A NOTE ON THE NORMALIZER PROBLEM *

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Abstract

In this paper the normalizer property of an integral group ring of a Frobenius group is investigated. Any element of the normalizer $\mathcal{N}_{\mathcal{U}_1}(G)$ of G is determined by a finite normal subgroup (see [8]). Using this we prove that the normalizer problem has a positive solution for Frobenius groups in general. Several criteria for the normalizer problem to have a positive solution are also given and we finish giving a short proof of a result due to Farkas and Linnell.

Resumo

Investigamos o problema do normalizador para um grupo de Frobenius G. Todo elemento de $\mathcal{N}_{\mathcal{U}_1}(G)$ é determinado por um subgrupo normal finito de G (veja [7]). Usando este fato, resolvemos por completo o problema do normalizador para um grupo de Frobenius em geral. Damos também vários critérios para que o referido problema tenha uma solução positiva e terminamos dando uma prova curtinha de um resultado devido a Farkas e Linell.

1. Introduction

Let G be a group and $\mathbb{Z}G$ its integral group ring. Denote by $\mathcal{U}_1 = \mathcal{U}_1(\mathbb{Z}G)$ the group of normalized units of $\mathbb{Z}G$. Until recently a long standing conjecture was the following (Problem 43 in [19]):

$$\mathcal{N}_{\mathcal{U}_1}(G) = G\mathcal{Z}(\mathcal{U}_1), \qquad \text{(NC)}$$

i.e. the normalizer of G in $\mathcal{U}_1$ is $\langle G, \mathcal{Z}(\mathcal{U}_1)\rangle$, where $\mathcal{Z}(\mathcal{U}_1)$ is the centre of $\mathcal{U}_1$.

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Coleman [1] showed that (NC) holds for all finite nilpotent groups. See also [4], [6], [12], [13], [14] and [15] for other families of finite groups for which a positive solution is known.

First Mazur [16] and then Jespers-Juriaans [7] and Hertweck [4, 5] proved that the isomorphism problem is strongly related to the normalizer problem.

In [9] an important representation theorem is proved for central units. The key observation to give an easy proof and also to generalize Mazur's result was to establish a link between this representation theorem and Mazur's result. This was done by Jespers and Juriaans in [7].

Any attempt to classify the groups for which the normalizer problem has a positive solution would have to deal with infinite groups. So it is natural to wonder if there also exists a representation theorem for normalizing units. One such attempt is made in [17] by Mazur. However his theorem does not seem strong enough to establish a positive solution for important families of groups. In [8] (working independently from Mazur) a strong representation theorem was proved (see the next section): *if $u \in \mathbb{Z}G$ is an normalizing element then $u = gw$ with $g \in G$ and $\langle \mathrm{supp}(w) \rangle$ is a finite normal subgroup of G.* This result generalizes the one in [9] on central units and is indeed strong enough to give a positive solution for the normalizer problem for many important families of groups (note that in case G is finite these results are classical). For example it is proved that the normalizing problem has a positive solution for locally nilpotent groups, groups with no two torsion, F.C.-groups whose commutator subgroup is a p-group (see also [17]) and torsion groups whose 2-elements form a subgroup. This representation theorem was also used to prove that the group $\mathrm{Out}_{\mathbb{Z}}(G)$ is always torsion; in fact it is an elementary abelian two group whose rank is not bigger than that of the torsion free rank of the centre of $\mathcal{U}_1(\mathbb{Z}G)$. In particular if it is finitely generated then it is finite. For example this is the case when the torsion subgroup of the fc-centre of G is finite. To prove these results it is proved that the map $f : \mathcal{N}_{\mathcal{U}_1}(G)/G \longrightarrow \mathcal{Z}(\mathcal{U})/\mathcal{Z}(G) : uG \longrightarrow uu^*\mathcal{Z}(G)$ is an embedding. This actually means that the first group is generated by symmetric units whose support contains one. Many other applications could be given but the focus, as it should be, was on the representation theorem.

