FIFTY YEARS OF LOCAL SOLVABILITY

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Abstract

This is an updated version of a lecture presented ten years ago in Córdoba, Argentina, delivered on the occasion of the X ELAM [27] to describe part of the outstanding progress carried out by several mathematicians to address the problem of telling apart solvable from non solvable equations after the considerable surprise caused in 1957 by Lewy's celebrated example. We have added a description of the progress achieved in the last decade, notably the solution of Treves's conjecture on the local solvability of systems of co-rank one.

1 Where it all begins

The most primitive question one can ask concerning a partial differential equation is whether there exists a solution, at least locally, and not subjected to any additional condition. For ordinary differential equations, very satisfactory theorems stating local existence of solutions under very mild hypotheses of regularity had been known since long ago, and it came as a surprise when Hans Lewy published in 1956 his now famous example of a first-order linear equation whose coefficients are polynomials of degree at most one, failing to have local solutions. Indeed, if $f \in C^\infty(\mathbb{R}^3)$ is conveniently chosen, the equation

$$(\partial_x + i\partial_y - (x+iy)\partial_z)u = f, \qquad (x,y,z) \in \mathbb{R}^3,$$

does not have distribution solutions in any open subset of $\mathbb{R}^3$ ([39]).

For elliptic equations local solvability was known and Hörmander had proved in his thesis that linear operators of real principal type were locally solvable. Let us recall some definitions and notation. A linear partial differential operator in an open subset Ω of $\mathbb{R}^n$ has the form

$$P(x,D)u = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u, \qquad u \in C_c^\infty(\Omega), \tag{1}$$

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where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ denotes a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ its length, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $D_j = \partial/i\partial x_j$. The principal symbol of $P(x, D)$ is the function

$$p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \qquad x \in \Omega, \qquad \xi \in \mathbb{R}^n.$$

Here, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. If we interpret $p_m(x, \xi)$ as defined on the cotangent bundle $T^*(\Omega)$ rather than on the modest cartesian product $\Omega \times \mathbb{R}^n$, the principal symbol becomes invariantly defined under change of coordinates. A linear partial differential operator is elliptic if its principal part has no real zeros $\xi \neq 0$ (of course, $p_m(x, 0) = 0$, $x \in \Omega$, because it is a homogeneous polynomial in ξ, but these zeros, being trivial, do not count). The main difficulty in finding a right inverse for a linear PDE comes from the real nontrivial zeros of the principal part. In the elliptic case those zeros simply do not occur. The best thing next to no zeros is simple zeros. An operator is said to be of principal type if

$$p_m(x, \xi) = 0, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{implies} \quad \nabla p_m(x, \xi) \neq 0.$$

At this point it is convenient to introduce a formal definition.

Definition 1 *A partial differential operator $P(x, D)$ in $\Omega \subset \mathbb{R}^n$ is locally solvable if every point $x_0 \in \Omega$ has a neighborhood $U \subset \Omega$ such that the equation*

$$P(x, D)u = f \tag{2}$$

can be solved in $\mathcal{D}'(U)$ for every $f \in C_c^\infty(U)$.

If the coefficients of p_m are real then the operator is said to be of real principal type and by the theorems of Hörmander [20] it is locally solvable. Notice that Lewy's operator $\partial_x + i\partial_y - (x + iy)\partial_z$ is of principal type but does not have real coefficients. A few years later Hörmander ([21], [22]) generalized Lewy's example in the following way. Denote by $\overline{P}(x, D)$ the operator obtained from $P(x, D)$ by replacing each coefficient by its complex conjugate and consider the commutator

$$\begin{aligned} C(x, D) &= [P(x, D), \overline{P}(x, D)] \\ &= P(x, D)\overline{P}(x, D) - \overline{P}(x, D)P(x, D) \end{aligned}$$

