# NILPOTENT ACTIONS ON NON-ABELIAN TENSOR PRODUCTS OF GROUPS

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**Abstract**

Let $G$ and $H$ be groups which act compatibly on one another. Papers [**5**], [**13**] and [**14**] consider a quotient $\eta(G,H)$ of the free product $G*H$ which is a group extension of the non-abelian tensor product $G\otimes H$. We prove that the group $\eta(G,H)$ is nilpotent if $G$ and $H$ are nilpotent groups which act nilpotently on each other. A couple of examples is given in detail to show that $\eta(G,H)$ fails to be nilpotent when at least one of the actions is non-nilpotent. We also determine the non-abelian tensor square of a finite group $G$ such that $G'$ is cyclic and $\gcd(|G'|,|G^{ab}|)=1$.

## 1. Introduction

The non-abelian tensor product of groups was introduced by R. Brown and J.-L. Loday [**2**] following work of C. Miller [**12**], K. Dennis [**3**] and A.S.-T. Lue [**11**]. It is defined for any pair of groups $G$ and $H$ where each one acts on the other (on right)

$$G\times H\to G,\ (g,h)\mapsto g^h;\quad H\times G\to H,\ (h,g)\mapsto h^g$$

and on itself by conjugation, in such a way that for all $g,g_1\in G$ and $h,h_1\in H$,

$$g^{(h^{g_1})}=\left(\left(g^{g_1^{-1}}\right)^h\right)^{g_1}\quad\text{and}\quad h^{(g^{h_1})}=\left(\left(h^{h_1^{-1}}\right)^g\right)^{h_1}. \tag{1}$$

In this situation we say that $G$ and $H$ act *compatibly* on each other. The *non-abelian tensor product* $G\otimes H$ is the group generated by all symbols $g\otimes h$, $g\in G,\ h\in H$, subject to the relations

$$gg_1\otimes h=(g^{g_1}\otimes h^{g_1})(g_1\otimes h)\quad\text{and}\quad g\otimes hh_1=(g\otimes h_1)(g^{h_1}\otimes h^{h_1})$$

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for all $g, g_1 \in G,\ h, h_1 \in H$.

In particular, as the conjugation action of a group $G$ on itself satisfies (1), the *tensor square* $G \otimes G$ of a group $G$ may always be defined.

Papers [**5**], [**13**] and [**14**] consider a group construction which is related to the non-abelian tensor product. In [**14**] it is defined as follows: let $G, H$ be groups acting compatibly on each other and $H^\varphi$ an extra copy of $H$, isomorphic through $\varphi : H \to H^\varphi,\ h \mapsto h^\varphi$, for all $h \in H$. Then we define the group

$$\eta(G,H) := \langle G, H^\varphi \mid [g, h^\varphi]^{g_1} = [g^{g_1}, (h^{g_1})^\varphi]\,,\ [g, h^\varphi]^{h_1^\varphi} = \left[g^{h_1}, (h^{h_1})^\varphi\right],$$
$$\text{for all } g, g_1 \in G,\ h, h_1 \in H\rangle.$$

If $G = H$ and all actions are conjugations then we write $\nu(G)$ for $\eta(G,H)$ (cf. [**15**]).

It follows from Proposition 1.4 in [**6**] that there is an isomorphism from the subgroup $[G, H^\varphi]$ of $\eta(G,H)$ onto the non-abelian tensor product $G \otimes H$, such that $[g, h^\varphi] \mapsto g \otimes h$, for $g \in G$ and $h \in H$. This isomorphism is useful to study the tensor product inside of $\eta(G,H)$. In [**15**], this approach has been used by the second author to settle a bound for the order of $G \otimes G$ when $G$ is a finite $p$-group. Also, working inside $\eta(G,H)$ the first author obtains in [**13**] (see also [**14**]) a bound for the orders of tensor squares of finite solvable groups. G. Ellis and F. Leonard [**5**] use a group isomorphic to $\eta(G,H)$ for giving a computer algorithm for determining tensor products (and Schur multipliers) of finite groups.

It has been proved in [**15**] that $\nu(G)$ preserves properties of $G$ such as finiteness, set of prime divisors, nilpotency and solvability. We observe that $[G, H^\varphi]$ is a normal subgroup of $\eta(G,H)$ and that $\eta(G,H) = [G, H^\varphi]GH^\varphi$. According to [**4**] the non-abelian tensor product of finite groups is finite. Also by [**18**] $G \otimes H$ is solvable if $G$ and $H$ are solvable. Consequently if $G$ and $H$ are finite (resp. solvable), then $\eta(G,H)$ is finite (resp. solvable). However, nilpotency of $G$ and $H$ does not imply nilpotency of $\eta(G,H)$. This can be seen for example by considering the cyclic groups $C_2$ and $C_p$ of orders 2 and p, respectively, where $p$ denotes an odd prime; if $C_p$ acts on $C_2$ trivially and this last group acts faithfuly on $C_p$ as an automorphism group of order 2, then the resulting group $\eta(C_p, C_2)$ involves the (non-nilpotent) dihedral group of order $2p$ (see Example 1 for details).