This paper is a natural continuation of [8]. First we prove that the normalizer problem holds for Frobenius groups in general. A particular case of our result (the case when G is a locally finite Frobenius group) was kept out of [8] (this paper was submitted in November 1999 and on request of the editor we concentrated on the representation theorem and only on some applications). Second we prove some sufficient conditions for the normalizer problem to have a positive solution which yield machinery to reach the final goal: *is the (NC) problem, in general, equivalent to the (NC) problem for finite groups?* Until now all our applications show that this should be the case. Third, we finish this paper by giving an easy and elegant proof of a result due to Farkas and Linnell which shows that not always a strong representation theorem is needed to obtain extensions of known results.

Finally it is interesting to notice that verification of (NC) is nearly always group theoretical in nature except for a group constructed by Marciniack and Roggenkamp for which Y. Li proved that (NC) holds (see [13, 15]). However, for infinite groups, ring theory is needed because our representation theorem is a ring theoretical result.

2. The Results

First we recall some results from [8]. The finite conjugacy centre of a group G is denoted by $\Delta(G)$ and its torsion subgroup by $\Delta^+(G)$.

We recall two essential and easy to prove properties of the normalizer (for proofs we refer to [19, Proposition 9.4 and Proposition 9.5]). Recall that by $*$ we denote the $\mathbb{Z}$-linear involution on the group ring $\mathbb{Z}G$ defined by $g^* = g^{-1}$, where $g \in G$.

Proposition 2.1 *Let G be an arbitrary group and $u \in U_1$.*

1. $u \in \mathcal{N}_{\mathcal{U}_1}(G)$ *if and only if* $u^*u \in \mathcal{Z}(\mathbb{Z}G)$.

2. *(Krempa) If $u \in \mathcal{N}_{U_1}(G)$ then $u^2 \in G\mathcal{Z}(\mathbb{Z}G)$, i.e. the automorphism on G determined by conjugation by u^2 is inner in G.*

Another easy to prove lemma is the following (see also Lemma 3.4 in [15]). Note that this result is proved in [8] but we include a proof.

Lemma 2.2 *Let G be an arbitrary group and $u \in \mathcal{N}_{\mathcal{U}_1}(G)$. If $u^n \in G$ for some positive integer n, then $u \in G$.*

Proof. Since $u \in \mathcal{N}_{\mathcal{U}_1}(G)$ we know that $u^*u = uu^*$. Because $u^n \in G$ we thus get $(u^*u)^n = (u^n)^*u^n = g^{-1}g = 1$. Hence u^*u is a periodic central unit and thus $u^*u \in \mathcal{Z}(G)$. Write $u = \sum_{g\in G} u_g g$ with each $u_g \in G$. So the coefficient of 1 of u^*u is $\sum_{g\in G} u_g^2 \neq 0$. It follows that $u^*u = 1$ and thus $u \in G$.

□

For an element $u = \sum_{g\in G} u_g g$ (each $u_g \in R$) in a group ring RG over a ring R we denote by $\text{supp}(u)$ the support of u, that is, the set $\{g \in G \mid u_g \neq 0\}$. In [9] it is shown that central units in $\mathbb{Z}G$, for G a finitely generated nilpotent group, have a presentation of the form gv with $g \in G$ and $v \in \mathcal{U}(\mathbb{Z}T(G))$, with $T(G)$ the torsion subgroup of G. As mentioned in [7] this result remains valid for groups G in which the torsion elements form a subgroup $T(G)$ and $G/T(G)$ is an ordered group and hence it holds for all groups in general.

