Then, F_3 has 21 fixed points, 12 of which are in $\mathbb{Y}_{13} \cap F_3$ and 2 belong to $(\mathbb{Y}_{11} \setminus \mathbb{Y}_{13}) \cap F_3$.

We also need to know the fibers of $\mathcal{T}\mathbb{X}'''$ over each fixed point.

Since $\mathbb{X}'''$ is a blowup of $\mathbb{X}$ along centers that lie over $\mathbb{Y}_{13} \cup \mathbb{Y}_{11}$, we have that $\mathcal{T}\mathbb{X}''' = \mathcal{T}\mathbb{X}$ restricted to $\mathbb{X} - (\mathbb{Y}_{13} \cup \mathbb{Y}_{11})$.

Therefore, for each of the 7 fixed points

$$\langle x_1^2, x_2^2\rangle, \langle x_1^2, x_2 \cdot x_3\rangle, \langle x_1^2, x_2 \cdot x_4\rangle, \langle x_1 \cdot x_3, x_2^2\rangle,$$

$$\langle x_1 \cdot x_4, x_2^2\rangle, \langle x_1 \cdot x_3, x_2 \cdot x_4\rangle, \langle x_1 \cdot x_4, x_2 \cdot x_3\rangle$$

the tangent space can be computed still in $\mathbb{X}$. Hence, for instance, at the fixed point $\langle x_1^2, x_2^2\rangle$, we have

$$\mathcal{T}_{\langle x_1^2, x_2^2\rangle} Gr(2, Q_{\langle x_1,x_2\rangle}) = (Q_{\langle x_1,x_2\rangle}/\langle x_1^2, x_2^2\rangle) \otimes \langle x_1^2, x_2^2\rangle^\vee$$

$$\begin{aligned} &= ((x_1^2+x_1\cdot x_2+x_1\cdot x_3+x_1\cdot x_4+x_2^2+x_2\cdot x_3+x_2\cdot x_4) - (x_1^2+x_2^2))\cdot((x_1^2)^{-1}+(x_2^2)^{-1}) \\ &= x_2/x_1 + x_3/x_1 + x_4/x_1 + x_3/x_2 + x_4/x_2 + x_1/x_2 \\ &\quad + x_2 \cdot x_3/x_1^2 + x_2 \cdot x_4/x_1^2 + x_1 \cdot x_3/x_2^2 + x_1 \cdot x_4/x_2^2 \end{aligned}$$

so that, adding up tangents of fiber and base, we get

$$\begin{aligned} \mathcal{T}_{(x_5,\langle x_1,x_2\rangle,\langle x_1^2,x_2^2\rangle)}\mathbb{X} = {} & x_1/x_5 + x_2/x_5 + x_3/x_5 + x_4/x_5 + x_1/x_2 \\ & + x_2/x_1 + 2 \cdot x_3/x_1 + 2 \cdot x_4/x_1 + 2 \cdot x_3/x_2 + 2 \cdot x_4/x_2 \\ & + x_2 \cdot x_3/x_1^2 + x_2 \cdot x_4/x_1^2 + x_1 \cdot x_3/x_2^2 + x_1 \cdot x_4/x_2^2. \end{aligned}$$

The term $2 \cdot x_i/x_j$ means that there exist two independent eigenvectors with the same character x_i/x_j in the decomposition.

We recall that the exceptional divisor $\mathbb{E}_1' \subset \mathbb{X}'$ of the blowup of $\mathbb{X}$ along $\mathbb{Y}_{13}$ is the projectivization $\mathbb{P}(\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}})$ of the normal bundle $\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}}$. The latter one is the quotient $\mathcal{T}\mathbb{X}/\mathcal{T}\mathbb{Y}_{13}$. Hence, over each fixed point $P \in \mathbb{Y}_{13}$, the fiber $\mathbb{E}'_{1(P)}$ is the projective space $\mathbb{P}(\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)})$.

