ON SOME GENERALIZATION OF A THEOREM OF B. H. NEUMANN

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Abstract

Some numerical characteristics of groups with polycyclic conjugacy classes have been obtained. These results could be considered as some extensions of a theorem belonging to B. H. Neumann and related to groups with boundedly finite conjugacy classes.

Let G be a group, S - its G-invariant subset (that is, $g^x \in S$ for every $g \in S$ and every $x \in G$). Then $C_G(S)$ is a normal subgroup of G. The factor-group $G/C_G(S) = Coc_G(S)$ is called *the cocentralizer of the set* S in the group G. The group $Coc_G(S)$ is isomorphic to some subgroup of $Aut(\langle S^G \rangle)$. The influence of cocentralizers on the structure of the group is a subject of study in many branches of Group Theory. Finite Group Theory provides many examples, in which cocentralizers of chief factors play an important role. It is enough to mention that a local formation is defined by restrictions on cocentralizers of chief factors. Classes of infinite groups defined by restrictions on cocentralizers of conjugacy classes, are also studied. Let $\mathcal{X}$ be a class of groups.

A group G is called an $\mathcal{X}C$ - group if $Coc_G(x^G) \in \mathcal{X}$ for each $x \in G$.

If $\mathcal{X} = \mathcal{F}$ is the class of all finite groups, then we obtain the well-known class of FC - groups. The theory of FC - groups has been developed intensively. B.H.Neumann [5] considered the following particularly interesting case: the FC-groups G, in which $|Coc_G(x^G)| \leq n$ for all elements $x \in G$ (the BFC - groups). He proved that the derived group of a BFC - group is finite.

A series of interesting articles, dedicated to bounds for the order of the derived group, have been published. The latest work here is the article of D. Segal and A. Shalev [7], in which the authors obtained the following estimation for the order of the derived group of a BFC - group $G: \ |[G,G]| \leq n^{\frac{1}{2}(13+log_2 n)}$.

Groups with the polycyclic-by-finite conjugacy classes (PC - groups), which are the groups G, such that $Coc_G(x^G)$ is a polycyclic-by-finite group for each element $x \in G$, have been considered in [1].

We shall obtain some extensions of Neumann's theorem on the class of PC - groups. First of all, we need to chose good numerical characteristics for polycyclic-by-finite groups, which characterize them similarly as the order characterizes finite groups. There are different numerical functions related to the polycyclic-by-finite groups G. For example, the minimal number of generating elements $d(G)$, the (special) rank $r(G)$, the Hirsch number $h(G)$, and so on. We will consider the following functions, specifically designed for polycyclic-by-finite groups:

Let G be a soluble-by-finite group, $S(G)$ its largest soluble normal subgroup. Let $is(G) = |G/S(G)|$.

Let G be a polycyclic-by-finite group. Then the subgroup $S(G)$ has a finite subnormal series with cyclic factors. In general, many such series exist.

Denote by $Ser(G)$ the set of all finite subnormal series with cyclic factors of the subgroup $S(G)$. In the set $Ser(G)$ choose a series Σ with the least length and denote the length of this series by $p(G)$.

This number $p(G)$ is a specific characteristic of a polycyclic-by-finite group. It is associated with special rank $r(G)$ of a group G, namely, $r(S(G)) \leq p(G)$, and therefore, $r(G) \leq p(G) + is(G)$. Clearly, $d(G) \leq p(G) + is(G)$. If G is an abelian finitely generated group, then $p(G) = r(G) = d(G)$.

We will also use other numerical parameters associated with a polycyclic group G. This is the Hirsch number $h(G)$, which is the number of all infinite cyclic factors in each subnormal series with cyclic factors. This concept can be extended in the following way.

Let G be a group. Say that G has finite 0 - rank (or torsion-free rank) $r_0(G) = r$, if G has a finite subnormal series with exactly r infinite cyclic factors, the remaining factors being periodic.

Every refinement of this series has only r infinite cyclic factors. Since every two finite subnormal series have isomorphic refinements, 0 - rank is independent from the choice of the subnormal series.

Note that if G is a polycyclic group, then $r_0(G)$ is exactly the Hirsch number.

A group G is said to be a $BPC-$ group (or G has the bounded polycyclic-by-finite classes) if there are numbers i, p, and *h with the following properties:*

(1) $Coc_G(x^G)$ is a polycyclic-by-finite group for each $x \in G$;

(2) $p(Coc_G(x^G)) \leq p$ for each $x \in G$;

(3) $is(Coc_G(x^G)) \leq i$ for each $x \in G$;

(4) $h(Coc_G(x^G)) \leq h$ for each $x \in G$.