Lemma 2.3 ([8]) *Let G be a group and let R be a ring with a unit. If $u \in \mathcal{N}_{\mathcal{U}(RG)}(G)$ then*

$$\sigma : G \to \text{Sym}(\text{supp}(u)) : g \mapsto \sigma_g$$

is a group homomorphism, where

$$\sigma_g(h) = ghu^{-1}g^{-1}u.$$

If, moreover, $1 \in \text{supp}(u)$, then

$$\text{Ker } \sigma = \mathcal{C}_G(\text{supp}(u)),$$

the centralizer of $\text{supp}(u)$ in G. Hence $G/\mathcal{C}_G(\text{supp}(u))$ is embedded in $\text{Sym}(\text{supp}(u))$ and so $\text{supp}(u) \subset \Delta(G)$.

We next recall the main theorem of [8]. For completeness' sake we include a proof.

Theorem 2.4 *Let G be a group and $u \in \mathcal{N}_{\mathcal{U}_1}(G)$. Then there exists a finite normal subgroup N of G so that*

$$u = gw$$

for some $g \in G$ and $w \in \mathbb{Z}N$. Moreover, w induces an automorphism ϕ of order a divisor of $2|N|$. If N has has odd order then ϕ is inner on G.

Proof. Let $X = \mathrm{supp}(u)$. To prove the result we may assume that $1 \in X$. Hence, because of the representation lemma of [8] , $X \subseteq \Delta(G)$. Hence X has only finitely many conjugates in G and thus $X \subseteq H$, with H a finitely generated normal subgroup of G contained in $\Delta(G)$. In particular, H is a finitely generated F.C.-group. Hence $N = T(H)$ is a finite invariant subgroup of H and H/N is free abelian and thus ordered. Since $u \in \mathcal{U}_1(\mathbb{Z}H)$ the lemma mentioned above implies that $u = hw$ for some $h \in H$ and $w \in \mathbb{Z}N$. As N is normal in G this shows the desired presentation for u. For the second part see [8].

□

The following result also appeared in [8] but, for completeness' sake, we also include a proof.

Theorem 2.5 *Let G be a finite group and suppose that F, the Fitting subgroup of G, has odd order. Let π be the set of prime divisors of $|F|$. If G is π-separable then any automorphism ϕ of G which induces the identity on F has odd order. In particular if ϕ is induced by a unit in $\mathcal{N}(\mathcal{U}_1(G))$ then ϕ is inner.*

Proof. Let $g \in G$ and $x \in F$. Then $g^{-1}xg = \phi(g^{-1}xg) = \phi(g)^{-1}x\phi(g)$. Hence $\phi(g)g^{-1}$ centralizes F. But since F is π-separable we have that it contains its centralizer in G. Hence for all $g \in G$, $\phi(g) = \psi(g)g$ with $\psi(g) \in \mathcal{Z}(F)$. Let

$n = exp(F)$ then $\phi^n(g) = \psi(g)^n g = g$. So the order of ϕ divides n and hence is odd. Krempa's result gives the final part.

□

Note that all we needed is a self centralizing normal subgroup of odd order. This result should be compared with one of Gross [3]. Also note that in case ϕ is induced by a normalizing unit then $\phi = \phi_g$ for some $g \in F$, where $\phi_g(x) = g^{-1}xg$ for all $x \in G$.

As a consequence we shall show that (NC) holds for Frobenius groups G. In first instance we show this result first for locally finite Frobenius groups. The main reason being that its proof is more illustrating. In case G is also finite, this result was proved by Polcino-Thierry in [18]. Even in this case our proof is much shorter.

Before proceeding with the proof we first make some observations on the F.C.-centre of a Frobenius group. Let G be a Frobenius group and H a Frobenius complement of G, i.e., H is disjoint from all its conjugates in G. If H is such a Frobenius complement of G we denote by $K = (G \setminus \bigcup_{g\in G} H^g)\bigcup\{1\}$. This is a normal subset of G which need not be a subgroup. If H has infinite order then K must also be infinite and hence the F.C.-centre of G is trivial. So if the F.C.-centre of G is non-trivial then H must be finite.

If G is a locally finite Frobenius group then a Frobenius kernel does exist and all complements are conjugate in G. Moreover K is this kernel and it is nilpotent (in the finite case this was proved by Thompson). We refer the reader to [11] for these results.

