ANALYTICALLY VARYING EIGENVECTORS AND THE STABILITY OF VISCOUS SHOCK WAVES

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Abstract

Given a matrix $A(\cdot)$ and a simple eigenvalue $\alpha(\cdot)$, both depending analytically on a complex parameter λ within a simply connected domain Λ, we present a simple algorithm, based on a classical result of Kato, for finding an associated eigenvector $V(\cdot)$ that likewise varies analytically with respect to λ. This is useful for numerical approximation of the Evans function/numerical determination of stability of traveling waves, as we demonstrate by an application to stability of viscous shock waves. Indeed, it extends to general traveling waves/systems of equations an efficient and robust 'shooting' method developed by Brin for the determination of shock stability in special cases where the relevant eigenvector $V(\cdot)$ is explicitly available.

1 The Evans Function and Numerical Investigation of Spectrum

Consider a single variable, N-component eigenvalue equation of the form

$$W'(x;\lambda) = A(x;\lambda)W(x;\lambda), \qquad W \in \mathbb{C}^N, \tag{1}$$

$\lambda \in \Lambda \subset \mathbb{C}$ simply connected, for which the matrix A has the properties

1. $\lim_{x\to\pm\infty} A(x;\lambda) = A_\pm(\lambda)$, with convergence at a sufficiently high (e.g., integrable) rate.

2. The two matrices $A_\pm(\lambda)$ have no center subspace and have stable/unstable subspaces of common dimensions k, $N-k$.

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Under these hypotheses, equation (1) has a k-dimensional stable and $N-k$-dimensional unstable manifold at both positive and negative spatial infinity; moreover, the stable manifold $\mathcal{S}^+$ at $+\infty$ and the unstable manifold $\mathcal{U}^-$ at $-\infty$ may be spanned by locally analytic (in λ) bases $W_+^1, \dots, W_+^k$ and $W_-^1, \dots, W_-^{N-k}$, respectively (see, e.g. [1, 4, 8]).

As discussed in [1, 4], such equations arise naturally in the study of stability of traveling wave solutions of general nonlinear evolution equations $u_t = \mathcal{F}(u)$, where (1) is the eigenvalue equation $(L-\lambda)w = 0$ for the linearized operator $L := D\mathcal{F}|_{\bar{u}}$ about an asymptotically constant traveling wave solution $u \equiv \bar{u}(x)$ (in coordinates so that its speed is zero), and Λ is typically the unstable open complex halfplane $\mathbb{C}^+ := \{\lambda : \, Re\,\lambda > 0\}$ or some portion of its closure; indeed, the boundary of the set on which assumption 2 holds is contained in the essential spectrum of L, hence encroachment of this boundary into the unstable open halfplane already implies a form of instability. The question of interest, then, is to determine the set of "eigenvalues" $\lambda \in \Lambda$ for which (1) possesses one or more bounded solutions (generalized eigenfunctions), i.e., those λ for which $\mathcal{S}^+$ and $\mathcal{U}^-$ have nontrivial intersection. Since solutions either decay or grow exponentially as $x \to \pm\infty$, these λ in applications to stability indeed correspond to eigenvalues of the operator L, and existence of such in the unstable half-plane corresponds to *linearized exponential instability.* Likewise, nonexistence on the closed, deleted unstable half-plane $\{\lambda : \, Re\lambda \geq 0\} \setminus \{0\}$ typically corresponds to *linearized and nonlinear orbital stability* [1]; in particular, this is true in the case of viscous shock waves, as shown in [8].

For the purpose of determining existence/nonexistence of eigenvalues in Λ, a useful tool is the *Evans function*

$$D(\lambda) := \det\Big(W_+^1(0;\lambda), \dots, W_+^{k^+}(0;\lambda), W_-^1(0;\lambda), \dots, W_-^{N-k}(0;\lambda)\Big), \qquad (2)$$

a Wronskian measuring the (solid) angle between $\mathcal{S}^+$ and $\mathcal{U}^-$; evidently, eigenvalues λ of (1) correspond to zeroes of $D(\cdot)$. The Evans function $D(\cdot)$ as defined in (2) is clearly locally analytic; with more care it can in fact be defined in *globally analytic* fashion on all of Λ; see, e.g., the discussion of Section 2, [4]. This suggests calculation of the winding number of $D(\cdot)$ around $\partial\Lambda$ or some appropriate subset as an efficient method for determining existence/nonexistence of eigenvalues. Such a method was recently carried out by Brin for a rather stiff class of systems in [2] using an efficient shooting method to numerically approximate the values of $D(\cdot)$. However, as specifically implemented in [2], this method made use of the additional property