which is an operator of order $2m-1$ with principal symbol $c_{2m-1}(x,\xi)$. If $P(x,D)$ is locally solvable in Ω then $c_{2m-1}(x,\xi)$ must vanish at every zero of $p_m(x,\xi)$ in $\Omega\times\mathbb{R}^n$. An operator satisfying the latter condition will be said to satisfy condition $(\mathcal{H})$. For the Lewy operator condition $(\mathcal{H})$ is violated at every point. If the coefficients of $P(x,D)$ are real or constant $c_{2m-1}(x,\xi)$ vanishes identically. This was a most remarkable advance because it explained a phenomenon that had appeared as an isolated example in terms of very general geometric properties of the symbol, an invariantly defined object. In spite of this success, it turned out that condition $(\mathcal{H})$ was not accurate enough to discriminate the solvable operators from the nonsolvable ones among some examples considered by Mizohata [41], which we now describe. Let k be a positive integer and consider the operator in $\mathbb{R}^2$ defined by

$$M_k=\frac{\partial}{\partial y}-iy^k\frac{\partial}{\partial x}.$$

If $k=1$ condition $(\mathcal{H})$ is violated at all points of the x axis so, in particular, M_1 is not locally solvable at the origin. For $k\geq 2$ condition $(\mathcal{H})$ is satisfied everywhere. On the other hand, it turns out by relatively simple arguments that M_k is locally solvable at the origin if and only if k is even ([16], [15]). The principal symbol of M_k is $m_1=-i(\eta-iy^k\xi)$. The crucial difference between k odd and k even is that in the first case the function y^k changes sign and in the second case it doesn't. Nirenberg and Treves [46] elaborated over these examples and identified a property that turned out to be the right condition for local solvability of operators of principal type. Let's write the principal symbol in terms of real and imaginary parts, $p_m(x,\xi)=a(x,\xi)+ib(x,\xi)$. A null bicharacteristic of $a(x,\xi)$ is a curve satisfying the system of ordinary differential equations

$$\begin{cases}\dot{x}=\nabla_\xi a(x,\xi), & x(0)=x_0\\ \dot{\xi}=-\nabla_x a(x,\xi), & \xi(0)=\xi_0,\end{cases}$$

with initial conditions verifying $a(x_0,\xi_0)=0$. The operator (1) satisfies condition $(\mathcal{P})$ if $b(x,\xi)$ does not change sign along any null bicharacteristic of $a(x,\xi)$ (if the projection of the bicharacteristic into Ω is singular one has to interchange the roles of a and b which amounts to multiplying $P(x,D)$ by i). This formulation is invariant by coordinate changes. When condition $(\mathcal{H})$ is violated the change of sign occurs at a simple zero of the function of one variable $t\mapsto b(x(t),y(t))$, where $(x(t),y(t))$ denotes some null bicharacteristic, while $(\mathcal{P})$

detects changes of sign at zeros of higher order, as in the Mizohata operators M_k for $k > 1$ and odd, including zeros of infinite order. For first-order operators of principal type with real analytic coefficients in the principal part they proved that $(\mathcal{P})$ is equivalent to local solvability. That was in 1963 and seven years later they were able to extend the same result to operators of arbitrary order m [47]. The proof of the sufficiency involved a microlocalization used to split the operator as a finite sum of simpler operators. Each simpler operator could be further reduced by factoring out an elliptic operator of order $m-1$. The remaining factor had order one and still verified condition $(\mathcal{P})$. It should be said, however, that the new operators introduced by microlocalization were no longer differential but pseudo-differential operators. Each first-order pseudo-differential factor was again simplified by conjugation with a Fourier integral operator. Property $(\mathcal{P})$ can be used at this stage to obtain *a priori* estimates. They are sufficiently strong to survive the microlocalization and can be patched together to obtain a local estimate for the original operators. This implies that the transpose of $P(x, D)$ is locally solvable (interchanging the roles of $P(x, D)$ and its transpose one proves that $P(x, D)$ itself is locally solvable). Three years later R. Beals and C. Fefferman [2] proved that smooth coefficients were enough to show that $(\mathcal{P})$ implies local solvability. They started from the reduced operators of Nirenberg and Treves and introduced a finer calculus of pseudo-differential operators that included the Calderón-Vaillancourt result that a pseudo-differential operator with symbol in the class $S^0_{1/2,1/2}$ is bounded in L^2 [12].