This paper is mainly concerned with nilpotency conditions on $\eta(G, H)$. In particular we prove

**Theorem A.** If $G$ and $H$ are finitely generated nilpotent groups such that the mutual actions of one another are Engelian, then $\eta(G, H)$ is nilpotent.

With the stronger assumption that the given actions are nilpotent we can improve the above result to obtain the following

**Theorem B.** If $G$ and $H$ are nilpotent groups which act nilpotently on each other, then $\eta(G, H)$ is nilpotent.

In [**1**] and [**10**] the tensor square of a finite split metacyclic group is determined. Here we make use of the present context to prove

**Theorem C.** If $G$ is a finite group such that the derived subgroup $G'$ is cyclic and $\gcd(|G'|, |G/G'|) = 1$, then $G \otimes G \cong G' \times (G/G' \otimes_{\mathbb{Z}} G/G')$.

This result shows that for a finite group $G$ with $G'$ cyclic and $\gcd(|G'|, |G/G'|) = 1$, the upper bound for $|G \otimes G|$ established in [**13**, Theorem 3.3] is sharp.

## 2. Nilpotent Actions

We use the following notation. For elements $x, y, z$ in a group $G$, the conjugate of $x$ by $y$ is $x^y = y^{-1}xy$; the commutator of $x$ and $y$ is $[x, y] = x^{-1}x^y$ and we use left normed notation for commutators, in particular $[x, y, z] = [[x, y], z]$. We write $\gamma_i(G)$ for the $i$-th term of the lower central series of $G$.

Throughout the paper we assume that the groups $G$ and $H$ act compatibly on one another. The following result is a consequence of [**2**, Proposition 2.3].

**Proposition 2.1** *The following relations hold for all $g, x \in G$ and $h, y \in H$:*

(a) $[g,h^\varphi]^{[x,y^\varphi]} = [g,h^\varphi]^{x^{-1}x^y} = [g,h^\varphi]^{(y^{-x}y)^\varphi}$;

(b) $[g,h^\varphi]^{[x,y^\varphi]^{-1}} = [g,h^\varphi]^{x^{-y}x} = [g,h^\varphi]^{(y^{-1}y^x)^\varphi}$;

(c) $[[g,h^\varphi],[x,y^\varphi]] = [g^{-1}g^h,(y^{-x}y)^\varphi]$;

(d) $[[g,h^\varphi],[x,y^\varphi]^{-1}] = [g^{-1}g^h,(y^{-1}y^x)^\varphi]$.

As observed in the Introduction, conditions like finiteness and solvability on the groups $G$ and $H$ are transferred to the group $\eta(G,H)$. However, nilpotency of $G$ and $H$ does not imply nilpotency of $\eta(G,H)$. This can be seen by the examples below.

**Example 1.** Let $p$ be an odd prime, $C_p = \langle a | a^p = 1\rangle$ be the cyclic group of order $p$ and $C_2 = \langle b | b^2 = 1\rangle$ be the cyclic group of order 2. Assume that $C_p$ acts on $C_2$ trivially and $C_2$ acts on $C_p$ by inversion, $a^b = a^{-1}$. It is an easy exercise to verify the compatibility of these actions. By these actions and the defining relations the group $\eta(C_p,C_2)$ may then be given by the presentation

$$\eta(C_p,C_2) = \langle a, b^\varphi \mid a^p = 1,\ (b^\varphi)^2 = 1,\ [a,b^\varphi]^a = [a,b^\varphi], [a,b^\varphi]^{b^\varphi} = [a^{-1},b^\varphi]\rangle.$$

Since $1 = [a^{-1}a,b^\varphi] = [a^{-1},b^\varphi]^a[a,b^\varphi] = [a^{-1},b^\varphi][a,b^\varphi]$, we have $[a^{-1},b^\varphi] = [a,b^\varphi]^{-1}$ and thus $[a,b^\varphi]^{b^\varphi} = [a,b^\varphi]^{-1}$. On the other hand $[a^2,b^\varphi] = [a,b^\varphi]^a[a,b^\varphi] = [a,b^\varphi]^2$ and a simple induction shows that $[a^n,b^\varphi] = [a,b^\varphi]^n$ for all integers $n$. Consequently, $[a,b^\varphi]$ has order dividing $p$. To see that this is a non-trivial element of $\eta(C_p,C_2)$ we use the above information to concretely construct a group of order $2p^2$ isomorphic to $\eta(C_p,C_2)$. In effect, let $V = \langle x, y \,|\, x^p,\ y^p,\ [x,y]\rangle$ ($\cong C_p \times C_p$) and let $\alpha : V \longrightarrow V$ be the involutory automorphism defined by $x \longmapsto xy, \quad y \longmapsto y^{-1}$. There is an isomorphism of $\eta(C_p,C_2)$ onto the semidirect product $V \rtimes \langle\alpha\rangle$ such that $a \mapsto x,\ b^\varphi \mapsto \alpha$ and $[a,b^\varphi] \mapsto y$. From this we deduce that $[C_p, C_2^\varphi] \cong C_p$ and $\eta(C_p,C_2)$ is non-nilpotent, since it actually involves the dihedral group of order $2p$ generated by the elements $[a,b^\varphi]$ and $b^\varphi$.