3. $A_\pm(\lambda)$ has a full set of eigenvalue–eigenvector pairs $\alpha^i_\pm(\lambda)$, $V^i_\pm(\lambda)$, with $\alpha^i_\pm(\cdot)$ and $V^i_\pm(\cdot)$ analytic on all of Λ, where $A_\pm$ is as defined in hypotheses 1–2. This additional hypothesis holds for viscous shock waves in the test case of 'artificial,' or scalar viscosity considered in [2], but not in general. For more realistic viscosities, or more general equations/traveling waves, it is overly restrictive.

The purpose of the present note is to describe an efficient and easily coded linear-algebraic subroutine augmenting the basic algorithm of Brin, with which the method extends to general equations/traveling waves, without need for any additional hypotheses beyond assumptions 1–2.

2 The Basic Algorithm of Brin

We now describe the basic algorithm of [2], as implemented under the additional hypothesis 3. Define the sets

$$E_+(\lambda) := \{\text{eigenpairs } (\alpha^i_+, V^i_+) \text{ of } A_+(\lambda) : \text{Re } \alpha^i_+ < 0, i = 1, \ldots, k\}$$
$$E_-(\lambda) := \{\text{eigenpairs } (\alpha^i_-, V^i_-) \text{ of } A_-(\lambda) : \text{Re } \alpha^i_- > 0, i = 1, \ldots, N-k\}.$$

Then, taking $M > 0$ sufficiently large, bounded solutions of (1) may be approximated for $x \geq M$ to any desired relative error by a linear combination of 'constant–coefficient' modes

$$\tilde{W}^i_+(x;\lambda) := e^{\alpha^i_+ x} V^i_+ \qquad i = 1, \ldots, k, \tag{3}$$

and similarly for $x \leq -M$; see, e.g., [1, 4, 8]. From the definition of D, it might seem, therefore, that the Evans function may be evaluated simply by solving (1) numerically once for each mode on the interval $x = \pm M$ to 0, initializing as

$$W^i_\pm(\pm M;\lambda) := e^{\pm\alpha^i_\pm M} V^i_\pm \tag{4}$$

for $M > 0$ sufficiently large, and computing the determinant (2).

However, in practice, the fastest growing mode dominates the numerical solution of the slower growing modes, and therefore the slower modes can not in practice be resolved directly [2]. Instead, the eigenvalue equation is lifted into wedge space, paralleling the analytic framework introduced in [1]. The wedge operator, $\wedge$, is defined [3, 4] on the standard basis vectors of $\mathbb{C}^m$ by the properties

$$e_i \wedge (e_j + xe_k) = e_i \wedge e_j + xe_i \wedge e_k$$
$$(e_i + xe_j) \wedge e_k = e_i \wedge e_k + xe_j \wedge e_k \qquad \text{(bilinearity)}$$

and

$$e_i \wedge e_j = -e_j \wedge e_i \qquad \text{(antisymmetry)}.$$

This definition extends to an operation

$$\wedge : \mathbb{C}^{\binom{m}{i}} \times \mathbb{C}^{\binom{m}{j}} \longrightarrow \mathbb{C}^{\binom{m}{i+j}} \tag{5}$$

by writing each vector as a linear combination of its basis vectors. A vector $v \in \mathbb{C}^{\binom{m}{i}}$ in this context is referred to as an i-form.

Given this definition, the Evans function can be reformulated as

$$D(\lambda) = \mathcal{W}_+(0;\lambda) \wedge \mathcal{W}_-(0;\lambda), \tag{6}$$

where

$$\mathcal{W}_+ := W_+^1 \wedge W_+^2 \wedge \cdots \wedge W_+^k$$

and

$$\mathcal{W}_- := W_-^1 \wedge W_-^2 \wedge \cdots \wedge W_-^{N-k}.$$

Moreover, $\mathcal{W}_\pm$ satisfy the lifted (linear) ordinary differential equations

$$\frac{d}{dx}\left[\mathcal{W}_\pm(x;\lambda)\right] = \mathcal{A}_\pm(x;\lambda)\mathcal{W}_\pm(x;\lambda) \tag{7}$$

for matrices $\mathcal{A}_\pm$ which can be determined by direct computation. It is a standard fact [1, 4, 2, 3] that the eigenvectors of $\mathcal{A}_\pm$ are given by all possible distinct wedge products formed from (possibly generalized) eigenvectors of $A_\pm$, with associated eigenvalues given by the sum of the eigenvalues of their factors.