Lemma 1. *Let G be a BPC- group.*

(1) If H is a subgroup of G then H is a BPC - group. Moreover, $p(Coc_H(x^H)) \leq p$, $is(Coc_H(x^H)) \leq i$, $h(Coc_H(x^H)) \leq h$.

(2) If H is a normal subgroup of G then G/H is a BPC - group. Moreover, $p(Coc_{G/H}((xH)^{G/H}) \leq p$, $is(Coc_{G/H}((xH)^{G/H}) \leq i$, $h(Coc_{G/H}((xH)^{G/H}) \leq h$.

(3) If H is a subgroup of G, L a normal subgroup of H then H/L is a BPC - group. Moreover, $p(Coc_{H/L}((xL)^{H/L}) \leq p$, $is(Coc_{H/L}((xL)^{H/L}) \leq i$, $h(Coc_{H/L}((xL)^{H/L}) \leq h$.

The lemma is obvious.

If G is a soluble group then we denote by $s(G)$ *the class of solubility of G* (or the derived length of G).

Lemma 2. *Let G be a BPC - group. Then*

(1) G is soluble-by-finite.

(2) $s(S(G)) \leq p + 1$.

(3) there is a function $f_1 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, such that $is(G) \leq f_1(p, i)$.

Proof. For every $x \in G$ put $C_x = Coc_G(x^G)$, $S_x = S(C_x)$. Let R_x be a preimage of S_x in the group G, and let $R = \bigcap_{x \in G} R_x$. Since $\zeta(G) = \bigcap_{x \in G} C_G(x^G)$, the Remak Theorem implies the imbedding

$$G/\zeta(G) \leq \prod_{x \in G} G/C_G(x^G) = \prod_{x \in G} C_x.$$

It follows that

$$G/R \leq \prod_{x \in G} C_x/S_x.$$

Since C_x/S_x is finite and $|C_x/S_x| \leq i$ for every $x \in G$, G/R is a periodic group. Moreover, the order of each of its elements divides the number i.

Furthermore, $R_x/C_G(x^G)$ possesses a finite subnormal series with cyclic factors, the length of which is at most p. It follows that $R_x/(RC_G(x^G))$ is finite and $|R_x/(RC_G(x^G))| \leq i^p$. Hence, $G/(RC_G(x^G))$ is finite and $|G/(RC_G(x^G))| \leq i^p i = i^{p+1}$. Since $C_{G/R}((xR)^{G/R}) \geq C_G(x^G)R/R$, $(G/R)/C_{G/R}((xR)^{G/R})$ is finite and $|(G/R)/C_{G/R}((xR)^{G/R})| \leq i^{p+1}$. In this case $[G/R, G/R]$ is finite (see, for example, [6, Theorem 4/35]). Moreover, there is a function g: $\mathbb{N} \longrightarrow \mathbb{N}$, such that $|[G/R, G/R]| \leq g(i^{p+1})$. Put $U/R = C_{G/R}([G/R, G/R])$. Then U/R is nilpotent and $(G/R)/(U/R)$ is finite. Also $|(G/R)/(U/R)| \leq (g(i^{p+1}))!$. The Remak Theorem implies the following imbedding for the group $R/\zeta(G)$:

$$R/\zeta(G) \leq \prod_{x \in G} (RC_G(x^G))/C_G(x^G) \leq \prod_{x \in G} R_x/C_G(x^G) = \prod_{x \in G} S_x.$$

Since $s(S_x) \leq p$ for each $x \in G$, $R/\zeta(G)$ is soluble and $s(R/\zeta(G)) \leq p$. The factor-group G/R is nilpotent-by-finite, so that G is soluble-by-finite. Moreover, $is(G) \leq (g(i^{p+1})! = f_1(p, i)$.

Let $S = S(G)$. Then $SC_G(x^G) \leq R_x/C_G(x^G)$ for any $x \in G$. Thus we obtain that $S \leq \bigcap_{x \in G} R_x = R$. Since R is soluble, $R \leq S$ and $R = S$. It follows that $s(S) \leq p + 1$.