Theorem 2.6 *Let G be a locally finite Frobenius group. Then the normalizer problem has a positive solution for G.*

Proof. Because of a theorem of [8], we may suppose that the 2-elements of G do not form a normal subgroup. Let $u \in \mathcal{N}_{\mathcal{U}_1}(G)$. Let $g \in G$ and $w \in \mathcal{U}_1(\mathbb{Z}N)$, with N a finite normal subgroup, so that $u = gw$. If N is trivial, then $u \in G$, as desired. So suppose N is not trivial, in particular $\Delta(G)$ is not trivial. Hence by the previous remarks, the Frobenius complement H is finite. Since N is a normal subgroup it is contained in the Frobenius kernel K of G. If $|N|$ is odd, then by Theorem 2.4 w and hence u induce an inner automorphism on G. If $|N|$ is even then K has 2-torsion. Since K is nilpotent, it follows that G has a normal Sylow 2-subgroup; a contradiction. This finishes the proof if G is infinite.

So we are left to deal with G a finite Frobenius group of even order. A well known result on Frobenius groups implies that the Frobenius kernel K is the Fitting subgroup $\mathrm{Fit}(G)$ of G. Since G does not have a normal Sylow 2-subgroup we get that the Frobenius kernel K is a nilpotent group of odd order. Hence it is well known that K is abelian. Let ϕ be induced by a unit u of $\mathcal{U}_1(G)$ normalizing G. We shall modify ϕ so that it is the identity on K. First, recall that Coleman's result states that $\mathcal{N}_{\mathcal{U}_1(G)}(P) = \mathcal{N}_G(P)\mathcal{C}_{\mathcal{U}_1(G)}(P)$ for any p-subgroup P of G, where p is a prime number. Hence for a prime divisor p of $|K|$, if P is a Sylow p-subgroup of K then there exists $g_p \in G$ so that $\phi(y) = g_p^{-1} y g_p$ for all $y \in P$. So conjugation by $g_p^{-1}u$ acts trivially on P; and we thus may assume that ϕ acts trivially on P. Suppose that q is another prime divisor of $|K|$ and let Q be a Sylow q-subgroup of K. Fix $1 \neq x \in P$. For a given $y \in Q$ we then have $\phi(xy) = \phi(x)\phi(y) = x\phi(y)$. As $\phi(xy) - xy = u^{-1}(xy)u - xy = [u^{-1}, xyu] \in [\mathbb{Z}G, \mathbb{Z}G]$ and because $\phi(xy) \in G$ we obtain (see [19, Lemma 7.2]) that there exists $g \in G$ (dependent on xy) so that $x\phi(y) = \phi(xy) = g^{-1}xyg = (g^{-1}xg)(g^{-1}yg)$. As N is the direct product of its Sylow subgroups we obtain $x = g^{-1}xg$ and $\phi(y) = g^{-1}yg$. Because G is a finite Frobenius group the first equality implies that $g \in K$. Hence $\phi(y) = y$ and thus ϕ is the identity on any Sylow subgroup of K, and thus also on K itself. Since $\mathrm{Fit}(G)$ is the Frobenius kernel of G it follows that G is π-separable, where π is the set of prime divisors of $|K|$ and so we can apply Theorem 2.5. Hence the finite case follows.

□

Note that the proof of the finite case also works for Coleman automorphisms which are defined in [15].