Now, to evaluate the Evans function is just a matter of solving the two equations (7) numerically, initializing as

$$\mathcal{W}_\pm^i(\pm M;\lambda) := e^{\pm\gamma_\pm^i M}\mathcal{V}_\pm^i, \tag{8}$$

where $\mathcal{V}_+ := V_+^i \wedge \cdots \wedge V_+^k$, $\mathcal{V}_- := V_-^i \wedge \cdots \wedge V_-^{N-k}$, $\gamma_+ := \alpha_+^1 + \cdots + \alpha_+^k$, and $\gamma_- := \alpha_-^1 + \cdots + \alpha_-^{N-k}$, for $M > 0$ sufficiently large, By the discussion above, $\mathcal{V}_+$ and $\mathcal{V}_-$ are the eigenvectors of $\mathcal{A}_+/\mathcal{A}_-$ with smallest (resp. largest) eigenvalue. Thus, only the fastest growing modes in (7) are needed in the computation of the Evans function, and so it can be resolved very accurately by numerical methods.

3 Global Eigenvectors: Extension to the General Case

In the lifted framework (7), what is actually required to implement the algorithm of the previous section is not assumption 3, but only that the eigenvalues γ_+ and γ_- of $\mathcal{A}_+/\mathcal{A}_-$ of smallest (resp. largest) real part be analytic on Λ, with associated analytic eigenvectors $\mathcal{V}_+/\mathcal{V}_-$. From simplicity of $\gamma_\pm$, a consequence of assumption 2, it follows immediately that $\gamma_\pm(\cdot)$ are analytic functions of λ, with uniquely determined associated analytic eigenprojections P_+ and P_-, each of which may readily be found numerically by standard algorithms. Evidently, then, the extension of Brin's algorithm to general systems reduces to the problem of finding *globally analytic eigenvectors* (on Λ) $\mathcal{V}_\pm$ associated with known analytic eigenvalues/eigenprojections.

This is easily done locally, but nonuniqueness of eigenvectors makes the global construction nontrivial. Nonetheless, the following beautiful result of Kato [7] yields existence in all cases. For reference below, we repeat the proof here.

Proposition 1 *(Kato) Let $P(z)$ be a projection acting on a finite dimensional vector space, analytic on a simply connected domain Ω. Then there exists an invertible operator $U(z)$, analytic on Ω, so that*

$$P(z) = U(z)P(0)U^{-1}(z).$$

Proof. By assumption, $P(z)$ is an analytic projection matrix, i.e., $P^2 = P$ and P is analytic. It follows that

$$P'P + PP' = P'. \tag{9}$$

Right multiplying (9) by P gives

$$PP'P = 0. \tag{10}$$

Let Q be the commutator

$$Q := P'P - PP'. \tag{11}$$

Combining (10) and (11) we find that

$$PQ = -PP' \text{ and } QP = P'P, \tag{12}$$

hence

$$P' = QP - PQ. \tag{13}$$

Define the operators U, V to be solutions of the respective ODE

$$U' = QU; \qquad U(0) = I, \tag{14}$$

and

$$V' = -VQ; \qquad V(0) = I. \tag{15}$$

(Local existence follows by Picard iteration, global existence by unique continuation/simple connectivity of Ω). Then, direct computation yields that $VU \equiv I$, i.e. U is invertible. But, also, PU and $UP(0)$, by (13), evidently satisfy the same analytic ODE $Y' = QY$ with the same initial value $Y(0) = P(0)$, hence are equal by uniqueness of solutions. □

Remark. The result of Proposition 1 holds also for projections on infinite-dimensional Banach spaces, as can be seen by the observation that UV and the identity operator I satisfy the same analytic ODE $Y' = QY - YQ$ with the same initial data $Y(0) = I$, hence $UV = I$ by uniqueness. (Finite-dimensionality was used only to conclude this from $VU = I$).