If in the decomposition of the normal space $\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)}$ the characters are all distinct, then in $\mathbb{P}(\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)})$ we get only a finite number of fixed points over the fixed point $P \in \mathbb{Y}_{13}$, namely, the dimension of $\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)}$.

We may compute the normal spaces for the following fixed points over the point that represents the line $\langle x_1, x_2\rangle$ in the 3-plane $x_5 = 0$,

$$\langle x_1^2, x_1\cdot x_2\rangle, \langle x_1^2, x_1\cdot x_3\rangle, \langle x_1^2, x_1\cdot x_4\rangle, \langle x_1\cdot x_2, x_1\cdot x_3\rangle,$$

$$\langle x_1\cdot x_2, x_1\cdot x_4\rangle, \langle x_1\cdot x_3, x_1\cdot x_4\rangle, \langle x_1\cdot x_2, x_2^2\rangle, \langle x_1\cdot x_2, x_2\cdot x_3\rangle,$$

$$\langle x_1\cdot x_2, x_2\cdot x_4\rangle, \langle x_2^2, x_2\cdot x_3\rangle, \langle x_2^2, x_2\cdot x_4\rangle, \langle x_2\cdot x_3, x_2\cdot x_4\rangle.$$

For instance, for the fixed point $\langle x_1^2, x_1\cdot x_2\rangle$ that belongs to $\mathbb{Y}_{13}\cap\mathbb{Y}_{11}$ we have

$$\begin{aligned}\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(\langle x_1^2, x_1\cdot x_2\rangle)} = \quad & \left(\begin{array}{c} x_2/x_1 + 2\cdot x_3/x_1 + 2\cdot x_4/x_1 + x_2^2/x_1^2 \\ +x_2\cdot x_3/x_1^2 + x_2\cdot x_4/x_1^2 + x_3/x_2 + x_4/x_2 \end{array}\right) \\ & -\big(x_2/x_1 + x_3/x_1 + x_4/x_1 + x_3/x_2 + x_4/x_2\big) \\ & = x_3/x_1 + x_4/x_1 + x_2^2/x_1^2 + x_2\cdot x_3/x_1^2 + x_2\cdot x_4/x_1^2.\end{aligned}$$

For the fixed point $\langle x_1\cdot x_3, x_1\cdot x_4\rangle$ that lies in $\mathbb{Y}_{13}\smallsetminus\mathbb{Y}_{11}$ we have

$$\begin{aligned}\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(\langle x_1\cdot x_3, x_1\cdot x_4\rangle)} = \quad & \left(\begin{array}{c} 2\cdot x_2/x_1 + x_1/x_3 + x_2/x_3 + x_1/x_4 \\ +x_2/x_4 + x_2^2/(x_1\cdot x_3) + x_2^2/(x_1\cdot x_4) \\ +x_2\cdot x_4/(x_1\cdot x_3) + x_2\cdot x_3/(x_1\cdot x_4) \end{array}\right) \\ & -\big(x_2/x_1 + x_1/x_3 + x_1/x_4 + x_2/x_3 + x_2/x_4\big)\end{aligned}$$

$$= x_2/x_1 + x_2^2/(x_1\cdot x_3) + x_2\cdot x_4/(x_1\cdot x_3) + x_2^2/(x_1\cdot x_4) + x_2\cdot x_3/(x_1\cdot x_4).$$

Computing at the other 10 points, we see that, in fact, the $\mathbb{C}^*$-action induced in $\mathbb{X}'$ has only finitely many fixed points.

We now proceed to the tangent space $\mathcal{T}_{P'}\mathbb{X}'$ at a fixed point P' of the exceptional divisor, in the fiber over $P \in \mathbb{Y}_{13}$.

This space is given by the decomposition

$$\mathcal{T}_{P'}\mathbb{X}' = \mathcal{L}_{P'} \oplus \mathcal{T}_P\mathbb{Y}_{13} \oplus \mathcal{T}_{[\mathcal{L}_{P'}]}\mathbb{P}(\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)})$$

where $\mathcal{L}_{P'}$ denotes the line represented by the point P' in the projective space $\mathbb{P}(\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)})$.