Theorem 2.7 *Let G be a Frobenius group. Then the normalizer problem has a positive solution for G.*

Proof. Let $u \in \mathbb{Z}G$ be a normalizing unit and denote by ϕ the automorphism it induces on G. Our representation theorem tells us that we may suppose that $\text{supp}(u)$ generates a finite normal subgroup N, say, of even order. Let H be a Frobenius complement. We saw above that H must be finite and hence NH is a finite Frobenius group with H as a Frobenius complement. Since N is normal it must be the Frobenius kernel and hence is nilpotent. Now modify u, with an element of N (see [8] Proposition 2.1), such that it acts as the identity on N. Now use theorem 2.6 to conclude that u acts as an inner automorphism on NH. But since it acts as the identity on N and N is the Frobenius kernel of NH it follows that we may modify u, with an element of N, so that it acts trivially on NH, i.e., u is central in NH and still we have that $\text{supp}(u) \subset N$. Now let $g \in G$ be any element. Then, since the support of u is contained in N, it is clear that $[u, g] \in N$ and so $g^{-1}ug \in \mathbb{Z}(NH)$. Since u is a central element of $\mathbb{Z}(NH)$ it follows that $g^{-1}ug$ is also a central element of $\mathbb{Z}(NH)$ and thus $[u, g]$ is a central element of the finite Frobenius group NH. Hence $[u, g] = 1$ and thus u is a central element of $\mathbb{Z}G$ which completes the proof.

□

We recall a definition from [10].

Definition: Let $\mathcal{F}$ be a family of groups and G an arbitrary group. We say that G is an $\mathcal{F}$-group if for every normal torsion free subgroup N of G we have that $G/N \in \mathcal{F}$.

We shall also say that a group G satisfies the *strong normalizer problem* if for any normalizing element $u \in \mathbb{Z}G$ there exists $g \in \langle supp(u) \rangle$ such that ug is a central element.

In [8] it is proved that nilpotent groups satisfy the strong normalizer problem. However it is not clear if this problem is equivalent to the original one.

Corollary 2.8 *The strong normalizer problem has a positive solution for all groups if and only if it has a positive solution for the class of finitely generated groups.*

Proof. Let G be any group and let u be a normalizing unit. From our representation theorem we know that we may suppose that the support of u generates a finite normal subgroup N, say, of G. Let M be the centralizer of N in G. Then M has finite index and so we may choose $\{g_1, \cdots, g_n\}$ a finite transversal for M in G. Let H be the group generated by N and this tranversal. Then H is finitely generated and hence it follows that $u = hw$ where $h \in N$ and $w \in \mathbb{Z}N$ is a central unit of $\mathbb{Z}H$. Now w commutes with all elements of M and all the $g_i's$ and so w commutes with all $g \in G$, i.e., w is a central unit.

□

Lemma 2.9 *Let G be a group with a torsion free subgroup H of finite index. If G/H satisfies the strong normalizer problem then G satisfies the normalizer problem. In particular if G is an $\mathcal{F}$-group then the normalizer problem has a positive solution for G provided the strong normalizer problem holds for every member of $\mathcal{F}$.*

Proof. Let $u \in \mathcal{N}_{\mathcal{U}_1}(G)$. Corollary 2.8 tells us that we may suppose that G is finitely generated and we may suppose that $\mathrm{supp}(u)$ generates a finite normal subgroup N, say. Denote by $\{g_1, \cdots, g_n\}$ a set of generators of G. Working modulo N it is clear that $[u, g_i] \in N$ for all i.

Let $\pi : \mathbb{Z}G \to \mathbb{Z}(G/H)$ denote the natural homomorphism. Hence π is injective on N and so the support of $\pi(u)$ is also a finite normal subgroup. Since, by assumption, G/H satisfies the strong normalizer problem and because $\beta = \pi(u)$ normalizes G/H, we get that $\beta = \pi(x)v$, for some central element v in $\mathbb{Z}(G/H)$ and $x \in N$. We claim that $z = ux^{-1}$ is central in $\mathbb{Z}G$. Indeed, for any $g \in G$,

$$\begin{aligned}\pi(z, g_i) = (\pi(z), \pi(g_i)) &= (\pi(u)\pi(x^{-1}), \pi(g_i)) \\ &= (\pi(x)v\pi(x^{-1}), \pi(g_i)) \\ &= (v, \pi(g_i)) = 1\end{aligned}$$

So $(z, g_i) \in H$. On the other hand, since $x \in N$, we have that

$$(z, g_i) \in N$$

Since N is periodic and H is torsion free we get that $(z, g_i) = 1$ and hence indeed z is central in $\mathbb{Z}G$. Hence, $u \in G\ \mathcal{Z}(\mathcal{U}_1)$, as desired.