Concerning the necessity of $(\mathcal{P})$ Moyer [42] removed in 1978 the analyticity hypothesis for operators in two variables and his technique was applied by Hörmander [23] to extend the result for operators in any number of variables with smooth coefficients. Finally, the conjecture of Nirenberg and Treves that $(\mathcal{P})$ was equivalent to local solvability for operators with smooth coefficients had been proved over a span of 15 years. The technique of Nirenberg-Treves and Beals-Fefferman actually gave more than solvability in the class of distributions: if one takes $f \in H^s_c(U)$ in (2), $s \in \mathbb{R}$, then one can choose u in H^{s+m-1}, where H^s denotes the L^2-based Sobolev space of order s. This is not enough to furnish smooth solutions when $f \in C^\infty_c(U)$ because, due to technical reasons, the diameter of U shrinks as s grows in the proofs. On the other hand, Hörmander [24] gave a different proof by studying the propagation of singularities of operators that verify $(\mathcal{P})$, which allowed to obtain semi-global solutions, i.e., solutions defined on a full compact set under the geometric assumption that bicharacteristics do not get trapped in the given compact set. Furthermore, the solutions can be taken smooth if f is smooth.

2 Nonlinear equations

A fully nonlinear operator of order m acting on functions u is of the form

$$u(x) \mapsto F(x, \partial^\alpha u(x)), \qquad |\alpha| \leq m. \tag{3}$$

Two different situations arise. If $F(x, \zeta)$ is a function defined in $\Omega \subset \mathbb{R}^n \times \mathbb{R}^N$ (where N is the number of α's $\in \mathbb{Z}_+^n$ such that $|\alpha| \leq m$), only real functions u are allowed in (3). On the other hand, if $\Omega \subset \mathbb{R}^n \times \mathbb{C}^N$ and $F(x, \zeta)$ is a holomorphic function of ζ, the natural functions $u(x)$ in (3) are complex valued. At any rate, the linearization of a nonlinear operator $F(u)$ at a function u is the linear differential operator $F'(u)$ defined by

$$F'(u)v = \frac{d}{dt} F(u + tv)\Big|_{t=0}.$$

In general, an equation such as $F(x, \partial^\alpha u(x)) = f(x)$ may fail to have a solution by very simple reasons: if the set of values of F and the set of values of f are disjoint (for instance, if $F > 0$ and $f \leq 0$), there can be no solution. So a better problem is the following: given u_0 and setting $f_0(x) = F(x, \partial^\alpha u_0(x))$, try solving

$$F(x, \partial^\alpha u(x)) = f(x) \tag{4}$$

for f sufficiently close to f_0 in some topology, if possible, with u close to u_0. Now this is an implicit function problem and a very powerful tool to handle it is the Nash-Moser implicit function theorem [45], [43], [19], [30]. In order to apply it one has to construct an operator $Q(u)v$ acting on some tame scale of spaces (typically, the scale of Sobolev spaces H^s) such that

$$F'(u)Q(u)v = v, \qquad v \in H^s, \quad s > s_0, \tag{5}$$

$$\|Q(u)v\|_s \leq C_s(\|u\|_{s+r}\|v\|_{s_0} + \|u\|_{s_0}\|v\|_{s+r}) \quad u, v \in H^{s+r}. \tag{6}$$

There are also other hypotheses in the Nash-Moser theorem but they are almost automatic when dealing with differential operators like (3). Note that (5) states that the linear differential operator $F'(u)$ has a right inverse, so if $F'(u)$ is locally solvable for each u we might be able to construct some right inverse $Q(u)$. It is with condition (6) that the trouble really comes: the coefficients of the linear operator $F'(u)$ depend on u and the norm of $Q(u)$ must grow in a specific way (tamely). In order to obtain tame estimates one usually needs explicit expressions, involving pseudo-differential and Fourier integral operators (the usual techniques to get *a priori* energy estimates do not seem useful when it comes to obtaining tame estimates). Fortunately, there exists a tame calculus for pseudo-differential and Fourier integral operators that makes of them an appropriate tool for the task when they can be used ([18], [1]). In the important case in which F is smooth in all arguments and $F'(u)$ is of real principal type for every u, this approach seems the natural one to solve (4) locally with u smooth if f is smooth, which would have interesting applications, particularly to nonlinear equations arising in Riemannian geometry, however the widely circulated preprint [18] was never published.