Thus, for instance, over the fixed point $\langle x_1^2, x_1x_2\rangle$ belonging to $\mathbb{Y}_{13}$ for which we have already gotten the expression

$$\mathcal{N}_{\mathbb{Y}_{13}/\mathbb{X}(P)} = x_3/x_1 + x_4/x_1 + x_2^2/x_1^2 + x_2\cdot x_3/x_1^2 + x_2\cdot x_4/x_1^2,$$

it follows that we have 5 fixed points, one for each eigenspace that appears in this decomposition. Taking for P' the point corresponding to the eigenspace with character x_3/x_1, we get

$$\begin{aligned}
\mathcal{T}_{P'}\mathbb{X}' = \quad & x_1/x_5 + x_2/x_5 + x_3/x_5 + x_4/x_5 + x_2/x_1 + 2\cdot x_3/x_1 \\
& +2\cdot x_3/x_2 + 2\cdot x_4/x_1 + 2\cdot x_4/x_2 + x_3/x_1 + \big(x_3/x_1 \\
& +x_4/x_1 + x_2^2/x_1^2 + x_2\cdot x_3/x_1^2 + x_2\cdot x_4/x_1^2 - x_3/x_1\big)\cdot(x_1/x_3) \\
= \quad & x_1/x_5 + x_2/x_5 + x_3/x_5 + x_4/x_5 + x_2/x_1 + 2\cdot x_3/x_1 \\
& +2\cdot x_3/x_2 + 2\cdot x_4/x_1 + 2\cdot x_4/x_2 + x_3/x_1 + x_4/x_3 \\
& +x_2^2/(x_1\cdot x_3) + x_2/x_1 + x_2\cdot x_4/(x_1\cdot x_3).
\end{aligned}$$

The reader interested in the effective calculation of all contributions may consult the MAPLE script available for download from [37].

4.3 Canonical curves in $\mathbb{P}^3$

We sketch in this final section the calculation of the number of canonical curves in $\mathbb{P}^3$ incident to 24 lines in general position. The result can be interpreted as a determination of an invariant of Gromov-Witten for $g = 4$, although the meaning of these invariants for positive genus is not clear yet.

We shall apply Bott's formula to a space of parameters for the family of canonical curves in $\mathbb{P}^3$, described in the article of *Jacqueline Rojas* and the 2nd author [35].

A variation of the method seems to work also for the case of the family of curves in $\mathbb{P}^3$ of genus 2 and degree 5. We hope to report on this elsewhere. In fact, the possibility of using this approach to determine the number of such curves contained in a general quintic of $\mathbb{P}^4$ sounds promising.

4.3.1 The component of Hilb

A canonical curve $C \subset \mathbb{P}^3$ is the image of a non-hyperelliptic curve of genus 4 by the canonical system. Its Hilbert polynomial is $6t + 1 - 4$.

Let $\mathbb{H}$ be the component of the Hilbert scheme of these curves. The canonical map embeds C as a curve of degree 6 in $\mathbb{P}^3$. The divisors cut out by quadrics on C are of degree 12. By Riemann-Roch, this linear system is of dimension 9 (=12+1-4). Hence, C is contained in a unique quadric surface. Similarly, C is contained in 5 independent cubics, 4 of which are multiples of the quadric. We see that C is the complete intersection of the quadric and a cubic. The dimensions of the vector spaces of forms of degrees $d = 4, ..., 7$ that vanish on C are listed below.

d	4	5	6	7
$\binom{d+3}{3} - (6 \cdot d + 1 - 4)$	14	29	51	81

(4.3-1)

The normal bundle of $C \subset \mathbb{P}^3$ is $\mathcal{N} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(3)$. We have $h^0(\mathcal{N}) = 24$, $h^1(\mathcal{N}) = 0$. It follows that C is a smooth point of $\mathbb{H}$ and $\dim \mathbb{H} = 24$.