□

If $\alpha \in \mathbb{R}$ then we denote the least integral number with the property $\alpha \leq s_{\mathbb{Z}}(\alpha)$ by $s_{\mathbb{Z}}(\alpha)$. Using the results of [7], we can obtain the following bound for the function g:

$$g(i^{p+1}) \leq (i^{p+1})^{\frac{1}{2}(13+log_2((i^{p+1})} = i^{\frac{1}{2}(p+1)(13+(p+1)log_2 i)}.$$

Hence

$$f_1(p,i) \leq s_{\mathbb{Z}}(i^{\frac{1}{2}(p+1)(13+(p+1)log_2 i)})!.$$

□

Lemma 3. *Let G be a BPC - group, A - a normal abelian subgroup of $G, B/C_G(A)$ a finitely generated abelian subgroup of $G/C_G(A), p(B/C_G(A)) = r$. Then $[A,B]$ is a finitely generated subgroup and $p([A,B]) \leq pr$, $h([A,B]) \leq hr$.*

Proof. Let

$$B/C_G(A) = \langle b_1 C_G(A)\rangle \times ... \times \langle b_r C_G(A)\rangle .$$

The mapping $\varphi_i : a \longrightarrow [a, b_i], a \in A$, is an endomorphism of A, such that Ker $\varphi_i = C_A(b_i)$, Im $\varphi_i = [A, b_i]$, that is

$$[A, b_i] = \text{ Im } \varphi_i \cong A/\text{Ker } \varphi_i = A/C_A(b_i).$$

Since $C_G(b_i^G) \cap A \leq C_A(b_i)$, $A/C_A(b_i)$ is finitely generated, moreover $p(A/C_A(b_i)) \leq p, h(A/C_A(b_i)) \leq h, 1 \leq i \leq r$. For all elements $u, v \in B$ we have $[a, uv] = [a, v][a, u]^v = [a, v][a^v, u^v]$. Since $B/C_G(A)$ is abelian, $u^v = uc$ for some $c \in C_G(A)$. Therefore $[a^v, u^v] = [a^v, uc] = [a^v, u]$. It follows that $[A, B] = [A, b_1]...[A, b_r]$, thus $[A, B]$ is finitely generated and $p([A, B]) \leq pr$, $h([A, B]) \leq hr$.

□

Corollary 1. *Let G be a BPC - group, A - a normal abelian subgroup of G, $B/C_G(A)$ a polycyclic subgroup of $G/C_G(A)$ and $p(B/C_G(A)) = r$. Then $[A, B]$ is finitely generated, moreover, $p([A, B]) \leq pr, h([A, B]) \leq hr$.*

Proof. The subgroup B has a finite subnormal series

$$C_G(A) = B_0 \leq B_1 \leq ... \leq B_r = B$$

with cyclic factors. By Lemma 3 the subgroup $[A, B_1]$ is finitely generated and $p([A, B_1]) \leq p, h([A, B_1]) \leq h$. Clearly, $[A, B_1]$ is B_2-invariant. We can apply now Lemma 3 to $A/[A, B_1]$ in order to obtain that $[A, B_2]$ is finitely generated and $p([A, B_2]) \leq 2p$, $h([A, B_2]) \leq 2h$. Using the similar arguments after r steps, we obtain that $[A, B_r] = [A, B]$ is finitely generated, moreover, $p([A, B]) \leq pr$, $h([A, B]) \leq hr$.

□

Corollary 2. *Let G be a BPC - group, A - a normal abelian subgroup of G, such that $G/C_G(A)$ is polycyclic-by-finite. Let $p(G/C_G(A)) = r, is(G/C_G(A)) = k$. Then $[A, G]$ is finitely generated, moreover, $p([A, G]) \leq p(r + k), h([A, G]) \leq h(r + k)$.*

Proof. Let $B/C_G(A) = S(G/C_G(A))$. Then $[A, B]$ is finitely generated and $p([A, B]) \leq pr, h([A, B]) \leq hr$ by Corollary 1. In the factor-group $\bar{G} = G/[A, B]$ the following inequality holds $C_{\bar{G}}(\bar{A}) \geq \bar{B} = B/[B, A]$, where $\bar{A} = A/[A, B]$. Let $\bar{X} = \{\bar{x}_1, ..., \bar{x}_k\}$ be a transversal to $\bar{B}$ in $\bar{G}$. Similarly as in Lemma 3, we can obtain that $[\bar{A}, \bar{x}_i]$ is finitely generated and $p([\bar{A}, \bar{x}_i]) \leq p, h([\bar{A}, \bar{x}_i]) \leq h$. Clearly, $[\bar{A}, \bar{G}] = [\bar{A}, \bar{x}_1]...[\bar{A}, \bar{x}_k]$, therefore $[\bar{A}, \bar{G}]$ is finitely generated and $p([\bar{A}, \bar{G}]) \leq pk, h([\bar{A}, \bar{G}]) \leq hk$. This means that $[A, G]$ is finitely generated and $p([A, G]) \leq p(r + k), h([A, G]) \leq h(r + k)$.