For the case of complex equations little is known. There are results only for two variables [14], [1] and the assumptions on $F'(u)$ are considerably more stringent than simply assuming that it is locally solvable. For instance, if $b(x,t)$ is a smooth real function in $\mathbb{R}^2$ such that $t \to b(x,t)$ does not change sign, then $L = \partial_t + ib(x,t)\partial_x$ is linear and verifies $(\mathcal{P})$ but it does not seem to be known if there exists a right inverse Q for L such that $\|Qf\|_s \leq C_s\|f\|_{s+r}$ for any positive s and f smooth and supported in a *fixed* neighborhood unless $b(x,t)$ does not change sign at all ([1]). In the general case, the parametrices constructed by the methods of Treves [48], [49] or even those which are valid in a more restricted set-up [1], [26], seem to verify only the weaker estimates $\|Qf\|_s \leq C_s\|f\|_{2s+r}$ which are not enough to guarantee the convergence of the Nash-Moser scheme. For the time being, general local solvability results are only known for semilinear equations and we describe them now.

Consider the semilinear equation

$$P(x,D)u = f(x,u,\ldots,D^\alpha u,\ldots) = f(x,Du), \quad |\alpha| \leq m-1, \tag{7}$$

where $P(x,D)$ is a linear partial differential operator of order m with smooth (i.e. C^∞) complex coefficients defined on $\mathbb{R}^n$ $f(x,\zeta)$ is a smooth complex function and the right hand side in (5) only involves derivatives of order $\leq m-1$. Assume that the linear part $P(x,D)$ satisfies the Nirenberg-Treves condition $(\mathcal{P})$. Hence, if f has the special form $f(x,\zeta) = \phi(x) + L(x)\zeta$ where $\zeta \mapsto L(x)\zeta$ is linear, the local existence theorem of Beals-Fefferman establishes that there exist local solutions of (7) with arbitrarily high finite regularity. More generally, the result of Beals and Fefferman can be extended to arbitrary smooth functions f, as shown by Santiago and Hounie ([32]) using a fixed point argument. This extends previous results of Dehman, Gramchev and Popivanov.

3 Operators with nonsmooth coefficients

If $P(x, D)$ is a principal type differential operator of order m with smooth coefficients satisfying condition $(\mathcal{P})$, the Beals-Fefferman theorem allows to locally solve the equation $P(x, D)u = f$, $f \in L^2$, with $u \in H^{m-1}$. This is the optimal regularity one can expect of u if $P(x, D)$ is not hypoelliptic. A natural question is: how much regularity must one demand on the coefficients of $P(x, D)$ to obtain the same result, if they are not smooth? Counting on the fingers the number of derivatives used in the known constructive proofs of solvability gives a large number that grows with the dimension. This has the following explanation. The microlocalization technique involves the continuity in Sobolev spaces of pseudo-differential and Fourier integral operators and it is a fact of life that the number of derivatives needed to control their norms grows proportionally to the dimension n of the surrounding space.

Consider, however, a differential operator of order one with smooth complex coefficients

$$\frac{\partial}{\partial t} + \sum_{j=1}^{n} a_j(x,t)\frac{\partial}{\partial x_j} + c(x,t) \tag{8}$$

defined in a neighborhood of the origin of $\mathbb{R}^{n+1}$. After a local change of variables, it can be put in the form

$$L = \frac{\partial}{\partial t} + i\sum_{j=1}^{n} b_j(x,t)\frac{\partial}{\partial x_j} + c(x,t) \tag{9}$$

with $b_j(x, t)$ smooth and real.

The solvability in L^2 (notice that $m = 1$ in this example) for the operator (9) verifying $(\mathcal{P})$ holds if the coefficients in the principal part are Lipschitz and $c(x, t)$ is measurable [25], independently of n. Since the straightening of bicharacteristics needed to put (8) into the form (9) consumes one derivative, the former is locally solvable in L^2 if the coefficients of the principal part have Lipschitz first-order derivatives. For operators in two variables, there is a way around the straightening-up of the characteristic, and L^2 solvability holds assuming that the coefficients of the principal part are just Lipschitz, as shown by Hounie and M. E. Moraes Melo ([28]).