4.3.2 Idea of the construction

The above discussion indicates how to produce a first approximation for $\mathbb{H}$. Let

$$X = \left\{ (f_2, f_3) \;\middle|\; \begin{array}{l} f_2 \in \mathbb{P}^9 \text{ denotes a quadric and} \\ f_3 \text{ is a cubic defined module } f_2 \end{array} \right\}. \tag{4.3-2}$$

We have a rational map

$$\begin{array}{cccc} \sigma : & X & \cdots \longrightarrow & \mathbb{H} \\ & (f_2, f_3) & \mapsto & f_2 \cap f_3, \end{array}$$

whose locus of indeterminacy appears where f_2 and f_3 admit a common component.

The main result of [35] says that there exists a sequence of 7 blowups $X^7 \to X^6 \to \cdots \to X^1 \to X$ along explicit, smooth centers, such that the rational map

$$\bar{\sigma} : X^7 \cdots \to \mathbb{H}$$

induced by σ is a birational morphism. We state it below, with the details needed for implementing the application of Bott's formula.

As we explain in the sequel, for the computation of that number it will suffice in fact to go up to X^5, avoiding the last two blowups.

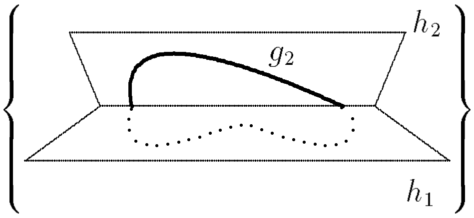
4.3.3 The 1st blowup

Fix homogeneous coordinates x_1, x_2, x_3, x_4 for the projective space $\mathbb{P}^3$. Let $\mathcal{F} = \langle x_1, x_2, x_3, x_4 \rangle$ be the vector space of linear forms.

Let Y_{11} be the subvariety of X (4.3-2) formed by pairs (f_2, f_3) with a common component, that is,

$$Y_{11} = \{(h_1 h_2, h_1 g_2) \mid h_1, h_2 \in \check{\mathbb{P}}^3,\, g_2 \in \mathbb{P}(\mathcal{S}_2(\mathcal{F}/\langle h_2 \rangle))\}. \tag{4.3-3}$$

This is the locus where the space of quartics $f_2 \cdot \mathcal{S}_2\mathcal{F} + f_3 \cdot \mathcal{F}$ has dimension less than 14. The first blowup $X^1 \to X$ is performed with center Y_{11}. The new variety, X^1, embeds in $X \times Gr(14, \mathcal{S}_4\mathcal{F})$. A general member of the exceptional divisor corresponds to the configuration indicated below.

$E^1_1 = \Bigg\{$ 

$\Bigg\}.$

The dotted quartic passes by the 2 points determinad by $h_1 \cap h_2 \cap g_2$.

By construction, X^1 is endowed with a vector subbundle $\mathcal{E}_4^{14} \subset \mathcal{S}_4\mathcal{F}$. The fiber of $\mathcal{E}_4^{14}$ over a general point of X^1 is a linear system of quartics with base locus a canonical curve.

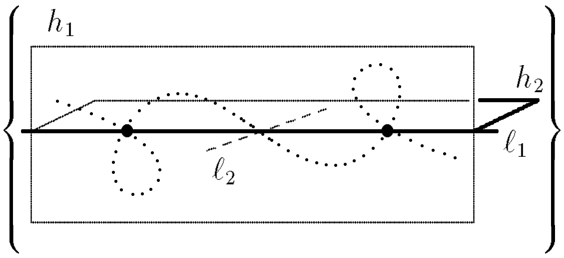
4.3.4 The 2nd blowup

Here on X^1, the next blowup center is the subvariety Y^1_{10} where the natural multiplication map $\mathcal{E}_{14} \otimes \mathcal{F} \to \mathcal{S}_5\mathcal{F}$, designed to produce quintics, has rank less than 29 (recall 4.3-1). The subvariety Y^1_{10} is contained in the exceptional divisor and is given generically by the condition that the distinguished conic g_2 (cf. picture above) contain the line $h_1 \cap h_2$. More precisely,