**Example 2.** Let us now consider the Klein four group $V_4 = \langle a, b \mid a^2, b^2, ab = ba\rangle$ acting trivially on the cyclic group of order 3, $C_3 = \langle c \mid c^3\rangle$, while we make $C_3$ act on $V_4$ by cyclically permuting the elements of order 2: $a^c = b,\ b^c = ab$. Again here the compatibility of the actions can be easily checked. From the defining relations of $\eta(V_4, C_3)$ we then get

$$[a^ib^j, c^\varphi]^a = [a^ib^j, c^\varphi] = [a^ib^j, c^\varphi]^b \text{ and } [a^ib^j, c^\varphi]^{c^\varphi} = [a^jb^{i+j}, c^\varphi]$$

for all $0 \le i, j \le 1$, where the sum $i + j$ is taken modulo 2. Consequently,

$$1 = [a^2, c^\varphi] = [a, c^\varphi]^a[a, c^\varphi] = [a, c^\varphi]^2,\ 1 = [b^2, c^\varphi] = [b, c^\varphi]^b[b, c^\varphi] = [b, c^\varphi]^2$$

and from $ab = ba$ we obtain $[ab, c^\varphi] = [a, c^\varphi][b, c^\varphi] = [b, c^\varphi][a, c^\varphi] = [ba, c^\varphi]$. This means that $[a, c^\varphi]$ and $[b, c^\varphi]$ are commuting elements of at most orders 2 in $\eta(V_4, C_3)$. On the other hand, $[a, (c^\varphi)^2] = [a, c^\varphi][a, c^\varphi]^{c^\varphi} = [a, c^\varphi][b, c^\varphi]$ gives

$$[a, (c^\varphi)^3] = [a, c^\varphi][(a, c^\varphi)^2]^{c^\varphi} = [a, c^\varphi]([a, c^\varphi][b, c^\varphi])^{c^\varphi} = [a, c^\varphi]^2[b, c^\varphi]^2,$$

which agrees with the relation $[a, (c^\varphi)^3] = 1$ arisen from $(c^\varphi)^3 = 1$. Hence $\eta(V_4, C_3)$ may be presented as

$$\begin{aligned}\eta(V_4, C_3) \;=\; &\langle a, b, c^\varphi \mid a^2, b^2, (c^\varphi)^3, [a, b] = 1, [a, c^\varphi]^a = [a, c^\varphi], [a, c^\varphi]^b = [a, c^\varphi],\\ &[a, c^\varphi]^{c^\varphi} = [b, c^\varphi], [b, c^\varphi]^a = [b, c^\varphi], [b, c^\varphi]^b = [b, c^\varphi], [b, c^\varphi]^{c^\varphi} = [ab, c^\varphi]\rangle.\end{aligned}$$

In order to describe the structure of $\eta(V_4, C_3)$ in the light of the above calculations we proceed as in the previous example. Here we start with an elementary abelian 2-group $W$ of order 16, say $W = \langle x, y, z, w\rangle$, and let $\beta$ be the automorphism of $W$ defined on these generators by $x \mapsto xz$, $y \mapsto yw$, $z \mapsto w$ and $w \mapsto zw$. Thus, $\beta$ has order 3 and the semi-direct product $W \rtimes \langle\beta\rangle$ has order 48. There is a mapping $\eta(V_4, C_3) \to W \rtimes \langle\beta\rangle$, such that $a \mapsto x$, $b \mapsto y$, $c^\varphi \mapsto \beta$, $[a, c^\varphi] = a^{-1}a^{c^\varphi} \mapsto z$ and $[b, c^\varphi] = b^{-1}b^{c^\varphi} \mapsto w$. This gives an isomorphism of $\eta(V_4, C_3)$ onto $W \rtimes \langle\beta\rangle$, under which the subgroup $[V_4, C_3^\varphi]C_3^\varphi$ is mapped onto the subgroup $\langle z, w, \beta\rangle$. This last group is readily seen to be isomorphic to the alternating group $A_4$. Consequently, our group $\eta(V_4, C_3)$ is non-nilpotent.

The reason why nilpotency of $G$ and $H$ does not imply nilpotency of $\eta(G, H)$ is that the terms in the lower central series depend upon the actions of $G$ on $H$ and of $H$ on $G$. Thus, let us impose some adicional condition on these actions.

For $g \in G, h \in H$ put $[g, {}_0h] := g^{-1}; [g, {}_1h] = g^{-1}g^h$ and, for $i \geq 1$, $[g, {}_{(i+1)}h] := [[g, {}_ih], h]$.

**Definition 2.2** *We say that the action of $H$ on $G$ is (right) $n$-Engelian if $[g, {}_nh] = 1$, for all $g \in G, h \in H$.*

In particular, if $G = H$ and the action is conjugation in $G$ then (right) $n$-Engelian action coincides with the (right) $n$-Engel condition on $G$.

Let $[G, {}_0H] := G$, $[G, {}_1H] := \langle g^{-1}g^h | g \in G,\ h \in H\rangle$ and, for $i \geq 1$, $[G, {}_{i+1}H] := [[G, {}_iH], H]$.

**Definition 2.3** *We say that the action of $H$ on $G$ is (right) nilpotent of class $c$ if $[G, {}_cH] = \{1\}$ and $[G, {}_{c-1}H] \neq \{1\}$.*

In particular, if $G = H$ and the action is conjugation in $G$ then (right) nilpotent action of class $c$ coincides with class-$c$ nilpotency of $G$.