This leads us to the following proposition regarding global eigenvectors.

Proposition 2 *For z in a simply connected domain Ω, let*

(i) $M(z)$ be an analytic matrix with simple eigenvalue $\mu(z)$.

(ii) $r(z)$ and $l(z)$ be right (column) and left (row) eigenvectors of $M(z)$ corresponding to $\mu(z)$.

(iii) $P(z) := (rl/lr)(z)$.

(iv) the matrix $Q(z)$ be defined as in (11) and the vector $R(z)$ (similarly as in (14)) be defined to satisfy

$$R' = QR; \qquad R(0) = r(0). \tag{16}$$

Then $R(z)$ is an analytically varying right eigenvector of $M(z)$ for all $z \in \Omega$.

By simplicity of μ, both μ and the associated eigenprojection P vary analyticially. As in the proof of the previous proposition, we thus find that both R and PR satisfy the same analytic ODE $Y' = QY$, with initial value $Y(0) = r(0)$, and are therefore analytic, equal, and nonvanishing on their mutual domain of existence, which by simple connectivity is all of Ω. But, $PR = R \neq 0$ implies that R is an eigenvector of M, as a nontrivial element of the one-dimensional eigenspace $Range(M)$. $\square$

Note that the proof of existence of R is constructive. It gives a method for computing the analytically varying right eigenvectors as the solution of the ordinary differential equation $R' = QR$, $R(0) = r(0)$, which can readily be solved numerically on the contour along which a winding number calculation is to be carried out. A simple λ-stepping procedure for ensuring the accuracy of the winding number calculation is discussed in [2, 3]. As for solving $R' = QR$, the top (resp. bottom) eigenvalue μ and associated eigenvectors l, r are readily calculated using the standard general complex matrix routine in EISPACK[1]. From these quantities, P and thereby Q may then be approximated (the latter by means of a difference quotient between the current value of P and the value of P one tenth the distance to the projected next step along the contour—this practice should probably be replaced by an adaptive step sizer in more general applications). The ODE $R' = QR$ is then stepped by an adaptive Runge-Kutta method. Note that the accuracy required in this computation is not high, essentially only the Rouché bound relative error < 1. Therefore, quick (yet sufficiently accurate) calculations can be made using a relatively large tolerance in the Runge-Kutta solver.

Remark. Though we did not implement this, accuracy might be improved by a final corrector step projecting $R \to PR$, to guarantee that R remains an eigenvector. Analytically, there is no difference between R and PR.

4 The Gap Lemma: Extension Into the Essential Spectrum

The modified algorithm described up to this point already suffices to treat most applications in stability of traveling waves, specifically, those for which

[1]Available at http://www.netlib.org/eispack

the linearized operator L about the traveling wave (see Section 1) has a *spectral gap* between its essential spectrum and the closed unstable complex halfplane $\overline{\mathbb{C}^+} = \{\lambda : Re\lambda \geq 0\}$. By the discussion in Section 1, assumptions 1–2 are then valid on the closure of $\Lambda := \mathbb{C}^+ \cup B(0,r)$ for $r > 0$ sufficiently small, allowing accurate numerical approximation of the winding number of $D(\cdot)$ about $\partial\Lambda$. This is always at least *one*, due to the presence of an eigenvalue at $\lambda = 0$ corresponding to translation invariance of the underlying PDE (also the reason we do not simply calculate winding number on $\partial\mathbb{C}^+$). If no other eigenvalues are detected, either on the boundary of Λ or its interior, then one may generally (in particular, for sectorial operator L) conclude linearized and nonlinear orbital stability: otherwise, linearized and (an appropriately defined version of) nonlinear *instability*.

However, in the application of our main interest, that of viscous shock waves, the spectral gap assumption typically *fails*, necessitating further discussion/elaboration of the method. Following [4], we divide into two cases. The first, simpler case is that of *totally compressive* shock waves in which all characteristics enter the shock: this includes both the scalar case and the 2×2 overcompressive case to be considered here. For our present purposes, the relevant property of this case is that, even though assumption 2 may fail at some point on the imaginary axis (namely, $\lambda = 0$ in the case of shock waves), i.e. some eigenvalue of $A_\pm(\lambda)$ becomes purely imaginary, a spectral gap is still maintained between the top k and the bottom $N - k$ eigenvalues (ordered by real part) of $A_\pm(\lambda)$. In this case, the eigenvalues $\gamma_\pm$ of $\mathcal{A}_\pm$ remain simple up to (and beyond) the offending point, and we may carry out the algorithm as described previously, with no further modification.