□

Lemma 4. *Let G be a BPC - group, $A-$ a subgroup of $\zeta(G)$ such that G/A is polycyclic. Let $p(G/A) = r$. Then $[G, G]$ is polycyclic subgroup and $p([G, G]) \leq p(r - 1), h([G, G]) \leq h(r - 1)$.*

Proof. Let

$$A = U_0 \lhd U_1 \lhd \ldots \lhd U_r = G$$

be a finite subnormal series and $U_{i+1}/U_i = \langle g_{i+1}U_i \rangle, 0 \leq i \leq r-1$. Since $A \leq \zeta(G), U_1$ is an abelian subgroup. Since U_2/U_1 is abelian, the mapping $u \longrightarrow [u, g_2]$, $u \in U_2$, is a $\mathbb{Z}U_2$ - endomorphism of U_1; that is, $[U_1, g_2] \cong U_1/C_{U_1}(g_2)$. It follows that $[U_1, g_2]$ is finitely generated and $p([U_1, g_2]) \leq p$, $h([U_1, g_2]) \leq h$. Since U_2/U_1 is cyclic, $U_2/[U_1, g_2]$ is abelian. This means that $[U_2, U_2] \leq [U_1, g_2]$, in particular, $[U_2, U_2]$ is polycyclic and $p([U_2, U_2]) \leq p, h([U_2, U_2]) \leq h$. Note that $[U_2, U_2]$ is U_3- invariant.

Using the similar arguments we obtain that $[U_r, U_r] = [G, G]$ is polycyclic and $p([G, G]) \leq p(r-1), h([G, G]) \leq h(r-1)$.

□

Corollary 1. *Let G be a BPC- group, A - a normal abelian subgroup of G such that G/A is polycyclic. Let $p(G/A) = r$. Then $[G, G]$ is a polycyclic subgroup and $p([G, G]) \leq p(2r-1), h([G, G]) \leq h(2r-1)$.*

Proof. Corollary 1 of Lemma 3 yields that $[A, G]$ is finitely generated and $p([A, G]) \leq pr, h([A, G]) \leq hr$. We can now apply Lemma 4 to the factor-group $G/[A, G]$ and obtain that $[G, G]$ is polycyclic and $p([G, G]) \leq p(2r - 1), h([G, G]) \leq h(2r-1)$.

□

Corollary 2. *Let G be a BPC - group, A - a normal abelian subgroup of G such that G/A is polycyclic-by-finite. Let $p(G/A) = r$, $is(G/A) = l$. Then $[G, G]$ is a polycyclic-by-finite subgroup, moreover $p([G, G]) \leq p(2r-1+l)$, $h([G, G]) \leq h(2r-1+l)$, $is([G, G]) \leq l^{\frac{1}{2}(log_2 l+1)}$.*

Proof. Let $S = S(G)$, then S/A is a polycyclic group and $p(S/A) = r$, $|G/S| = l$. Corollary 2 yields that $[S, S]$ is polycyclic and $p([S, S]) \leq p(2r-1)$, $h([S, S]) \leq h(2r-1)$. Consider now the factor-group $G/[S, S]$. Corollary 2 of Lemma 3 implies that $[S/[S,S], G/[S,S]]$ is finitely generated and $p([S/[S,S], G/[S,S]]) \leq pl$, $h([S/[S,S],\ G/[S,S]]) \leq hl$.

The center of $G/([S,S][S,G])$ includes $S/([S,S][S,G])$, therefore by Schur Theorem (see, for example, [6, Theorem 4.12]) we obtain that $[G/([S,S][S,G]), G/([S,S][S,G])]$ is finite. Moreover, $|[G/([S,S][S,G]), G/([S,S][S,G])]| \leq l^{\frac{1}{2}(log_2 l+1)}$ [8].

It follows that $[G,G]$ is a polycyclic-by-finite subgroup, moreover $p([G,G]) \leq p(2r-1+l)$, $h([G,G]) \leq h(2r-1+l), is([G,G]) \leq l^{\frac{1}{2}(log_2 l+1)}$.