Let us denote the subgroup $[G, H^\varphi]$ of $\eta(G, H)$ by $\tau(G, H)$.

**Proposition 2.4** *If the action of $H$ on $G$ is $n$-Engelian (resp. nilpotent of class $c$) then $[G, H]$ satisfies the $n$-Engel condition (resp. is nilpotent of at most class $c$). Here $[G, H]$ denotes $[G, {}_1H]$.*

**Proof.** It follows from Proposition 2.3 in [**2**] that there are group homomorphisms

$$\lambda : \tau(G, H) \to G, \qquad \mu : \tau(G, H) \to H$$

such that $([g, h^\varphi])\lambda = g^{-1}g^h$ and $([g, h^\varphi])\mu = h^{-g}h$. Further, by Proposition 2.5 in [**13**], for any elements $g \in G,\ h \in H$ and $t \in \tau(G, H)$ we have

$$g^{(t)\lambda} = g^{(t)\mu} \quad \text{and} \quad h^{(t)\lambda} = h^{(t)\mu} \tag{2}$$

Thus, if $g, x \in [G, H]$ then there exists a $t \in \tau(G, H)$ such that $x = (t)\lambda$. Hence by (2)

$$[g, x] = [g, (t)\lambda] = g^{-1}g^{(t)\lambda} = g^{-1}g^{(t)\mu} = [g, (t)\mu],$$

and by induction $[g, {}_i x] = [g, {}_i((t)\mu)]$. This implies the result.

□

M. Visscher [**18**] showed that if $[G, H]$ satisfies the n-Engel condition (resp. is nilpotent of class $c$), then $G \otimes H$ satisfies the (n+1)-Engel condition (resp. is nilpotent of class at most $c+1$). This together with Proposition 2.4 leads to the following

**Corollary 2.5** *If the action of $H$ on $G$ is n-Engelian (resp. nilpotent of class $c$), then $G \otimes H$ satisfies the (n+1)-Engel condition (resp. is nilpotent of class at most $c+1$).*

**Remark.** The converse of Proposition 2.4 is false. In Example 1, for instance, the subgroup $[C_p, C_2]$ of $C_p$ is obviously nilpotent but the given action of $C_2$ on $C_p$ is not Engelian.

**Lemma 2.6** *Let $g \in \gamma_i(G)$, $h \in \gamma_j(H), x \in G, y \in H$ and write*

$$C_{ij} = [\gamma_{i+1}(G), \gamma_j(H^{\varphi})][\gamma_i(G), \gamma_{j+1}(H^{\varphi})].$$

*Then*

*(i)* $[[g, h^{\varphi}], [x, y^{\varphi}]] \in C_{ij}$ ;

*(ii)* $[g, h^{\varphi}]^x \equiv [g, h^{\varphi}]\,[g, [h, x]^{\varphi}] \pmod{C_{ij}}$;

*(iii)* $[g, h^{\varphi}]^{y^{\varphi}} \equiv [g, h^{\varphi}]\,[[g, y], h^{\varphi}] \pmod{C_{ij}}$;

*(iv)* $[g, h^{\varphi}]^{xy^{\varphi}} \equiv [g, h^{\varphi}]\,[g, [h, x]^{\varphi}]\,[[g, y], h^{\varphi}]\,[[g, y^{\varphi}], [x, h^{\varphi}]^{-1}] \pmod{C_{ij}}$.

**Proof.** By Proposition 2.1 , identities (2) and commutator calculus we obtain

$$\begin{aligned}
[g,h^{\varphi}]^{[x,y^{\varphi}]} &= \left[g^{x^{-1}x^{y}},(h^{y^{-x}y})^{\varphi}\right] \\
&= \left[[x^{-1}x^{y},g^{-1}]\,g,[y^{-x}y,h^{-1}]^{\varphi}\,h^{\varphi}\right] \\
&= [[x^{-1}x^{y},g^{-1}],h^{\varphi}]^{g}\,[g,h^{\varphi}]\left[[x^{-1}x^{y},g^{-1}],[y^{-x}y,h^{-1}]^{\varphi}\right]^{gh^{\varphi}}\cdot \\
&\cdot\left[g,[y^{-x}y,h^{-1}]^{\varphi}\right]^{h^{\varphi}}
\end{aligned}$$

From this it follows that

$$\begin{aligned}
[[g,h^{\varphi}],[x,y^{\varphi}]] &= [g,h^{\varphi}]^{-1}[g,h^{\varphi}]^{[x,y^{\varphi}]} \\
&= [[x^{-1}x^{y},g^{-1}],h^{\varphi}]^{g[g,h^{\varphi}]}\left[[x^{-1}x^{y},g^{-1}],[y^{-x}y,h^{-1}]^{\varphi}\right]^{gh^{\varphi}}\cdot \\
&\cdot\left[g,[y^{-x}y,h^{-1}]^{\varphi}\right]^{h^{\varphi}}
\end{aligned}$$

Now as $C_{ij}\trianglelefteq\eta(G,H)$ (by Proposition 2.1 in [**13**]), it follows that $[[g,h^{\varphi}],[x,y^{\varphi}]]\in C_{ij}$, proving (i).