It is not known whether it is possible (or impossible) to bound the regularity of the coefficients for higher order operators with a constant independent of n in order to obtain solvability. A related question is to determine the mimimal number of derivatives of a positive symbol (say classical of order one) needed to guarantee that the corresponding pseudo-differential operator verifies a Gårding inequality.

4 Solvability in L^p

Consider again a principal type differential operator $P(x, D)$ of order m with smooth coefficients satisfying condition $(\mathcal{P})$. For a given $1 < p < \infty$ we say that $P(x, D)$ is locally solvable in L^p if we may locally solve the equation $P(x, D)u = f$, $f \in L^p$, with $u \in H_p^{m-1} = (I - \Delta)^{-(m-1)/2} L^p$. Here Δ denotes Laplacian so H_p^{m-1} is the L^p-based Sobolev space of order $m - 1$. This is true if $p = 2$ by the Beals-Fefferman theorem but false, in general, for $p \neq 2$ [17] even if $P(x, D)$ is subelliptic. A standard duality argument shows that the local solvability in L^p of the operator P implies a local *a priori* estimate of the form $\|u\|_{p',m-1} \leq C\|{}^tPu\|_{p'}$, for all test functions u supported in some neighborhood, where ${}^tP$ indicates the formal transpose of P. On the other hand, Littman [40] showed that an *a priori* estimate

$$\|u\|_{p,1} \leq C\|Du\|_p, \qquad u \in C_c^\infty(\mathbb{R}^n)$$

cannot hold as long as $p > 2n/(n - 1)$, if D denotes the wave operator (which is of principal type and —having constant coefficients— satisfies $(\mathcal{P})$). Then, Kenig and Tomas ([37], [38]) showed that the *a priori* estimate

$$\|u\|_p \leq C\|(D + i)u\|_p, \qquad u \in C_c^\infty(\mathbb{R}^3)$$

holds if and only if $p = 2$. For local versions of these negative results see [36] and the references given therein. Later P. Guan [17] considered the operator on $\mathbb{R}_t \times \mathbb{R}^n$

$$P = \frac{\partial^2}{\partial t^2} - \Delta_x - it^{2k}\Delta_x, \tag{10}$$

where Δ_x is the Laplace operator acting on $\mathbb{R}_x^n$ and k is a positive integer. Then, given $p > 2$, one can choose n large enough, so that P given by (10) is not solvable in L^p.

On the other hand, there is solvability in L^p for operators in two variables (any order) or of order one (any number of variables) satisfying $(\mathcal{P})$. The case of order one is due to Perdigão and Hounie, for smooth coefficients and to Maria Eulália Moraes Melo e Hounie for Lipschitz coefficients (to be precise: under the same conditions that hold when $p = 2$) ([31], [29]).

The main "explanation" for this phenomena is technical. The reduction technique of Nirenberg-Treves (present in some form in all proofs of solvability) involves a Fourier integral operator which is bounded in L^2 but not necessarily bounded in L^p for $p \neq 2$. When the order is one or the number of variables is two the use of Fourier integral operators can be avoided.

While it is true that for any locally solvable vector field L and $1 < p < \infty$ the equation $Lg = f$ can locally be solved in L^p if f is in L^p ([31], [29]), this is false, in general, for $p = \infty$ ([35]) This difficulty can be dealt with by introducing the space $X = L^\infty(\mathbb{R}_t; \mathrm{bmo}(\mathbb{R}_x))$ of measurable functions $u(x,t)$ such that, for almost every $t \in \mathbb{R}$, $x \mapsto u(x,t) \in \mathrm{bmo}(\mathbb{R})$ and $\|u(t,\cdot)\|_{\mathrm{bmo}} \leq C < \infty$ for a.e. $t \in \mathbb{R}$, where $\mathrm{bmo}(\mathbb{R})$ is a space of bounded mean oscillation functions, dual to the semi-local Hardy space $h^1(\mathbb{R})$ of Goldberg. It was shown in [3] that for a substantial subclass of the class of locally solvable vector fields L, the equation $Lu = f$ can be locally solved with $u \in X$ if $f \in L^\infty$. This result was recently improved by Evandro da Silva in his Ph. D. thesis [13] by showing that for any locally solvable vector field L the equation $Lu = f$ can be locally solved with $u \in X$ for any $f \in X$. This can be regarded as an ersatz for $p = \infty$ of the L^p local solvability valid for $1 < p < \infty$. While the original proof used pseudo-differential operators and required smooth coefficients, using instead other tools of harmonic analysis (singular integrals, commutators estimates, dyadic decompositions) it is possible [33] to extend the solvability theorem to vector fields L with C^{1+r} coefficients, $0 < r < 1$.