□

Lemma 5. *Let G be a BPC - group, $x_1, ..., x_n \in G, X = \langle x_1 \rangle^G ... \langle x_n \rangle^G$. Then $Coc_G(X)$ is a polycyclic-by-finite group and $p(Coc_G(X)) \leq pn, h(Coc_G(X)) \leq hn$, $is(Coc_G(X)) \leq i^n$.*

In fact, it follows from imbedding that

$$Coc_G(X) \leq Coc_G(\langle x_1 \rangle^G) \times ... \times Coc_G(\langle x_n \rangle^G).$$

Lemma 6. *Let G be a BPC - group, A - a normal abelian torsion-free subgroup, such that G/A is abelian. Then $r_0([A,G])$ is finite, moreover, there exists a function $f_2 : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, such that $r_0([A,G]) \leq f_2(p,h)$.*

Proof. Since G/A is abelian, for every element $g \in G$, the mapping $\varphi_g : a \longrightarrow [a,g]$, $a \in A$, is a $\mathbb{Z}G$- endomorphism of A. This means that $Im\varphi_g = [A,g]$ and $Ker\varphi_g = C_A(g)$ are G - invariant subgroups of A and $[A,g] \cong A/C_A(g)$. Since $C_G(g^G) \cap pA \leq C_A(g)$, $[A,g]$ is finitely generated and $p([A,g]) \leq p$. But $[A,g]$ is torsion-free, hence $p([A,g]) = h([A,g])$, and $p([A,g]) \leq h$.

Suppose that $A \nleq \zeta(G)$. Among the elements of G with a property $[A,x] \neq \langle 1 \rangle$, we choose the element x_1 such that a number $h([A,x_1])$ is the largest. Put $B_1 = [A,x_1]$, $A_1 = C_A(x_1)$. Then A/A_1 is finitely generated, torsion-free, and $h(A/A_1) \leq h$. Let $A/A_1 = \langle a_1 A_1 \rangle \times \langle a_2 A_1 \rangle \times ... \times \langle a_l A_1 \rangle, l \leq h$. Put $E_1 = \langle a_1, ..., a_l \rangle^G$. Then $A = E_1 A_1$. Lemma 5 yields that $G/C_G(E_1)$ is a polycyclic group and $p(G/_G(E_1)) \leq pl \leq ph$. Let $G_1 = C_G(E_1) \cap C_G(x_1^G)$. Then G/G_1 is a polycyclic group and $p(G/G_1) \leq ph + p = p(h+1)$.

Consider the subgroups $[A_1, x]$ where $x \in G_1$. Assume that there exists an element $x_2 \in G_1$ such that the pure envelope of B_1 (in A) does not include $[A_1, x_2]$. In other words, there is an element $u \in [A_1, x_2]$ such that $\langle u \rangle \cap B_1 = \langle 1 \rangle$. Let a be an element of A_1, such that $u = [a, x_2]$. For each element $e \in E_1$ we have

$$[e, x_1x_2] = [e, x_2][e, x_1]^{x_2} = [e^{x_2}, x_1^{x_2}] = [e, x_1]$$

, so that $B_1 = [A, x_1] = [E_1A_1, x_1] = [E_1, x_1] = [E_1, x_1x_2]$. Furthermore, $[a, x_1x_2] = [a, x_2][a, x_1]^{x_2} = [a, x_2] = u$. It follows that $\langle u, B_1 \rangle \leq [A, x_1x_2]$. In particular, $h([A, x_1x_2]) > h([A, x_1])$. But this is a contradiction to the choice of x_1. This contradiction shows that the pure envelope of B_1 includes $[A_1, x]$ for each $x \in G_1$. This means that it includes $[A_1, G_1]$. In particular, $r_0([A_1, G_1]) \leq h$. Using now Corollary 1 of Lemma 3, we obtain that and $r_0([A, G])$ is finite, moreover, $r_0([A, G]) \leq h + h(ph + p) = h + hp + (h^2)p$.

□

Lemma 7. *Let G be a BPC - group, A - a torsion-free subgroup of $\zeta(G)$ such that G/A is abelian. Then $r_0([G, G])$ is finite. Moreover, there is a function f_3 : $\mathbb{N} \longrightarrow \mathbb{N}$, such that $r_0([G, G]) \leq f_3(h)$.*

Proof. Since $[G, G] \leq \zeta(G)$ then $C_G(g^G) = C_G(g)$. Moreover, $G/C_G(g) \cong [g, G] \leq A$, in particular, $G/C_G(g)$ is a torsion-free finitely generated abelian group, that is $p(G/C_G(g)) = h(G/C_G(g)) \leq h$ for each element $g \in G$.

Suppose that G is non-abelian. Among the elements g of G with the property $[g, G] \neq \langle 1 \rangle$ we choose the element g_1 such that the number $p([g_1, G]) = h([g_1, G])$ is the largest. Put $L_1 = [g_1, G], G_1 = C_G(g_1)$. Then $G/G_1 = \langle u_1G_1 \rangle \times \langle u_2G_1 \rangle \times ... \times \langle u_lG_1 \rangle$ where $l \leq h$. Let $U_1 = \langle u_1, ..., u_l \rangle^G$. Then $G = U_1G_1$. Lemma 5 yields that $G/C_G(U_1)$ is finitely generated and $p(G/C_G(U_1)) \leq hl \leq h^2$. Let $V_1 = C_G(U_1) \cap G_1$. Then G/V_1 is finitely generated and $p(G/V_1) = h(G/V_1) \leq h + h^2$.

