KURAMOTO-SIVASHINSKY APPROXIMATION FOR THE TWO-PHASE FLOW OF A DUSTY GAS

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Abstract

Initial and initial-boundary value problems for the Kuramoto-Sivashinsky model, describing two-phase flows of a dusty gas, are considered. Existence and uniqueness of global strong solutions for small initial data are proved.

1 Introduction

Historically, interest in two-phase flows of a dusty gas (i.e., gas or fluid with suspended particles) dates from the 1940s. There were engineering troubles concerned with the loss of jet thrust of missiles constructed using the solid fuel. Flame fronts, containing dust particles after fuel combustion, created (by hypothesis) some kind of barriers which made smooth flow in an engine nozzle doubtful. Later, in laboratory experiments, it was observed that adding dust to a gas flowing turbulently through a pipe results in a reduction in the pressure gradient required to maintain the flow at its original rate, [8, 20]. In the following years, there were a number of mathematical works related to this phenomenon, [3,4,5,6,9,12,15,16,18]. Unfortunately, a complete mathematical analysis was lacking, namely global-in-time solvability questions of the corresponding mathematical models. Such analysis seems indispensable because nowadays multiphase models of "medium - solid particles" type include not only dusty gases, but also other important phenomena of physical or engineering interest such as enhanced oil recovery, ecological catastrophes, creation of new materials and technologies, and others, [15].

In this paper we propose to model the carrier phase of a dusty gas by the Kuramoto-Sivashinsky (KS) equation (2.1), which is widely used in the theory

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of viscous turbulent flow and in the study of propagation of flame fronts, [10, 19]. The latter ones provide classical examples of dusty media. The particles are modelled by the system of first-order nonlinear hyperbolic equations (2.2) and (2.3) coupled with (2.1) by the right-hand side term. Thus we have a nonlinear parabolic-hyperbolic system modelling flow of a viscous liquid with solid particles.

Generally speaking, it is known that nonlinear hyperbolic equations, do not possess solutions that are regular globally in time, [17]. On the other hand, the presence of a linear damping makes it possible to prove existence of global regular solutions for small initial data, [14]. The system (2.1)-(2.3) also contains an implicit linear damping in (2.2), namely the term $K(u-v)$, that guarantees global solvability of the mixed problem and of the Cauchy problem for (2.1)-(2.3) provided the initial data are sufficiently small in norms of (2.6). This stipulates that the constant λ in (2.6) is less then $K/7$, where $K>0$ defines the dissipativity of (2.2). To illustrate this crucial condition, let us consider the following Cauchy problem (see also [7]):

$$u_t + Ku = u^2, \qquad K > 0,$$

$$u(0) = u_0 > 0.$$

If $K = 0$, the solution is

$$u(t) = u_0/(1 - u_0 t)$$

which blows up at $t = 1/u_0$. If K is a positive constant, then

$$u(t) = Ke^{-Kt}u_0/(K - (1 - e^{-Kt})u_0),$$

and it can be seen that for $K \geq u_0$, $u(t)$ is smooth for all $t > 0$,. However, for $K < u_0$ at instant $t = (1/K)ln\big(u_0/(u_0 - k)\big)$ the solution blows up.

From the physical point of a view, growth of the initial data means destabilization of the "fluid - particles" system which is stabilized by viscosity (the term νu_{xxxx}) and by friction (the terms $K(u-v)$ and αu). The dissipative term αu in (2.1) is needed only for the Cauchy problem. For the mixed problem one can omit it, replacing by a relation between ν and μ which is due to the destabilizing effect of μu_{xx} and by the dissipativeness of νu_{xxxx}.

There are few published results concerning the mathematical correctness of dusty gas models. We chose the KS equation for a liquid phase in order to prove existence of global strong solutions, because for a fluid governed by the Euler equations, we proved in [5] that the Cauchy problem does not have even solutions local in time within classes of functions with a finite number of derivatives. For liquid flow governed by the Navier-Stokes system, the Cauchy problem for the liquid-particles model admits existence of strong local solutions, [4]. We do not know any published results in the theory of two-phase flow of dusty gas which used the KS equation to model the liquid phase.

Our goal is to study well-posedness of the mixed problem and the Cauchy problem for this model. Here we prove the existence and uniqueness of strong global solutions for small initial data. The method of successive approximations, compactness arguments and continuation of local solutions are used.