For part (ii) we observe that modulo $C_{ij}$,

$$\begin{aligned}
[g,h^{\varphi}]^{x} &\equiv [g^{x},(h^{x})^{\varphi}] && \text{(by defining relations of } \eta(G,H)) \\
&\equiv [[x,g^{-1}]\,g,(h^{x})^{\varphi}] \\
&\equiv [[x,g^{-1}],(h^{x})^{\varphi}]^{g}\,[g,(h^{x})^{\varphi}] \\
&\equiv [g,(h^{x})^{\varphi}] \\
&\equiv [g,(hh^{-1}h^{x})^{\varphi}] \\
&\equiv [g,(h^{-1}h^{x})^{\varphi}]\,[g,h^{\varphi}]^{(h^{-1}h^{x})^{\varphi}} \\
&\equiv [g,[h,x]^{\varphi}]\,[g,h^{\varphi}]^{[x,h^{\varphi}]^{-1}} && \text{(by Proposition 2.1 (b))} \\
&\equiv [g,[h,x]^{\varphi}]\,[g,h^{\varphi}]\,[g,h^{\varphi}]^{-1}\,[g,h^{\varphi}]^{[x,h^{\varphi}]^{-1}} \\
&\equiv [g,h^{\varphi}]\,[g,[h,x]^{\varphi}]\,[[g,[h,x]^{\varphi}],[g,h^{\varphi}]]\,[[g,h^{\varphi}],[x,h^{\varphi}]^{-1}] \\
&\equiv [g,h^{\varphi}]\,[g,[h,x]^{\varphi}] && \text{(by (i)).}
\end{aligned}$$

The proof of part (iii) is analogous to (ii), while part (iv) follows from (ii), (iii) and Proposition 2.1 (d).

□

**Proof of Theorem A.** In order to take into account nilpotency classes and action lengths of the groups involved, we formulate Theorem A as

**Theorem 2.7** *Let $G$ and $H$ be nilpotent groups of classes $a$ and $b$ respectively. Suppose the action of $H$ on $G$ is $l$-Engelian and the action of $G$ on $H$ is $k$-Engelian. Then $\dfrac{\eta(G,H)}{\tau(G,H)'}$ satisfies the $n$-Engel condition for $n = c + (2c-1)m$, where $c = \max\{a,b\}$ and $m = \max\{l,k\}$. In particular, if furthermore $G$ and $H$ are finitely generated groups , then $\eta(G,H)$ is nilpotent.*

**Proof.** Firstly we note that by Proposition 2.1 (ii) in [**13**], $[\gamma_i(G), \gamma_j(H)^{\varphi}] \trianglelefteq \eta(G,H)$, for all $i,j \geq 1$. Let $u,v$ be elements in $\eta(G,H)$ and put $c = \max\{a,b\}$, $m = \max\{l,k\}$. Let us prove by induction on $s \geq 0$ that

$$[u,\ _{(c+sm)}v] \equiv 1 \left(\mathrm{mod}\ \tau(G,H)' \prod_{i=1}^{s+1} [\gamma_i(G), \gamma_{s+2-i}(H)^{\varphi}]\right). \tag{3}$$

For $s = 0$ we observe that the equality $\eta(G,H) = \tau(G,H)GH^{\varphi}$ implies

$$\gamma_i\left(\eta(G,H)/\tau(G,H)\right) \cong \gamma_i(G)\gamma_i(H^{\varphi}), \quad \text{for all } i.$$

Thus $\gamma_{c+1}\left(\eta(G,H)/\tau(G,H)\right) = 1$ and therefore

$$[u,\ _{c}v] \equiv 1 \bmod (\tau(G,H)).$$

Now suppose $s \geq 0$ and that (3) is true. Then $[u,\ _{(c+sm)}v]$ has the form

$$[u,\ _{c+sm}v] \equiv \prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t}, h_{i_t}^{\varphi}\right] (\mathrm{mod}\ \tau(G,H)')\,,$$

where $g_{i_t} \in \gamma_i(G), h_{i_t} \in \gamma_{s+2-i}(H), 1 \leq t \leq r_i, 1 \leq i \leq (s+1)$. We can write the element $v$ of $\eta(G,H)$ as $v = zxy^{\varphi}$, where $z \in \tau(G,H), x \in G$ and $y \in H$. Thus, since $\tau(G,H)' \trianglelefteq \eta(G,H)$, by commutator calculus we obtain, modulo $\tau(G,H)'$,

$$\begin{aligned}
[u,\ _{(c+sm+1)}v] &\equiv \left[\prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t}, h_{i_t}^{\varphi}\right], zxy^{\varphi}\right] \\
&\equiv \left[\prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t}, h_{i_t}^{\varphi}\right], xy^{\varphi}\right]\left[\prod_{i=1}^{s}\prod_{t=1}^{r_i}\left[g_{i_t}, h_{i_t}^{\varphi}\right], z\right]^{xy^{\varphi}} \\
&\equiv \left[\prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t}, h_{i_t}^{\varphi}\right], xy^{\varphi}\right] \quad (\text{since } z \in \tau(G,H)) \\
&\equiv \prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t}, h_{i_t}^{\varphi}\right]^{-1}\left[g_{i_t}, h_{i_t}^{\varphi}\right]^{xy^{\varphi}}
\end{aligned}$$