□

Note that this result enables us to give an induction argument in some cases. For example if G is a polycyclic-by-finite group then we may use induction on the Hirsch length where possible.

Let $u \in ZG$ be a normalizing element whose support generates a finite normal subgroup of G and suppose that the normalizer problem has a positive solution for N. Using Krempa's result, it is easy to see that we may modify u such that its support is still contained in N and it induces a derivation $\rho : G \longrightarrow O_2(\mathcal{Z}(\mathcal{N}))$ given by $\rho(g) = [u, g^{-1}]$ (see the proof of theorem 2.1 of [8]). Denoting by M the centralizer of N in G we have that G/M is a finite group and so ρ induces an element of $H^1(G/M, O_2(\mathcal{Z}(\mathcal{N})))$. So if $H^1(G/M, O_2(\mathcal{Z}(\mathcal{N}))) = 0$, e.g. if $[G : M]$ is odd, then ρ is inner and hence u induces an inner automorphism on G.

Corollary 2.10 *Let G be a group such that the normalizer of finite normal subgroups of G has odd index. Then the normalizer problem has a positive solution for G.*

So now we actually know what the obstruction is to achieve the main goal mentioned in the introduction. From here we could try to establish a positive solution for several other classes and families of groups but of course this would not shed any light on things.

In [2] D. Farkas and P. Linnell give a short proof of a result due to Marciniak and Sehgal. We finish this paper giving an even shorter proof of the result of Farkas and Linnell.

Theorem 2.11 ([2], Theorem 1) *Let G be an arbitrary group and let U be a subgroup of $\mathcal{U}_1(\mathbb{Z}G)$. If U contains G and $[U:G]$ is finite, then $U = G$.*

Proof. It is easy to see that we may suppose that G is a finitely generated F.C.-group (see [2] page 1). In particular G is a residually finite group. Now let $u \in U$ then, since G is residually finite, there exists a normal subgroup $N \subset G$ of finite index in G such that the canonical homomorphism $\pi : \mathbb{Z}G \longrightarrow \mathbb{Z}(G/N)$ is injective on $\mathrm{supp}(u)$ (see [10]). Since $[U:G]$ is finite it follows that $\pi(U)$ is finite. In this case it is well known that $\pi(U) = G/N$ (see [19]). It follows that $|\mathrm{supp}(\pi(u))| = 1$ and since π is injective on $\mathrm{supp}(u)$ it follows that $|\mathrm{supp}(u)| = 1$ and thus $u \in G$.

$\square$

It is of course enough to suppose that U is torsion over G, i.e., for all $u \in U$ there is an integer n, depending on u, such that $u^n \in G$.

Note that if $u \in \mathbb{Z}G$ is a central unit then its support generates a finitely generated F.C.-group. So the same proof of the former theorem and a classical result due to Berman-Higman, gives us the following: Let $u \in \mathbb{Z}G$ be a torsion central unit, then $u \in G$. This result has been proved and reproved by many authors but, as just mentioned, it follows directly from Berman-Higman's classical theorem of the finite case. In [8] (see lemma 2.2 of the present paper) it is proven that if $u \in \mathbb{Z}G$ is a normalizing unit which is torsion over G then $u \in G$. Note that, using lemma 2.3, the same proof of theorem 2.11 and the known finite case, we also obtain this result. What we want to stress here is that for some results a representation theorem is not needed but for others it seems necessary to have one. Note also that most results mentioned here remain valid for group rings RG with R a $G-$adapted ring, that is, R is a commutative integral domain of characteristic zero such that no rational prime p, say, dividing the order of G is invertible in R. We leave this for the reader to check.

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