2 Main results

For $T > 0$, let $Q = \{(x,t) : \ x \in \Omega, \ t \in (0,T)\}$ where $\Omega \subseteq \mathbb{R}$ is either the interval $\Omega = (0,1)$ for the mixed problem or the line $\Omega = \mathbb{R}$ for the Cauchy problem. In Q we consider the following problem:

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxxx} + \alpha u = mK(v-u), \tag{2.1}$$

$$v_t + vv_x = K(u-v), \tag{2.2}$$

$$m_t + (mv)_x = 0, \tag{2.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad m(x,0) = m_0(x) \geq 0, \tag{2.4}$$

$$\left.\begin{array}{l} u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0, \\ v(0,t) = v(1,t) = 0, \end{array}\right\} \text{ if } \Omega = (0,1). \tag{2.5}$$

Here u and v are velocities of the medium and of solid particles respectively; m is the concentration of particles; μ, ν and α are positive constant viscosity and friction coefficients, and $K > 0$ is the constant coefficient of phase interaction. The force $mK(u-v)$ is equivalent to the net effect of the dust on the gas. For example, if spherical dust particles of radius R are used in the model, K is given by the Stokes drag formula as $6\pi\mu R$.

Remark 1 *In the case of the bounded domain $\Omega = (0,1)$, we use for the viscous fluid in (2.1) the sticking condition: $u = 0$ at the walls $x = 0, 1$, which is natural for viscous flows. On the other hand, putting $u = 0$ in the hyperbolic equation (2.2), it can be seen that the lines $x = 0, 1$ are the characteristics of*

(2.2), and along these lines we have $v(0,t) = v(1,t) = 0$. Physically, this fact means that if a solid particle meets the walls $x = 0, 1$ with velocity equal to zero, it remains at the same place and has zero velocity for all $t > 0$. The condition $u_{xx}(0,t) = u_{xx}(1,t) = 0$ is common for this type of problems (see, for instance, [13]), and it can be replaced by $u_x(0,t) = u_x(1,t) = 0$. This justifies the formulation of the mixed problem (2.1)-(2.5).

We define a real λ as follows:

$$\begin{aligned} \lambda = & \left[\|v_0'\|^2 + K(\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2)/2\alpha + K(\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2)/2\nu\right]^{1/2} \\ & + \left[\|v_0''\|^2 + K(\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2)/\nu\right]^{1/2}. \end{aligned} \tag{2.6}$$

Hereafter all the norms $\|\cdot\|$ are in $L^2(\Omega)$.

The main results of this paper are the following.

Theorem 1 *Let $\Omega = (0,1)$, $0 < \mu < \min\{\alpha, \nu\}$, $K > 0$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $m_0 \in H^1(\Omega)$. If $\lambda < K/7$, then for all $T > 0$ the problem (2.1)-(2.5) has a unique strong solution:*

$$\begin{aligned} & u \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0,T; H^4(\Omega) \cap H_0^1(\Omega)), \\ & u_t \in L^2(0,T; L^2(\Omega)), \\ & v \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)), \\ & v_t \in L^\infty(0,T; H^1(\Omega)), \\ & m \in L^\infty_{loc}(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^1(\Omega)), \\ & m \geq 0, \\ & m_t \in L^\infty_{loc}(0,T; L^2(\Omega)). \end{aligned} \tag{2.7}$$

Theorem 2 *Let $\Omega = \mathbb{R}$, $0 < \mu < \min\{\alpha, \nu\}$, $K > 0$, $u_0 \in H^2(\Omega)$, $v_0 \in H^2(\Omega)$ and $m_0 \in H^1(\Omega) \cap L^1(\Omega)$. If $\lambda < K/7$, then the Cauchy problem (2.1)-(2.4) has a unique strong solution satisfying (2.8) for all $T > 0$.*

Scheme of the Proof. We prove these theorems simultaneously in the following way. First, by exploiting the method of successive approximations, we construct a local solution. Then, using a priori estimates and small initial data (in norm (2.6)), we extend the local solution to the whole interval $(0,T)$.

3 Preliminary results

Lemma 1 *For any $w_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f(x,t) \in L^2(0,T;H^2(\Omega))$ there is a $t_1 > 0$ such that there exists a unique solution $w \in L^\infty(0,t_1;H^2(\Omega))$ of the problem*

$$\begin{aligned} &w_t + ww_x = K(f - w), \\ &w(x,0) = w_0(x), \\ &w(0,t) = w(1,t) = 0, \quad if\ \Omega = (0,1) \end{aligned}$$

which satisfies the inequality

$$\|w\|^2_{H^2(\Omega)}(t) \leq C_v \left(\|w_0\|^2_{H^2(\Omega)} + K \int_0^t \|f\|^2_{H^2(\Omega)}(\tau)\, d\tau \right),\ t \in (0,t_1). \tag{3.1}$$

Proof. The proof can be found in [4], p. 954.