$$Y^1_{10} = \{(h_1 h_2, h_1 g_2; h_1(\mathcal{S}_3\mathcal{F})_{(\ell_2, k_2)})\}$$

where g_2 denotes the quadratic forma defining the pair of lines ℓ_1, ℓ_2 in the plane h_2 and $(\mathcal{S}_3\mathcal{F})_{(\ell_2,k_2)}$ stands for the subspace of cubic forms cutting in the plane h_2 a scheme union of the line ℓ_2 with the doublet of points given by k_2 over the line ℓ_1. The generic member of the exceptional divisor of the blowup $X^2 \to X^1$ along of Y^1_{10} is depicted below.

$E^2 =$



.

One shows that X^2 is endowed with a vector subbundle $\mathcal{E}_5^{29} \subset \mathcal{S}_5\mathcal{F}$ and embeds in $X^1 \times Gr(29, \mathcal{S}_5\mathcal{F})$. The dotted curve is a quintic in the plane h_1, singular at the doublet of distinguished points $k_2 \subset \ell_1$ and contains the point $\ell_1 \cap \ell_2$. The canonical curve appears as the scheme union of the line ℓ_2 and a quintic with the two embedded points indicated by $\bullet$ at the two points of k_2.

4.3.5 The 3rd and the 4th blowups

The 2 previous blowups were meant to get a bundle of quartics and another of quintics with correct ranks (see table (4.3-1)). Quite in contrast now, for the locus in X^2 where sextics "fail", the picture is more subtle. We find here 3 interesting subvarieties, Y_8^2, $Y_{14}^2 \supset Z_8^2$, which will serve for the next three blowup centers. Precisely, $X^3 \to X^2$ is the blowup along

$$Z_8^2 = \{(h^2, hg_2; h \cdot V_3; h \cdot (\mathcal{S}_4\mathcal{F})_{\ell_2,\mathrm{q}})\}.$$

Here, $g_2 = \ell_1\ell_2$ corresponds to a pair of lines in the plane h. The space of quartics is h times the space of cubics of the form

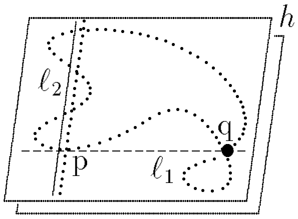
$$V_3 = h\mathcal{S}_2\mathcal{F} + g_2\mathcal{F} + \langle f_3' \rangle$$

with $f_3' = \ell_2 c$, denoting by c the equation of the divisor determined by a pair of points, p and q, over the line ℓ_1. The space of quintics is also h times a space of quartics, of the form

$$(\mathcal{S}_4\mathcal{F})_{\ell_2,\mathrm{q}} = h\mathcal{S}_3\mathcal{F} + \ell_2(\mathcal{S}_3\overline{\mathcal{F}})_{\mathrm{q}}$$

where $(\mathcal{S}_3\overline{\mathcal{F}})_{\mathrm{q}}$ denotes the system of cubics in the plane h that vanish at the point q. It turns out that each fiber of Z_8^2 over $h \in \check{\mathbb{P}}^3$ is isomorphic to the subvariety denoted by Y^1 in [35], corresponding to the plane configurations given by a pair of lines ℓ_1, ℓ_2 and a pair of points p,q such that $\mathrm{p} \in \ell_1 \cap \ell_2$ and $\ell_2 \ni \mathrm{q}$. The exceptional divisor now has the following aspect,