5 Nondetermined systems of vector fields

The solvability theorems valid for one equation of principal type can be extended to determined systems (principal type for a system means that the determinant of the matrix principal part has at most simple (nontrivial) real zeros). For nondetermined systems the theory is at a primitive stage of development and only special equations involving vector fields and related to the familiar gradient and divergence equations can be dealt with in some generality. Important classes of such equations arise naturally in the theory of holomorphic functions of several variables ($\overline{\partial}$ and $\overline{\partial}_b$ equations).

Let L_j, $j = 1, \dots, n$ be linearly independent complex vector fields defined on the open set $\Omega \subset \mathbb{R}^N$, i.e.,

$$L_j = \sum_{k=1}^{N} a_{jk}(x) \frac{\partial}{\partial x_k}, \qquad a_{jk} \in C^\infty(\Omega).$$

$$\begin{cases} LU_j &= 0, \\ U_j(x,0) &= x_j, \end{cases}$$

The basic question is to determine when an overdetermined or underdetermined equations like

$$\begin{cases} L_1 u &= f_1, \\ L_2 u &= f_2, \\ \quad\vdots & \\ L_n u &= f_n, \end{cases} \qquad f_1, \dots, f_n \in C^\infty(\Omega), \tag{11}$$

or

$$L_1 u_1 + \cdots + L_n u_n = f, \qquad f \in C^\infty(\Omega), \tag{12}$$

can be solved for any choice of the right hand side satisfying the proper compatibility conditions. For instance, if the vectors L_j commute pairwise, the compatibility conditions will be $L_j f_k = L_k f_j$ for the system (11) and void for (12). In the case of a single vector field, ($n = 1$), (11) and (12) coincide and the answer depends on condition ($\mathcal{P}$) of Nirenberg and Treves ([46]). Since it is possible to obtain equivalent equations by replacing each L_j by a linear combination of the vectors L_j, so that the corresponding matrix is non-singular, we realize that the relevant geometric object will be the vector bundle $\mathcal{L} \subset \mathbb{C} \otimes T\Omega$ generated by $L_1, \dots, L_n$. We say that a sub-bundle of $\mathcal{L} \subset \mathbb{C} \otimes T\Omega$ is involutive (or formally integrable) if it satisfies the Frobenius condition

$$[\mathcal{L}, \mathcal{L}] \subseteq \mathcal{L}. \tag{13}$$

In this case, if $\mathcal{L}^\perp$ denotes the orthogonal of $\mathcal{L}$ relative to the duality between tangent vectors and differential forms of degree 1 ($\mathcal{L}^\perp$ is a sub-bundle of the complexified cotangent bundle), the exterior derivative defines, by passage to the quotient, a complex of differential operators

$$E^p \xrightarrow{\delta_p} E^{p+1}, \qquad p = 0, 1, \dots, n-1, \tag{14}$$

where we have written

$$E^p = C^\infty(\Omega, \Lambda^p(\mathbb{C} \otimes T^*\Omega/\mathcal{L}^\perp))$$

In local coordinates, equations $\delta_0 u = f$ and $\delta_{n-1} u = f$ can be expressed in the form (11) and (12) respectively.