□

Lemma 2 *Let $L > 0$ be fixed. For any $p_0(x) \in H^2(-L,L) \cap H_0^1(-L,L)$ and $f(x,t) \in L^2\big((-L,L) \times (0,T)\big)$, there exists a unique solution of the problem*

$$p_t + pp_x + \mu p_{xx} + \nu p_{xxxx} + \alpha p = f(x,t), \tag{3.2}$$

$$p(x,0) = p_0(x), \tag{3.3}$$

$$p(-L,t) = p(L,t) = p_{xx}(-L,t) = p_{xx}(L,t) = 0 \quad if\ \Omega = (-L,L) \tag{3.4}$$

such that $\forall\, t \in (0,T)$

$$\|p\|^2_{H^2(\Omega)}(t) + \int_0^t \|p\|^2_{H^4(\Omega)}(\tau)\, d\tau \leq C_u \left(\|p_0\|^2_{H^2(\Omega)} + \int_0^t \|f\|^2_{L^2(\Omega)}(\tau)\, d\tau \right). \tag{3.5}$$

Proof. The proof of Lemma 2 can be found in [2]. The estimate (3.5) does not depend on L, so we consider the problem (3.2)-(3.4) in any interval $(-L,L)$, $L > 0$ and then, passing to the limit as $L \to \infty$, we obtain a solution of the Cauchy problem (3.2), (3.3) (see, for instance, [11]).

□

Lemma 3 *Let*

$$\begin{aligned} &a(x,t) \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right), \\ &b(x,t) \in L^\infty\left(0,T;H^1(\Omega)\right), \\ &f(x,t) \in L^2\left(0,T;H^1(\Omega)\right), \\ &q_0(x) \in H^1(\Omega). \end{aligned}$$

Then there exists a unique solution

$$q(x,t) \in L^\infty\left(0,T; H^1(\Omega)\right) \tag{3.6}$$

of the following problem:

$$q_t + aq_x + bq = f, \tag{3.7}$$

$$q(x,0) = q_0(x), \tag{3.8}$$

$$q(0,t) = q(1,t) = 0 \quad if \quad \Omega = (0,1). \tag{3.9}$$

Proof. In fact, multiplying (3.7) by q, integrating over $Q_t = \Omega \times (0,t)$ and using Gronwall's lemma, we obtain

$$\|q\|^2(t) \le \left(\|q_0\|^2 + \|f\|^2_{L^2(Q)}\right) e^{C_1 T},\ t \in (0,T), \tag{3.10}$$

where C_1 is a positive constant.

Differentiating (3.7) with respect to x, multiplying by q_x and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|q_x\|^2(t) + \int_\Omega \left(\frac{1}{2}a_x q_x^2 + \frac{1}{2}(aq_x^2)_x + b_x q q_{xx} + bq_x^2\right) dx = \int_\Omega f_x q_x \, dx. \tag{3.11}$$

Notice that

$$\begin{aligned}\left|\int_\Omega b_x q q_x \, dx\right| &\le \sup_\Omega |q|(t)\|b_x\|(t)\|q_x\|(t) \\ &\le C\sqrt{\|q\|^2 + \|q_x\|^2} \cdot \|q_x\| \le C(\|q\|^2 + \|q_x\|^2),\end{aligned}$$

where the constant $C > 0$ does not depend on t. Therefore, integrating (3.11) over $(0,t)$ and using the Cauchy and Gronwall inequalities, we conclude that

$$\|q_x\|^2(t) \le \left(\|q_0'\|^2 + \|f_x\|^2_{L^2(Q)}\right) e^{C_2 T}, \quad t \in (0,T). \tag{3.12}$$

Estimates (3.10) and (3.12) imply (3.6).