If there is an element $g_2 \in V_1$ such that the pure envelope of L_1 in A does not include $[g_2, V_1]$, then there is an element $w \in [g_2, V_1]$ with the property $\langle w \rangle \cap L_1 = \langle 1 \rangle$. Let v be an element of V_1such that $w = [g_2, v]$. For each element $u \in U_1$, we have $[g_1g_2, u] = [g_1, u][g_2, u] = [g_1, u]$, so that $L_1 = [g_1, G] = [g_1, U_1G_1] = [g_1, U_1] = [g_1g_2, U_1]$. Furthermore, $[g_1g_2, v] = [g_1, v][g_2, v] = [g_2, v]$, thus $\langle w, L_1 \rangle \leq [g_1g_2, G]$. This means that $p([g_1g_2, G]) > p([g_1, G])$. But it contradicts to the choice of g_1. This contradiction proves that the pure envelope of L_1 in A includes $[g, V_1]$ for every $g \in V_1$. Thus, it includes $[V_1, V_1]$. In particular, $r_0([V_1, V_1]) \leq h$. Using now Corollary 1 to Lemma 4 we obtain that $r_0([G, G])$ is finite and $r_0([G, G]) \leq h + h(2(h + h^2) - 1) = 2h^2 + 2h^3$.

□

Lemma 8. *Let G be a metabelian BPC - group, $g \in G$. Then $[g, G]$ is a polycyclic group and $p([g, G]) \leq p^2 + p, h([g, G]) \leq h(p + 1)$.*

Proof. Let $A = [G, G]$. Since G/A is abelian, then the mapping $\varphi_g : a \longrightarrow [g, a]$, $a \in A$, is a $\mathbb{Z}G$ - endomorphism. In particular, $Im\varphi_g = [g, A]$ and $Ker\varphi_g = C_A(g)$ are G - invariant subgroups and $[g, A] \cong A/C_A(g)$. As shown above, it follows that $[g, A]$ is finitely generated and $p([g, A]) \leq p, h([g, A]) \leq h$. In the factor-group $\bar{G} = G/[g, A]$ the subgroup$\langle g, A \rangle / [g, A]$ is abelian, hence $\bar{B} = B/[g, A] = \langle g \rangle^G [g, A]/[g, A]$ is also abelian. Furthermore, $Coc_{\bar{G}}(\bar{B})$ is finitely generated and $p(Coc_{\bar{G}}(\bar{B})) \leq p$, $h(Coc_{\bar{G}}(\bar{B})) \leq h$. Corollary 1 of Lemma 3 yields that $[\bar{B}, \bar{G}]$ is finitely generated and $p([\bar{B}, \bar{G}]) \leq p^2$, $h([\bar{B}, \bar{G}]) \leq hp$. It follows that $[g, G]$ is polycyclic and $p([g, G]) \leq p^2 + p$, $h([g, G]) \leq hp + h$.

□

Lemma 9. *Let G be a BPC - group, $K = [G, G]$ an elementary abelian q - subgroup for some prime q. Then K is finite and $|K| \leq q^{f_4(p)}$ for some function $f_4 : \mathbb{N} \longrightarrow \mathbb{N}$.*

Proof. Assume that G is non-abelian. Then there are the elements u_1, v_1 such that $[u_1, v_1] = a_1 \neq 1$. Let $G_1 = C_G(u_1^G) \cap C_G(v_1^G)$. By Lemma 5 G/G_1 is polycyclic and $p(G/G_1) \leq 2p$. If $[u,v] \in \langle a_1 \rangle$ for each pair $u, v \in G_1$ then $[G_1, G_1] \leq \langle a_1 \rangle$. By Corollary 1 of Lemma 4 $[G,G]$ is finite and $p([G,G]) \leq p(4p-1)+1$. Hence we can suppose that there are the elements $u_2, v_2 \in G_1$, such that $a_2 = [u_2, v_2] \notin \langle a_1 \rangle$. In other words, $\langle a_1, a_2 \rangle = \langle a_1 \rangle \times \langle a_2 \rangle$. Using the similar arguments, we may choose the elements $u_1, ..., u_l; v_1, ..., v_l$ with the following properties:

$$u_{j+1}, v_{j+1} \in G_j = \bigcap_{1 \leq m \leq j} C_G(u_m^G) \cap C_G(v_m^G), 1 \leq j \leq l-1;$$
$$\text{if } [u_j, v_j] = a_j \quad \text{then} \quad \langle a_1, ..., a_l \rangle = \langle a_1 \rangle \times ... \times \langle a_l \rangle .$$