The second, and more physically relevant case, as described in [4, 8] is the one for which at least one characteristic leaves the shock: from the present, abstract point of view, the case for which at least one of the eigenvalues $\gamma_\pm$ collides with another eigenvalue of $\mathcal{A}_\pm$ as λ approaches some point of the imaginary axis ($\lambda = 0$, in the case of viscous shock waves). Nonetheless, it was shown in [4] using the "Gap Lemma" together with a bifurcation analysis of coalescing eigenvalues of $A_\pm(\lambda)$ that, for viscous shock waves, both $\gamma_\pm$ and their associated (lifted) eigenvectors $\mathcal{V}_\pm$ may be extended analytically through such points to determine an analytically extended Evans function. And, as shown in [8], it is this, extended function that is relevant to orbital stability of the wave.

Thus, this case too should be easily treatable in the same way, this time via *analytic extrapolation* of $\gamma_\pm$, $\mathcal{V}_\pm$ using values already calculated in the interior of Λ (resolvent set of L). The numerical approximation of D given $\gamma_\pm$, $\mathcal{V}_\pm$ is numerically well posed, by the same estimate used to prove the Gap Lemma; see [2, 3]. However, a simpler approach, and one which appears to give good results, is to instead compute the winding number around $\mathbb{C}^+$ of the "deflated" Evans function $\tilde{D}(\lambda) := D(\lambda)/\lambda^k$, for which the zero at the origin has been removed (taking care to avoid the origin). It is this, second method that we have used in practice for the general case. For the 2×2 overcompressive case discussed in the following section, we have carried out computations both ways, obtaining consistent results. For comparison with [2, 3], we present here the "undeflated" calculation, computed around the boundary of $\Lambda := \mathbb{C}^+ \cup B(0, r)$ (r small).

5 Viscous Shock Wave Calculation

In this section, we summarize the main results of [1, 2, 3, 4, 6, 8] necessary for the numerical determination of the stability of viscous shock waves. Careful details are only given in the case of the energy estimate involved since this is a necessary extension of previous calculations [2].

The associated linearized operator of the general viscous conservation law

$$u_t + f(u)_x = (B(u)u_x)_x \qquad u, f \in \mathbb{R}^l \tag{17}$$

is

$$Lw = -(\tilde{A}(x)w)' + (B(x)w')' \tag{18}$$

for a certain matrix $\tilde{A}(x)$ [4] which depends on f and B. As a first order system, the eigenvalue equation $Lw = \lambda w$ is given by

$$W'(x;\lambda) = A(x;\lambda)W(x;\lambda) \tag{19}$$

for another matrix $A(x;\lambda)$ [4] which depends on $\tilde{A}$ and B. The Evans function associated with (19) is well-defined [1].

In [8], it is shown that viscous shock wave solutions, i.e., traveling wave solutions with $\lim_{\xi\to\pm\infty} u(\xi) = u_\pm$, of (17) are L^p orbitally stable if and only if the Evans function associated with (19) has precisely k zeroes in $\{Re\lambda \geq 0\}$ where k is the dimension of the stationary manifold of viscous shock wave solutions.

In order to make use of this result, an energy estimate is used to find a bounded region, $\Omega \subseteq \{Re\lambda \geq 0\}$, where the eigenvalues with nonnegative real part must lie. Then, the number of zeroes of the Evans function in Ω is found by applying the argument principle numerically, and is then compared to k. There are certain surmountable technical difficulties involved in this program [3], but generally the numerical determination of viscous shock wave stability is dependable.