□

4 Local solution

Let $u^0 = 0$. For $n \ge 1$, $n \in \mathbb{N}$, we define approximations u^n, v^n and m^n as solutions of the following problem:

$$u_t^n + u^n u_x^n + \mu u_{xx}^n + \nu u_{xxxx}^n + \alpha u^n = m^n K(v^n - u^{n-1}), \tag{4.1}$$

$$v_t^n + v^n v_x^n = K(u^{n-1} - v^n), \tag{4.2}$$

$$m_t^n + (m^n v^n)_x = 0, \tag{4.3}$$

$$u^n(x,0) = u_0(x), \quad v^n(x,0) = v_0(x), \quad m^n(x,0) = m_0(x), \tag{4.4}$$

$$\left.\begin{array}{l} u^n(0,t) = u^n_{xx}(0,t) = u^n(1,t) = u^n_{xx}(1,t) = 0, \\ v^n(0,t) = v^n(1,t) = 0, \end{array}\right\} \text{ if } \Omega = (0,1). \qquad (4.5)$$

By Lemma 1, we conclude that for any $v_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ there exists $t_1 > 0$ such that for all $0 < t < t_1$, equation (4.2) with initial and boundary conditions (4.4) and (4.5) (in the case of the mixed problem) has a unique solution such that

$$\|v^n\|^2_{H^2(\Omega)}(t) \le C_v \left(\|v_0\|^2_{H^2(\Omega)} + K \int_0^t \|u^{n-1}\|^2_{H^2(\Omega)}(\tau)\, d\tau \right), \qquad (4.6)$$

where the constant C_v does not depend on v^n.

The approximations $m^n(x,t)$ for $t \in (0,t_1)$ can be found by formula [17], p. 141:

$$m^n(x,t) = m_0\big(y^n(0;x,t)\big) \exp\left\{ -\int_0^t \frac{\partial v^n}{\partial x}\big(y^n(\tau;x,t),\tau\big)\, d\tau \right\} \ge 0, \qquad (4.7)$$

where $y^n(\tau;x,t)$ is a solution of the Cauchy problem

$$\frac{dy}{d\tau} = v^n(y,\tau); \quad y(t;x,t) = x,$$

which is defined for every $v^n \in L^\infty(0,t_1;H^2(\Omega))$.

Setting $f(x,t) = Km^n(v^n - u^{n-1})$ in (3.2) and taking into account Lemma 2, (4.6) and (4.7), we conclude from (4.1) that $\forall t \in (0,t_1)$:

$$\begin{aligned} &\|u^n\|^2_{H^2(\Omega)}(t) + \int_0^t \|u^n\|^2_{H^4(\Omega)}(\tau)\, d\tau \\ &\qquad \le C_u \left(\|u_0\|^2_{H^2(\Omega)} + \int_0^t \|Km^n(v^n - u^{n-1})\|^2_{L^2(\Omega)}(\tau)\, d\tau \right), \qquad (4.8) \end{aligned}$$

where the constant C_u does not depend on u^n.

Thus, all the approximations u^n, v^n and m^n are defined on $(0,t_1)$ and (4.6)-(4.8) hold.

The next step is to show that these approximations are uniformly bounded in $n \in \mathbb{N}$ on some interval $(0,t_2)$. Inequality (4.8) implies

$$\begin{aligned} \|u^n\|^2_{H^2(\Omega)}(t) &\le \|u_0\|^2_{H^2(\Omega)} + \int_0^t \int_\Omega K^2 (m^n)^2 |v^n - u^{n-1}|^2\, dx\, d\tau \\ &\le \|u_0\|^2_{H^2(\Omega)} + 2K^2 \int_0^t \sup_\Omega |m^n|^2 \big(\|v^n\|^2 + \|u^{n-1}\|^2\big)(\tau)\, d\tau \\ &\le 2K^2 \int_0^t A^{n-1}(\tau) B^{n-1}(\tau)\, d\tau + \|u_0\|^2_{H^2(\Omega)}, t \in (0,t_1), \qquad (4.9) \end{aligned}$$

where

$$A^{n-1}(\tau) = \|m_0\|^2_{H^1(\Omega)} \exp\left\{2C_v \int_0^\tau \left(\|v_0\|^2_{H^2(\Omega)} + K\int_0^s \|u^{n-1}\|^2_{H^2(\Omega)}\, d\xi\right)^{1/2} ds\right\}$$

and

$$B^{n-1}(\tau) = \|v_0\|^2 + K\int_0^\tau \|u^{n-1}\|^2(s)\, ds + \|u^{n-1}\|^2(\tau).$$