Put $g_l = u_1...u_l$. Then $[g_l, v_j] = [u_j, v_j] = a_j$ for each $j, 1 \leq j \leq l$. It follows that $\langle a_1, ..., a_l \rangle \leq [g_l, G]$. Lemma 8 yields that $[g_l, G]$ is finite and $p([g_l, G]) \leq p^2 + p$. It follows that $l \leq p^2 + p$. The process of the choosing of elements u_j, v_j are finite, hence $[G_l, G_l] \leq \langle a_1, ..., a_l \rangle$ for some l. Lemma 5 yields that G/G_l is polycyclic and $p(G/G_l) \leq 2pl$. Now we can apply Corollary 1 of Lemma 4.

□

Lemma 10. *Let G be a BPC - group, $K = [G,G]$ a divisible abelian q-subgroup for some prime q. Then K is a Chernikov subgroup and $r(K) \leq f_5(p)$ for some function $f_5 : \mathbb{N} \longrightarrow \mathbb{N}$.*

Proof. We can again assume that G is non-abelian, and there are the elements u_1, v_1 such that $[u_1, v_1] = a_1 \neq 1$. Since K is divisible, there are the elements $\{b_n | n \in \mathbb{N}\}$ with the properties: $a_1 = b_1, b_{n+1}^p = b_n, n \in \mathbb{N}$. In other words, there exists a Prüfer q - subgroup $A_1 = \langle b_n | n \in \mathbb{N} \rangle$, such that $\langle a_1 \rangle \leq A_1$. Put $G_1 = C_G(u_1^G) \cap C_G(v_1^G)$. By Lemma 5 G/G_1 is polycyclic and $p(G/G_1) \leq 2p$. If $[u,v] \in A_1$ for each pair $u, v \in G_1$, then $[G_1, G_1] \leq A_1$. By Corollary 1 of Lemma 4 $[G,G]$ has finite rank and $r([G,G]) \leq p(4p-1)+1$. Hence we may assume that there are the elements $u_2, v_2 \in G_1$, such that $[u_2, v_2] = a_2 \notin A_1$. Then $\langle A_1, a_2 \rangle = A_1 \times \langle d_1 \rangle$ for some element d_1 (see, for example, [2, Theorem 21.2]). In particular, $p(\langle a_1, a_2 \rangle) = 2$. Using the similar arguments we may choose the elements $u_1, ..., u_l; v_1, ..., v_l$ with the following properties:

$$u_{j+1}, v_{j+1} \in G_j = \bigcap_{1 \le m \le j} C_G(u_m^G) \cap C_G(v_m^G), 1 \le j \le l-1;$$

$$\text{if } [u_j, v_j] = a_j, \quad \text{then} \quad \langle a_1, ..., a_l \rangle = \langle a_1 \rangle \times ... \times \langle a_l \rangle .$$

Using the arguments from the previous lemma we can prove that $l \le p^2 + p$. In fact, this means that $r([G_l, G_l]) \le l$. Now we can use Corollary 1 of Lemma 4.

□

Corollary 1. *Let G be a BPC - group, $K = [G, G]$ - an abelian q-subgroup for some prime q. Then K is a Chernikov subgroup and $r(K) \le f_6(p)$ for some function $f_6 : \mathbb{N} \longrightarrow \mathbb{N}$.*

Proof. By Lemma 9 K/K^q is finite. In this case $K = D \times F$ where the subgroup F is finite and D is divisible [4, Lemma 7]. It follows that $|K/K^q| = |F/F^q|$ and $p(F) = log_q(|F/F^q|) = log_q(|K/K^q|)$. Using Lemma 9 to the factor-group G/D we obtain that $p(F) \le f_4(p)$. By Lemma 8 $H = F^G$ being finite, $K/H = DH/H \cong D/(D \cap H) \cong D$, in particular, $r(D) = r(K/H)$. By Lemma 10, $r(K/H) \le f_5(p)$. Hence we may put $f_6 = f_4 + f_5$.

□

Corollary 2. *Let G be a BPC- group, $K = [G, G]$ - an abelian periodic subgroup. Then K has finite special rank and $r(K) \le f_6(p)$.*

Indeed, $K = \times_{q \in \Pi(K)} K_q$ where K_q is a Sylow q - subgroup of K. Put $L_q = \times_{s \neq q} K_s$. Now we can apply Corollary 1 to the factor-group G/L_q.