From this and by using Lemma 2.6 (iv) we obtain

$$\begin{aligned}[u,\,_{(c+sm+1)}v] \equiv & \prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t},h_{i_t}^{\varphi}\right]^{-1}\left[g_{i_t},h_{i_t}^{\varphi}\right]\left[g_{i_t},\left[h_{i_t},x\right]^{\varphi}\right]\cdot\\ & \cdot\left[\left[g_{i_t},y\right],h_{i_t}^{\varphi}\right]\left[\left[g_{i_t},y^{\varphi}\right],\left[x,h_{i_t}^{\varphi}\right]^{-1}\right]\\ & \left(\bmod\ \tau(G,H)'\prod_{i=1}^{s+1}\left(\left[\gamma_{i+1}(G),\gamma_{s+2-i}(H)^{\varphi}\right]\left[\gamma_i(G),\gamma_{s+2-i+1}(H)^{\varphi}\right]\right)\right),\end{aligned}$$

that is,

$$\begin{aligned}[u,\,_{(c+sm+1)}v] \equiv & \prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t},\left[h_{i_t},x\right]^{\varphi}\right]\left[\left[g_{i_t},y\right],h_{i_t}^{\varphi}\right]\\ & \left(\bmod\ \tau(G,H)'\prod_{i=1}^{s+2}\left[\gamma_i(G),\gamma_{s+3-i}(H)^{\varphi}\right]\right)\end{aligned}$$

Now using, step by step, similar arguments as in the proof of the preceding identity and Proposition 2.1 (d) we show (by induction on $j \geq 1$) that

$$\begin{aligned}[u,\,_{(c+sm+j)}v] \equiv & \prod_{i=1}^{s+1}\prod_{t=1}^{r_i}\left[g_{i_t},\left[h_{i_t},\,_jx\right]^{\varphi}\right]\left[\left[g_{i_t},\,_jy\right],h_{i_t}^{\varphi}\right]\\ & \left(\bmod\ \tau(G,H)'\prod_{i=1}^{s+2}\left[\gamma_i(G),\gamma_{s+3-i}(H)^{\varphi}\right]\right)\end{aligned}$$

In particular, if $j = m$ then

$$[u,\,_{(c+(s+1)m)}v] \equiv 1 \left(\bmod\ \tau(G,H)'\prod_{i=1}^{s+2}\left[\gamma_i(G),\gamma_{(s+1)+2-i}(H)^{\varphi}\right]\right)$$

since $[h,\,_mg] = [g,\,_mh] = 1$, for all $g \in G, h \in H$. Hence, (3) occurs for all $s \geq 0$.

Finally we observe that if $s = 2c-1$ then $s+2-i \geq c+1$ whenever $i \leq c$. Consequently $[u,\,_{(c+(2c-1)m)}v] = 1$ and therefore $\eta(G,H)/\tau(G,H)'$ satisfies the $n$-Engel condition for $n = c+(2c-1)m$. In addition $\eta(G,H)$ is a solvable group. Thus, if $G$ and $H$ are also finitely generated groups, then $\eta(G,H)/\tau(G,H)'$ is a finitely generated solvable Engel group. Therefore, by Theorem 1 in [**8**], $\eta(G,H)/\tau(G,H)'$ is a nilpotent group. As $\tau(G,H)$ is also nilpotent (cf. [**18**], see also [**13**]), it follows from Corollary VI.6.g of [**17**] that $\eta(G,H)$ is nilpotent.

**Proof of Theorem B.** Let $C_{r,2}$ denote the binomial coefficient $r(r-1)/2$, for all positive integer $r$. We rephrase Theorem B more specifically as

**Theorem 2.8** *Let $G$ and $H$ be nilpotent groups of classes $a$ and $b$ respectively, such that the action of $H$ on $G$ is nilpotent of length $l$ and the action of $G$ on $H$ is nilpotent of length $k$. Put $c = \max\{a, b\}$, $m = \max\{l, k\}$, $n = c + (2c-1)m$ and $s = 1 + \min\{a, b, k, l\}$. Then $\eta(G, H)$ is a nilpotent group and its class is not more than $nC_{s+1,2} - C_{s,2}$.*

**Proof.** Using an argument similar to that employed in the previous result we prove that $\eta(G,H)/\tau(G,H)'$ is nilpotent of at most class $n$. By [**13**, Corollary 2.6] and Corollary 2.5 above, $\tau(G, H)$ is nilpotent of at most class $s$. Therefore $\eta(G, H)$ is nilpotent, and its class is not more than $nC_{s+1,2} - C_{s,2}$ [**17**, Corollary VI.6.g].