In [2], an energy estimate for the case of identity viscosity (i.e., $B = I$) is shown. Extending this result, let $B(u)$ be a nonidentity (possibly nonconstant) matrix. In keeping with [4], let $B(u)$ be positive definite, i.e. $B(u) = (P^t P)(u)$. Then rewrite the eigenvalue equation as $-(Bw')' + \lambda w = -(Aw)'$, multiply on the left by w^* and integrate over $\mathbb{R}$:

$$\begin{aligned} -\int w^*(Bw')' + \lambda \int w^* w &= -\int w^*(Aw)' \\ \int (w^*)' Bw' + \lambda \|w\|_{L^2}^2 &= \int (w^*)' Aw \\ \int (Pw')^*(Pw') + \lambda \|w\|_{L^2}^2 &= \int (w^*)' Aw \\ \|Pw'\|_{L^2}^2 + \lambda \|w\|_{L^2}^2 &= \int (w^*)' Aw. \end{aligned}$$

Taking real parts on both sides and the magnitude on the right gives

$$\begin{aligned} \|Pw'\|_{L^2}^2 + Re(\lambda)\|w\|_{L^2}^2 &\leq \left| \int (w^*)' Aw \right| \\ &\leq \int |(w^*)' Aw| \\ &\leq \int |A|_{\ell^2} |(w^*)'| \cdot |w| \\ &\leq \|(|A|_{\ell^2})\|_{L^\infty} \cdot \|w'\|_{L^2} \cdot \|w\|_{L^2}. \end{aligned}$$

Therefore,

$$Re(\lambda)\|w\|_{L^2}^2 \leq \|(|A|_{\ell^2})\|_{L^\infty} \cdot \|w'\|_{L^2} \cdot \|w\|_{L^2} - \|Pw'\|_{L^2}^2. \tag{20}$$

Taking imaginary parts and magnitude on both sides plus multiplying by $c > 0$ (similarly) gives

$$c|Im(\lambda)| \cdot \|w\|_{L^2}^2 \leq c\|(|A|_{\ell^2})\|_{L^\infty} \cdot \|w'\|_{L^2} \cdot \|w\|_{L^2}. \tag{21}$$

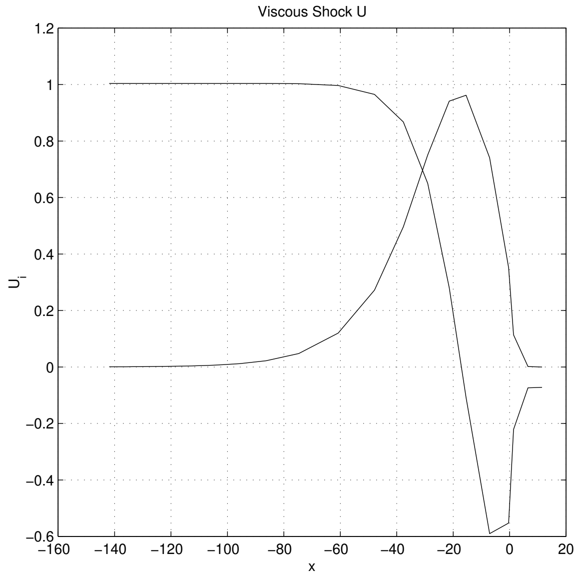

Figure 1: Viscous shock wave for the modified cubic model.

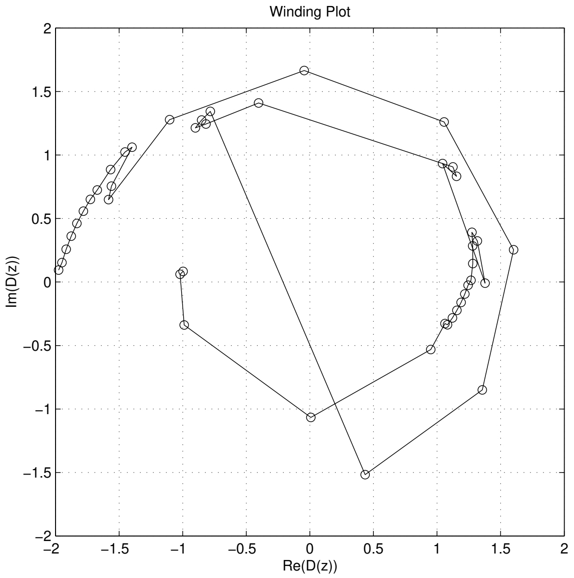

Figure 2: $D(\partial\Omega)$ normalized to show the winding number.