For arbitrary $\mathcal{R} > 2\|u_0\|^2_{H^2(\Omega)}$, we have put $u^0 = 0 < \mathcal{R}$. The inductive hypothesis

$$\sup_{0\le\tau\le t_1} \|u^{n-1}\|^2_{H^2(\Omega)}(\tau) < \mathcal{R}$$

implies

$$\sup_{0\le\tau\le t_1} \|u^n\|^2_{H^2(\Omega)}(\tau) \le \|u_0\|^2_{H^2(\Omega)} + 2K^2\int_0^{t_1} F(\mathcal{R},\tau)G(\mathcal{R},\tau)\, d\tau,$$

where

$$F(\mathcal{R},\tau) = \|m_0\|^2_{H^1(\Omega)} e^{2C_v\tau\sqrt{\|v_0\|^2_{H^2(\Omega)} + K\mathcal{R}\tau}}$$

and

$$G(\mathcal{R},\tau) = \|v_0\|^2_{H^2(\Omega)} + K\mathcal{R}\tau + \mathcal{R}.$$

Consequently, there exists a real $t_2 > 0$ such that for all $n \in \mathbb{N}$

$$\sup_{0\le\tau\le t_2} \|u^n\|^2_{H^2(\Omega)}(\tau) < \mathcal{R}$$

and the estimates (4.6) and (4.7) imply that v^n and m^n are bounded uniformly in n on $(0, t_2)$.

Now we prove convergence of the approximations. The functions $U^n = u^n - u^{n-1}$, $V^n = v^n - v^{n-1}$ and $M^n = m^n - m^{n-1}$ satisfy the following problem:

$$\begin{aligned} &U^n_t + u^n_x U^n + u^{n-1}U^n_x + \mu U^n_{xx} + \nu U^n_{xxxx} + \alpha U^n \\ &= K\left(m^n V^n + v^{n-1}M^n - u^{n-1}M^n - m^{n-1}U^{n-1}\right), \end{aligned} \tag{4.10}$$

$$V^n_t + v^n V^n_x + v^{n-1}_x V^n = K(U^{n-1} - V^n), \tag{4.11}$$

$$M^n_t + (v^n M^n + m^{n-1}V^n)_x = 0, \tag{4.12}$$

$$U^n(x,0) = V^n(x,0) = M^n(x,0) = 0, \tag{4.13}$$

$$\left.\begin{aligned} &U^n(0,t) = U^n_{xx}(0,t) = U^n(1,t) = U^n_{xx}(1,t) = 0, \\ &V^n(0,t) = V^n(1,t) = 0, \end{aligned}\right\} \text{ if } \Omega = (0,1). \tag{4.14}$$

Considering (4.11) and (4.12) as equations of type (3.7) with f depending on U^{n-1}, we obtain from (4.10) by (3.5) that there exist $t_3 > 0$ and $0 < \beta < 1$ such that

$$\sup_{0\le t\le t_3} \|U^n\|_{H^2(\Omega)}(t) \le \beta \sup_{0\le t\le t_3} \|U^{n-1}\|_{H^2(\Omega)}(t).$$

Let $t_0 = \min(t_i)$, $i = 1, 2, 3$. Then a subsequence of $\{u^n\}$ converges in $L^\infty(0, t_0; H^2(\Omega))$ which implies convergence of $\{v^n\}$ and $\{m^n\}$ in $L^\infty(0, t_0)$; $H^2(\Omega))$ and in $L^\infty(0, t_0; H^1(\Omega))$ correspondingly.

The fact that the limit of the approximations is the required solution of (2.1)-(2.5) is established in the usual way, see [11].

Lemma 4 *A solution of (2.1)-(2.5), satisfying (2.8), is unique.*

Proof. To prove uniqueness of the solution obtained, we consider two solutions (u, v, m) and $(\widetilde{u}, \widetilde{v}, \widetilde{m})$ of (2.1)-(2.5). The functions $U = u - \widetilde{u}$, $V = v - \widetilde{v}$ and $M = m - \widetilde{m}$ satisfy the following problem:

$$\begin{aligned} &U_t + uU_x + \widetilde{u}_x U + \mu U_{xx} + \nu U_{xxxx} + \alpha U \\ &\quad = K\big[m(V - U) + M(\widetilde{v} - \widetilde{u})\big], \end{aligned} \tag{4.15}$$

$$V_t + vV_x + \widetilde{v}_x V = K(U - V), \tag{4.16}$$

$$M_t + vM_x + \widetilde{m}_x V + mV_x + \widetilde{v}_x M = 0, \tag{4.17}$$

$$U(x, 0) = V(x, 0) = M(x, 0) = 0, \tag{4.18}$$

$$\left.\begin{aligned} &U(0,t) = U_{xx}(0,t) = U(1,t) = U_{xx}(1,t) = 0, \\ &V(0,t) = V(1,t) = 0, \end{aligned}\right\} \text{ if } \Omega = (0,1). \tag{4.19}$$

It is easy to see that equations (4.16) and (4.17) have type (3.7) with f depending on U. Therefore, applying (3.5) to (4.15), we get

$$\|U\|^2_{H^2(\Omega)}(t) \le CK \int_0^t \|U\|^2_{H^2(\Omega)}(\tau)\, d\tau.$$

This implies that $U = 0$ and, consequently, $V = 0$ and $M = 0$.