**Remark.** The bound for the nilpotency class of $\nu(G)$ $(= \eta(G, G))$ that we obtain by applying Theorem 2.8 to this special case is greater than that established by the second author in [**15**]. This happens because certain relations holding in $\nu(G)$ do not hold in these more general groups $\eta(G, H)$, with $G \neq H$, such as (cf. [**15**]):

$$[x, y^{\varphi}, z] = [x, y, z^{\varphi}] = [x, y^{\varphi}, z^{\varphi}], \quad \text{for all } x, y, z \in G.$$

**Definition 2.9** *We say that a subgroup $M$ of $G$ is a $H$-*subgroup *(resp. $H$-*trivial*) if $m^h \in M$ (resp. $m^h = m$), for all $m \in M$ and $h \in H$.*

**Proposition 2.10** *Let $G = P \times M$ and $H = Q \times N$ be direct products of their normal subgroups $P, M$ and $Q, N$, respectively. Suppose $P, M$ are $H$-subgroups of $G$ and $Q, N$ are $G$-subgroups of $H$, so that $G$ and $H$ induce actions of $P$ on $Q$, $Q$ on $P$, $M$ on $N$ and of $N$ on $M$. Then*

*(i) If $Q$ is $M$-trivial and $P$ is $N$-trivial, then we have*

$$\eta(G, H) = \langle P, Q^{\varphi}\rangle\, [P, N^{\varphi}][M, Q^{\varphi}]\, \langle M, N^{\varphi}\rangle\,; \quad \langle P, Q^{\varphi}\rangle \cong \eta(P, Q)$$

*(ii) If $G$ and $H$ are finite groups such that* $\gcd(|Q|, |M|) = \gcd(|P|, |N|) = 1$, *$Q$ and $M$ act trivially on each other and similarly do $P$ and $N$ act, then*

*(a)* $\eta(G,H) = \langle P, Q^{\varphi}\rangle \langle M, N^{\varphi}\rangle$ ; $\langle P, Q^{\varphi}\rangle \cong \eta(P,Q)$, $\langle M, N^{\varphi}\rangle \cong \eta(M,N)$;

*(b)* $\tau(G,H) \cong \tau(P,Q) \times \tau(M,N)$.

**Proof.** Part (i) is a routine extension of Proposition 3.5 in [**15**]. Part (ii) follows from (i) by observing that $[P, N^{\varphi}]$ and $[M, Q^{\varphi}]$ are epimorphic images of $P \otimes N$ and $M \otimes Q$, respectively, and that $P \otimes N \cong P^{ab} \otimes_{\mathbb{Z}} N^{ab}$, $M \otimes Q \cong M^{ab} \otimes_{\mathbb{Z}} Q^{ab}$ (cf. [**2**], Proposition 2.4).

$\square$

Let $C = C_H(G) = \{h \in H \mid g^h = g, \text{ for all } g \in G\}$. We have that $H/C$ is a group of automorphisms of $G$. Thus, if $G$ is a finite $p$-group and $H$ a $p'$ -group then (cf. [**7**] , Theorem 3.6) $[G, {}_2H/C] = [G, H/C]$. In particular, if $[G, {}_nH/C] = \{1\}$ for some $n \geq 1$, then $H/C = \{1\}$. Consequently, we get the following

**Lemma 2.11** *If $G$ is a finite $p$-group and $H$ is a finite $p'$-group such that the action of $H$ on $G$ is nilpotent then $H$ acts trivially on $G$.*

Now suppose that $G$ and $H$ are finite nilpotent groups and that the actions of $H$ on $G$ and of $G$ on $H$ are nilpotent. We can write $G$ and $H$ as

$$G = P_1 \times \ldots \times P_r \times P_{r+1} \times \ldots \times P_m, \quad H = Q_1 \times \ldots \times Q_r \times Q_{r+1} \times \ldots \times Q_n, \tag{4}$$

where $P_i$ is the Sylow $p_i$-subgroup of $G$, $i = 1, 2, \ldots, m$, $Q_j$ is the Sylow $q_j$-subgroup of $H$ , $j = 1, 2, \ldots, n$, $p_k = q_k$ for $k = 1, 2, \ldots, r$ and $p_k \neq q_l$, $r < k \leq m$, $r < l \leq n$. It's clear that each $P_i$ is a normal $H$-subgroup of $G$ and that every $Q_j$ is a normal $G$-subgroup of $H$. Hence, the action of $G$ on $H$ and that of $H$ on $G$ induce actions of $P_i$ on $Q_j$ and of $Q_j$ on $P_i$ , for all $i, j$. Further, by Lemma 2.11, $P_i$ and $Q_j$ act trivially on each other whenever $p_i \neq q_j$. Thus, using Proposition 2.10 (ii) and induction we obtain the

**Proposition 2.12** *Let $G$ and $H$ be finite nilpotent groups ( like in* (4)*) such that the actions of $G$ on $H$ and of $H$ on $G$ are nilpotent. Then*

$$G \otimes H \cong (P_1 \otimes Q_1) \times \ldots \times (P_r \otimes Q_r).$$

*In particular, if* $\gcd(|G|, |H|) = 1$ *then* $G \otimes H = \{1\}$.

**Proof of Theorem C.** Using Lemma 3.2 in [**13**] we obtain an exact sequence

$$1 \longrightarrow [G', G^{\varphi}] \longrightarrow \tau(G,G) \longrightarrow \tau\left(G^{ab}, G^{ab}\right) \longrightarrow 1 \tag{5}$$

where $[G', G^{\varphi}] \leq \tau(G,G)$. We observe that as $G'$ is cyclic, it is follows from [**13**, Corollary 2.8] that $\tau(G,G)$ is abelian. Since $\tau\left(G^{ab}, G^{ab}\right) \cong G^{ab} \otimes G^{ab}$ and $G^{ab}$ is abelian we have by [**2**, Proposition 2.4] that $\tau\left(G^{ab}, G^{ab}\right) \cong G^{ab} \otimes_{\mathbb{Z}} G^{ab}$. Let us prove that $[G', G^{\varphi}] \cong G'$. This together with (5) and the fact that $\gcd(|G'|, |G^{ab}|) = 1$ will imply the result.