Lemma 11. *Let G be a BPC$-$ group, $A-$ an abelian normal subgroup such that G/A is abelian. Then $[G, G]$ has finite special rank, moreover, $r([G, G]) \le f_7(p, h)$ and $r_0([G, G]) \le f_8(p, h)$ for some functions $f_7 : \mathbb{N} \longrightarrow \mathbb{N}$, $f_8 : \mathbb{N}.\times\mathbb{N} \longrightarrow \mathbb{N}$.*

Proof. Let T be the periodic part of the subgroup A. By Lemma 6, $r_0([A/T, G/T])$ is finite and $r_0([A/T, G/T]) \le f_2(p, h)$. Let B/T be a pure envelope of $[A/T, G/T]$

in A/T. Then A/B is torsion-free and $A/B \leq \zeta(G/B), r_0(B/T) = r_0([A/T, G/T])$. Lemma 7 proves that $L/B = [G/B, G/B]$ has finite 0- rank, moreover, $r_0(L/B) \leq f_3(h)$. It follows that $r_0(L/T)$ is finite and $r_0(L/T) \leq f_2(p,h) + f_3(h)$. Let be $\{c_1T, ..., c_lT\}$a maximal $\mathbb{Z}$ - independent set of elements of $L/T, l \leq f_2(p,h) + f_3(h)$. By Lemma 8, the subgroup $C = \langle c_1\rangle^G ... \langle c_l\rangle^G$ is finitely generated and $p(C) \leq (p^2 + p)l + l, h(C) \leq h(p+1)l + l$. Furthermore, L/C is periodic and G/L is abelian. Now we can apply Corollary 2 of Lemma 10 to the factor-group G/C.

$\square$

Theorem 1. *Let G be a BPC - group. Then $[G, G]$ is a soluble-by-finite group of finite special rank, moreover, $r([G,G]) \leq f_9(p,h,i)$, $r_0([G,G]) \leq f_{10}(p,h)$, $is([G,G]) \leq f_{11}(p,i)$ for some functions $f_9 : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \longrightarrow N$, $f_{10} : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$, $f_{11} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$.*

Proof. By Lemma 2 G is soluble-by-finite. Let $S = S(G)$. Then Lemma 2 yields that $s(S) \leq p+1$. Using Lemma 11 after p steps we obtain that the subgroup $[S,S]$ has finite special rank, moreover, $r([S,S]) \leq pf_7(p,h), r_0([S,S]) \leq pf_8(p,h)$. Lemma 2 yields that $|G/S| = l \leq f_1(p,i)$. Now we can apply to the factor-group $G/[S,S]$ Corollary 2 of Lemma 4.

$\square$

In connection with the theorem of B. H. Neumann and the result proven above the following question arises: is the subgroup $[G,G]$ polycyclic-by-finite? There is an example of the group G_p satisfying the following properties: G_p is a p-group, p is prime, $\zeta(G_p) = [G_p, G_p]$ is a Prüfer p - subgroup, G_p is an FC - group [3, Example 2.11]. For every element $g \in G$ we have $G_p/C_{G_p}(g) \cong [g, G_p]$. Since $[g, G_p]$ is a finite subgroup of $[G_p, G_p]$, it is cyclic. Hence $Coc_{G_p}(g^{G_p}) = G_p/C_{G_p}(g)$ is a cyclic p - group, so that G_p is BPC - group. But its derived group is not polycyclic. Moreover, the derived subgroup of a BPC - group can be non-minimax. In fact, let $G = \times_{p\in\mathbb{P}} G_p$. Then $Coc_G(g^G)$ is cyclic for each element $g \in G$, but $[G,G]$ is locally cyclic subgroup with the infinite set

$\Pi([G,G])$ $(\Pi([G,G]) = \mathbb{P})$. Here, $\mathbb{P}$ is the set of all primes. These examples show, that Theorem 1 cannot be improved.

Theorem 2. *The following relations hold:*

$$f_9(p,h,i) \leq 4p^5 + p^4(h^2+h+4) + p^3(2h^3+3h^2+2h+1)$$
$$+p^2(2h^3+3h^2+2h) + p(2h^3+2h^2+h) + pt + t^{\frac{1}{2}(log_2 t+1)};$$

$$f_{10}(p,h) \leq 2h^4(p^2+p) + h^3(p^3+3p^2+4p) + h^2(p^3+3p^2+3p) + h(p^2+p) + ht;$$

$$f_{11}(p,i) \leq t^{\frac{1}{2}(log_2 t+1)}$$

$$where \quad t = s_{\mathbb{Z}}(i^{\frac{1}{2}(p+1)(13+(p+1)log_2 i)})!.$$

We will omit the proof here.

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