$$E^3 = \left\{ \quad \right\}.$$



The canonical curve appears as the scheme union of a nonplanar structure of degree 2 over a line ℓ_2 and a plane quartic g_4 (dotted in the picture above) singular and with embedded point at the point q, e tangent to the line ℓ_1 in the point p. The scheme structure over the double line ℓ_2 is of arithmetic genus -5. It is given by an ideal of the form $\langle h^2, h\ell_2, \ell_2^2, f_6\rangle$, with $f_6 = hg_5 + \ell_2\ell_{\mathrm{p}}g_4$ where $\ell_{\mathrm{p}} \in \mathbb{P}(\mathcal{F}/\langle h, \ell_1\rangle)$ marks the point p over the line ℓ_1 and the plane quintic $g_5 \in \mathbb{P}\big(\mathcal{S}_5(\mathcal{F}/\langle h\rangle)_{\mathrm{q}}/(\ell_2\mathcal{S}_4(\mathcal{F}/\langle h\rangle)_{\mathrm{q}})\big)$ vanishes at the point q and is nonzero module ℓ_2 times the system of plane quartics containing q.

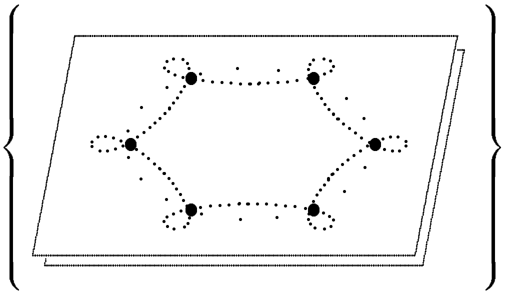
The 4th blowup center is the strict transform Y_{14}^3 of Y_{14}^2. We proceed to describe it. Travelling back to $X^1 \subset X \times Gr(14, \mathcal{S}_4\mathcal{F})$, we find the subvariety

$$Y_{14}^1 = \left\{ \big(h^2, hg_2; h \cdot (\langle g_3\rangle + h \cdot \mathcal{S}_2\mathcal{F} + g_2 \cdot \mathcal{F})\big) \;\middle|\; \begin{array}{l} g_2 \in \mathbb{P}\big(\mathcal{S}_2(\overline{\mathcal{F}})\big), \\ g_3 \in \mathbb{P}\big(\mathcal{S}_3(\overline{\mathcal{F}})/(g_2 \cdot \overline{\mathcal{F}})\big) \end{array} \right\},$$

where $\overline{\mathcal{F}} = \mathcal{F}/\langle h\rangle$. The fiber over each plane $h \in \check{\mathbb{P}}^3$ is isomorphic to the variety X formed by pairs (g_2, g_3) as above. The latter variety[2] was investigated in [35], as a first approximation for a space of parameters for the family of

[2]Note the subtle change of font type, X instead of X; ditto for Y's versus Y right ahead!

conical sextuplets. Moreover, the intersection of Y^1_{14} with the current blowup center (Y^1_{10}), yields fiberwise the first blowup center, Y, used in the strategy for flattening that family of sextuplets. Similarly, as already mentionned above, the fibers of the inclusion $Z^2_8 \subset Y^2_{14}$ are isomorphic to $\mathrm{Y}^1 \subset \mathrm{X}^1$. It follows that Y^3_{14} has fibers / $\check{\mathbb{P}}^3$ isomorphic to X^2. We perform the blowup $X^4 \to X^3$ with center Y^3_{14}. The exceptional divisor is illustrated below.

$E^4 =$  .

The canonical curve is generically given by a singular plane sextic with embedded points at the conical sextuplet given by $Z(g_2, g_3)$, indicated by $\bullet$.