Conjugation in $G$ gives rise to actions of $G$ on $G'$ and of $G'$ on $G$. Let $G^{\psi}$ be an extra copy of $G$, isomorphic through $\psi : G \longrightarrow G^{\psi}$, and consider the group $\tau(G', G)$ $(\leq \eta(G', G))$. It is clear that the correspondence $[x, g^{\psi}] \mapsto [x, g^{\varphi}]$, $x \in G'$, $g \in G$ induces an epimorphism $\alpha$ from $\tau(G', G)$ onto $[G', G^{\varphi}]$ $(\leq \tau(G,G))$. The subgroup $S = \langle [x, x^{\psi}] | x \in G' \rangle$ is normal in $\tau(G', G)$ and it is contained in the $\ker(\alpha)$ since, in $\tau(G,G)$, $[x, x^{\varphi}] = 1$, for all $x \in G'$ (by Proposition 2.1 (c)). Thus, $\alpha$ induces an epimorphism

$$\beta : \tau(G', G)/S \longrightarrow [G', G^{\varphi}]. \tag{6}$$

Now consider the group $\tau\left(G', G^{ab}\right)$, where $G^{ab}$ is $G'$-trivial and the action of $G^{ab}$ on $G'$ is induced by that of $G$ via the natural epimorphism:

$$c^{G'g} = c^{g} \,, \qquad c \in G',\ g \in G.$$

By [**13**, Lemma 3.1] there is an exact sequence

$$1 \longrightarrow [G', (G')^{\psi}] \xrightarrow{inc} \tau(G', G) \xrightarrow{\gamma'} \tau\left(G', G^{ab}\right) \longrightarrow 1 \tag{7}$$

where $[G', (G')^{\psi}] \leq \tau(G', G)$ and $\gamma'$ is the homomorphism given by $([x, g^{\psi}])\gamma' = [x, (G'g)^{\phi}]$, for all $x \in G',\ g \in G$. We remark that $S \subset \ker\gamma'$ and $S \subset [G', (G')^{\psi}]$. Hence, the following sequence is exact

$$[G', (G')^{\psi}]/S \longrightarrow \tau(G', G)/S \longrightarrow \tau\left(G', G^{ab}\right) \longrightarrow 1. \tag{8}$$

The map $\theta : G' \otimes G' \longrightarrow \tau(G', G)/S$ defined on the generators by $(x \otimes y)\theta = S[x, y^{\psi}]$ extends to a homomorphism and $x \otimes x \in \ker\theta$, for all $x \in G'$. Thus, $\theta$ induces a homomorphism

$$\theta_1 : G' \wedge G' \longrightarrow \tau(G', G)/S$$

such that $\mathrm{Im}\theta_1 = [G', (G')^{\psi}]/S$. Here $G' \wedge G'$ is the exterior square of $G'$. Now, as $G'$ is cyclic, $G' \wedge G' = \{1\}$. Hence $[G', (G')^{\psi}]/S = \{1\}$ and, by (8) and (6), there is an epimorphism

$$\beta_1 : \tau(G^{'}, G^{ab}) \longrightarrow [G^{'}, G^{\varphi}]. \tag{9}$$

Using Proposition 5.1 in [**9**] we obtain a homomorphism

$$\delta : G^{ab} \otimes G' \longrightarrow G', \tag{10}$$

such that $\ker(\delta) \cong H_1(G^{ab}, G')$, where $H_1(G^{ab}, G')$ denotes the first homology group of $G^{ab}$ with coefficients in $G'$. We observe that $\tau(G', G^{ab}) \cong G^{ab} \otimes G'$ (cf. [**1**], Proposition 1). It is not hard to check that the exponent of $G^{ab} \otimes G'$ divides $|G'|$. By [**16**, Theorem 10.26], $\exp(H_1(G^{ab}, G'))$ divides $|G^{ab}|$. As $\gcd(|G'|, |G^{ab}|) = 1$, it follows that $H_1(G^{ab}, G') = 0$. Thus $\delta$ is a monomorphism and, by (9) ,

$$|[G', G^{\varphi}]| \leq \left|G^{'}\right|. \tag{11}$$

But $\left|G^{'}\right|$ divides $|\tau(G, G)|$, since the correspondence $[g, h^{\varphi}] \mapsto [g, h], g, h \in G$, induces an epimorphism from $\tau(G, G)$ onto $G^{'}$. As $\gcd\left(\left|G^{'}\right|, \tau(G^{ab}, G^{ab})\right) = 1$ it follows from (5) and (11) that $\left|G^{'}\right| = \left|[G^{'}, G^{\varphi}]\right|$. This together with (9) and (10) yields $[G^{'}, G^{\varphi}] \cong G^{'}$, as required.

$\square$

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