Adding (20) and (21) gives

$$
\begin{aligned}
Re(\lambda) + c|Im(\lambda)| &\leq \frac{(1+c)\|(|A|_{\ell^2})\|_{L^\infty} \cdot \|w'\|_{L^2} \cdot \|w\|_{L^2}}{\|w\|_{L^2}^2} - \frac{\|Pw'\|_{L^2}^2}{\|w\|_{L^2}^2} \\
&\leq \frac{\frac{1+c}{2}\|(|A|_{\ell^2})\|_{L^\infty}\left(d\|w'\|_{L^2}^2 + \frac{1}{d}\|w\|_{L^2}^2\right)}{\|w\|_{L^2}^2} - \frac{\|Pw'\|_{L^2}^2}{\|w\|_{L^2}^2}
\end{aligned}
$$

for any value $d > 0$. For the specific choices $c = 0$ and

$$d = \frac{2\|Pw'\|_{L^2}^2}{\|(|A|_{\ell^2})\|_{L^\infty}\|w'\|_{L^2}^2},$$

we have

$$Re(\lambda) \leq \left(\frac{\|(|A|_{\ell^2})\|_{L^\infty}\|(|P^{-1}|_{\ell^2})\|_{L^\infty}}{2}\right)^2. \tag{22}$$

And for the specific choices $c = 1$ and

$$d = \frac{\|Pw'\|_{L^2}^2}{\|(|A|_{\ell^2})\|_{L^\infty}\|w'\|_{L^2}^2},$$

we have

$$Re(\lambda) + |Im(\lambda)| \leq \left(\|(|A|_{\ell^2})\|_{L^\infty}\|(|P^{-1}|_{\ell^2})\|_{L^\infty}\right)^2. \tag{23}$$

Inequalities (22) and (23) place the zeroes (with nonnegative real part) in a truncated wedge [3]. For simplicity, though, we take Ω to be a rectangle in the right half plane, the left side of which lies on the imaginary axis and passes through the origin. However, the Evans function has a zero at the origin with order equal to the dimension of the viscous shock wave manifold, k [4]. So one can not apply the argument principle to $D(\lambda)$ over this domain. To circumvent this issue, we include a neighborhood of the origin in Ω, a move that brings up technical difficulties overcome in [4, 6].

As a sample calculation, take the modified cubic model

$$u_t + (|u|^2 u)_x = \begin{pmatrix} 1 & 0.05 \\ 0.1 & 1 \end{pmatrix} u_{xx} \tag{24}$$

for which figure 1 shows a particular viscous shock wave. Note that only 1 of every 30 points in the discretization of the viscous shock wave are shown in the figure making it look nondifferentiable when of course it is smooth. Figure 2 shows the image $D(\partial\Omega)$ normalized to highlight the winding number. From the figure, it is clear that the winding number is 2. Since the dimension k of the viscous shock wave manifold for this model is also 2, this particular viscous shock wave is stable. These calculations were done using the Riemann Problem Package [5].

The results, and calculations may be compared with those carried out in [2, 3] for the same model with artificial viscosity:

$$u_t + (|u|^2 u)_x = u_{xx}. \tag{25}$$

As expected, all computed values remain quite close, despite the difference in the underlying algorithms.

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References

[1] Alexander, J.; Gardner, R.; Jones, C. K. R. T., *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math., 410 (1990), 167–212.

[2] Brin, L. Q., *Numerical Testing of the Stability of Viscous Shock Waves.* Ph.D. dissertation, Indiana University, May 1998.

[3] Brin, L. Q., *Numerical testing of the stability of viscous shock waves*, Math. Comp., 70 (2001), 1071-1088.

[4] Gardner, R.; Zumbrun, K., *The gap lemma and geometric criteria for instability of viscous shock profiles*, Comm. Pure Appl. Math, 51(7) (1998),797-855,

[5] Isaacson, I.; Marchesin, D.; Plohr, B., *Riemann problem package*, Unpublished software, available at `ftp://ftp.ams.sunysb.edu/pub/rp`.

[6] Kapitula, T.; Sandstede, B., *Stability of bright solitary-wave solutions to perturbed nonlinear Schrodinger equations*, Physica D, 15 (1998), 2757-2762.

[7] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, (1985).

[8] Zumbrun, K.; Howard, P., *Pointwise semigroup methods and stability of viscous shock waves*, Indiana U. Math. J., 47(3) (1998), 741–871.

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