□

5 Global solution

We need global a priori estimates to extend the local solution to the whole interval $(0, T)$. First, we define the energy function

$$E(t) = \|u\|^2(t) + \|\sqrt{m}v\|^2(t).$$

Then, multiplying (2.1) by u, (2.2) by mv, adding up the results, integrating over Q_t and taking into account (2.3), we obtain the first estimate:

$$E(t) + \int_0^t \left[\nu\|u_{xx}\|^2(\tau) + \alpha\|u\|^2(\tau) + 2K\|\sqrt{m}(u-v)\|^2(\tau)\right] d\tau \leq E(0). \quad (5.1)$$

Multiplying (2.2) by v and integrating over Q_t, we get

$$\|v\|^2(t) + K\int_0^t \|v\|^2(\tau)\,d\tau \leq \|v_0\|^2 + K\int_0^t \|u\|^2(\tau)\,d\tau. \quad (5.2)$$

Differentiating (2.2) twice with respect to x, multiplying the results by v_x and v_{xx} and integrating over Q_t, we obtain the inequalities

$$\begin{aligned}\|v_x\|^2(t) + \int_0^t (K - \sup_\Omega |v_x|)\|v_x\|^2(\tau)\,d\tau &\leq \|v_0'\|^2 + K\int_0^t \|u_x\|^2(\tau)\,d\tau \\ &\leq \|v_0'\|^2 + \frac{K}{2}\int_0^t (\|u\|^2 + \|u_{xx}\|^2)(\tau)\,d\tau \end{aligned} \quad (5.3)$$

and

$$\|v_{xx}\|^2(t) + \int_0^t (K - 7\sup_\Omega |v_x|)\|v_{xx}\|^2(\tau)\,d\tau \leq \|v_0''\|^2 + K\int_0^t \|u_{xx}\|^2(\tau)\,d\tau. \quad (5.4)$$

Lemma 5 *If $\lambda < K/7$, then*

$$\sup_\Omega |v_x(x,t)| < K/7 \quad \textit{for all } t > 0. \quad (5.5)$$

Proof. Indeed, when $t = 0$, $\sup_\Omega |v_x| \leq \|v_0'\| + \|v_0''\| \leq \lambda < K/7$. Suppose that $\sup_\Omega |v_x(x,t_*)| = K/7$ for some $t = t_* > 0$. Then (5.1)-(5.4) yield

$$\begin{aligned} K/7 = \sup_\Omega |v_x|(t_*) &\leq \|v_x\|(t_*) + \|v_{xx}\|(t_*) \\ &\leq \sqrt{\|v_0'\|^2 + KE(0)/2\alpha + KE(0)/2\nu} + \sqrt{\|v_0''\|^2 + KE(0)/\nu} = \lambda < K/7. \end{aligned}$$

This contradiction proves Lemma 5.

□

It follows from (2.3) that the concentration $m(x,t)$ satisfies the conservation law

$$\int_\Omega m(x,t)\,dx = \int_\Omega m_0(x)\,dx = C \quad \text{for all } t > 0.$$

Moreover, (4.7) implies that $m(x,t) \geq 0$.

Thus, if $\lambda < K/7$, then

$$
\begin{aligned}
&u \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap L^2(0,T;H^4(\Omega)\cap H_0^1(\Omega)),\\
&u_t \in L^2(0,T;L^2(\Omega)),\\
&v \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\\
&v_t \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&m \in L^\infty_{loc}(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)),\\
&m \geq 0,\\
&m_t \in L^\infty_{loc}(0,T;L^2(\Omega))
\end{aligned}
$$

and these inclusions do not depend on $T > 0$. This allows us to extend the local solution for all $t > 0$ (see [1] for details). Uniqueness of the global solution follows directly from Lemma 4. Theorems 1 and 2 are thereby proved.

□

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