4.3.6 The 5th blowup

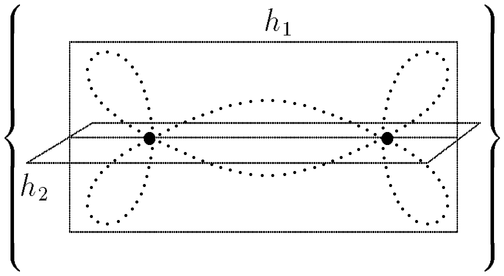
The next center is the subvariety $Y^4_8 \subset X^4$ described below. It is born back in X^2, denoted Y^2_8. It is actually disjoint from the blowup center Z^2_8, whence $Y^3_8 \cong Y^2_8$. Now the blowup center Y^3_{14} intersects Y^3_8 in a divisor of the latter. For this reason, we have $Y^3_8 \cong Y^4_8$. The local calculations reveal at last that

$$Y^2_8 = \{(h_1, h_2, \ell, k_2)\}$$

where ℓ denotes a line in the intersection of the planes h_1, h_2 and k_2 denotes a doublet of points on ℓ. The embedding in X^2 is obtained setting $f_2 = h_1 h_2$, $f_3 = h_1 \ell^2$, $f_4 = h_1 \ell k_2$ and $f_5 = h_1 k_2^2$. We have perpetrated the abuse of thinking of ℓ both as a line and as defining linear form in $\mathbb{P}(\mathcal{F}/\langle h_2 \rangle)$ and similarly for f_4, f_5.

The exceptional divisor of $X^5 \to X^4$ can be pictured thus.

$E^5 =$



The canonical curve is given by a sextic in the plane h_1, with two triple and embedded points in k_2, indicated by $\bullet$.

At this stage, one shows that X^5 is endowed with a vector subbundle $\mathcal{E}_6^{51} \subset \mathcal{S}_6\mathcal{F}$. The subscheme of $\mathbb{P}^3$ defined as the base locus of any fiber of $\mathcal{E}_6^{51}$ is a curve. The general such one is a canonical curve.

However, this is not a flat family yet. In fact, there exists a closed subscheme $Y^5 \subset X^5$ such that, for each $\mathrm{y} \in Y^5$ the subscheme of $\mathbb{P}^3$ defined by the base locus of $(\mathcal{E}_6^{51})_{\mathrm{y}}$, though of dimension 1, has the wrong Hilbert polynomial $(7t-9)$. One finds out that Y^5 is a union of two smooth components, Y_{16}^5, Y_{17}^5. Blowing up first Y_{17}^5 and then the strict transform Y_{16}^6, obtains X^7. The latter variety is endowed with a vector subbundle $\mathcal{E}_7^{81} \subset \mathcal{S}_7\mathcal{F}$ that produces a flat, complete family of canonical curves, as well as a birational morphism $\overline{\sigma} : X^7 \to \mathbb{H}$.

The last two blowups, $X^7 \to X^6 \to X^5$, are in fact irrelevant for the enumerative calculation. Indeed, let $D \subset \mathbb{H}$ be the hypersurface formed by the points of $\mathbb{H}$ that correspond to canonical curves incident to a given line. We wish to compute $\int_{\mathbb{H}} D^{24}$. One shows that this number is the same as $\int_{X^7} (\overline{\sigma}^{-1} D)^{24}$. Now $\widetilde{D} = \overline{\sigma}^{-1} D$ is the strict transform of its image $\overline{D} \subset X$. Let $\pi^i : X^i \to X^{i-1}$ be the blowup map. Let $D^{(i)}$ the strict transform of $\overline{D}$ in X^i. It can be shown that the total and strict transforms coincide at the last two stages, $\widetilde{D} = D^{(7)} = (\pi^7)^{-1} D^{(6)}$ and $D^{(6)} = (\pi^6)^{-1} D^{(5)}$. This is so by the simple reason that they do not contain the blowup centers. By the projection formula we may write $\int_{X^7} \widetilde{D}^{24} = \int_{X^7} (\pi^7)^* (D^{(6)})^{24} = \int_{X^5} (D^{(5)})^{24}$.

This last number is the one we are able to compute using Bott's formula. A script in MAPLE for the explicit computation can be found in [37]

Remark: scripts for MAPLE

There are two scripts available for download at [37]. They have been adapted by the first author profiting from P. Meurer [32]. The first one computes the number of conics in the cubic surface. The other one finds the number of twisted cubics in the quintic

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