The cohomology groups of the sequence (14)

$$\mathcal{H}^q(\Omega, \mathcal{L}), \qquad q = 0, 1, \ldots, n$$

are called the cohomology groups associated to $\mathcal{L}$ on Ω and a natural problem of the theory is to determine when they vanish for a given structure $\mathcal{L}$. The localized problem is also relevant. If we fix a point $A \in \Omega$ we may consider the associated complex

$$C^\infty(A, \Lambda^p(\mathbb{C} \otimes T^*\Omega/\mathcal{L}^\perp)) \xrightarrow{\delta_p} C^\infty(A, \Lambda^{p+1}(\mathbb{C} \otimes T^*\Omega/\mathcal{L}^\perp)), \tag{15}$$

where $0 \le p \le n-1$ and $C^\infty(A, \Lambda^p(\mathbb{C} \otimes T^*\Omega/\mathcal{L}^\perp))$ denotes the space of germs of sections of the vector bundle $\Lambda^p(\mathbb{C} \otimes T^*\Omega/\mathcal{L}^\perp))$ at the point A. The cohomology groups of (15) are denoted by

$$\mathcal{H}^q(A, \mathcal{L}), \qquad q = 0, \ldots, n. \tag{16}$$

Results of a general nature are known only under the assumption that $\mathcal{L}$ is *locally integrable*, i.e., when $\mathcal{L}^\perp$ is locally generated by the differentials of $m = N - n$ functions of class C^∞. Little is known so far about the groups (16); there are complete descriptions in the following cases: i) when $\mathcal{L}$ defines an elliptic structure ([50]), ii) when the structure $\mathcal{L}$ is real analytic and the Levi form is non degenerate ([52]), iii) when $n = 1$ ([46]), iv) when the structure $\mathcal{L}$ is real analytic, $m = 1$ and $q = 1$ ([53]) or $q = n$ ([6]). The problem of finding solutions to (12) with compact support is also of interest ([5], [34]) and related to the extension property of Hartogs for solutions of the homogeneous equation and may shed some light on the computation of cohomology groups for other structures, like tubes ([51],[54]).

In the locally integrable case of co-rank one, i.e., when $m = 1$ and there is smooth function defined in a neighborhood of A

$$Z : U_A \longrightarrow \mathbb{C}$$

whose differential generates $\mathcal{L}^\perp/U_A$, the solvability conjecture of Treves can be stated as follows: *Given* $1 \le q \le n$, *exactness at level* q *in the complex (15), i.e.,*

$$(\bullet)_q \qquad \mathrm{Ker}\ \delta_q = \mathrm{Im}\ \delta_{q-1},$$

is equivalent to:

$(\star)_{q-1}$ *Given any open neighborhood* V *of* A *there is another open neighborhood* V' *of* A *such that, for every regular value* $z_0 \in \mathbb{C}$ *of* $Z : V \to \mathbb{C}$, *either* $Z^{-1}\{z_0\} \cap V' \doteq \mathcal{F}(z_0, V') = \emptyset$ *or else the homomorphism*

$$\tilde{H}_{q-1}(\mathcal{F}(z_0, V')) \longrightarrow \tilde{H}_{q-1}(\mathcal{F}(z_0, V))$$

induced by the inclusion map

$$\mathcal{F}(z_0, V') \subset \mathcal{F}(z_0, V)$$

vanishes identically.

Several articles have been published towards the verification of this conjecture. In [9] it is proved that $(\bullet)_q \Rightarrow (\star)_{q-1}$ for all $q = 1, \ldots, n$ while in [44] it is proved that $(\star)_0 \Rightarrow (\bullet)_1$ and in [7] that $(\star)_{n-1} \Rightarrow (\bullet)_n$. If we furthermore make the extra assumption that the structure $\mathcal{L}$ is real-analytic then the work [4] gives a complete proof that $(\star)_{q-1} \Rightarrow (\bullet)_q$ for all $q = 1, \ldots, n$, a result that had been previously proved in the cases $q = 1$ [48] and $q = n$ [6]. We also mention that it is possible to give a meaning for hyperfunction solvability of the complex $\mathcal{L}$ (which generalizes the natural concept within the real-analytic category) and to prove that condition $(\star)_{q-1}$ is in fact equivalent to the hyperfunction solvability at the origin in degree q, for every $q = 1, \ldots, n$. On this subject we refer to [10] and [11]. Finally, the proof of $(\star)_{q-1} \Rightarrow (\bullet)_q$ for all $q = 1, \ldots, n$ for the C^∞ category was very recently proved in [8], thus completing the proof of the Treves conjecture.

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