RESPONSE TO RESONANT AND NEAR-RESONANT FORCING IN A SIMPLE MODEL FOR NONLINEAR ENERGY CASCADES

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Abstract

The response to resonant and near–resonant forcing is studied in a simple model for nonlinear energy cascades. The model is the forced inviscid Burgers equation $u_t + \left(\frac{u^2}{2}\right)_x = f$. The nonlinear term represents simultaneously energy transfer and dissipation mechanisms in more complex systems. The force f is tuned to mimic resonant, near resonant or far from resonant forcing, due either to sources external to the system, or to nonlinear interaction with degrees of freedom not represented by the variable u. A rich phenomenology is found as f switches from resonant to non–resonant, including sharp phase transitions and intermittent events associated with enhanced energy transfer.

1 Introduction

Large systems in Nature, such as the Ocean and the Atmosphere, have their energy content distributed among many internal degrees of freedom or modes, such as waves and eddies. Energy typically flows into the system from external sources, which act preferably on some of its degrees of freedom, gets distributed throughout the system via nonlinear interactions, and is eventually dissipated, often through nonlinear mechanisms, such as breaking waves and turbulence. Thus the system can be said to be in a statistically steady state, though not one of thermodynamical equilibrium, but rather one characterized by permanent energy transfer among scales.

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The description of such forced and dissipated systems involves a number of mathematical challenges. A significant one is our incomplete understanding of resonant energy exchange. When a set of modes has various linear frequencies of oscillation, its modes can exchange energy efficiently only if the subsets of modes providing energy and those receiving it have approximately equal combined frequencies. Otherwise, their relative phase would oscillate rapidly, and the effective energy exchange would be greatly diminished. Perfect frequency match is denoted resonance; near–resonance and non–resonance are defined accordingly.

Two problems appear though: that the boundary between near–resonant and non–resonant behavior is somewhat vague –and dependent on the level of nonlinearity present–, and that the combined effect of many near–resonances is hard to evaluate. In this work, we describe a simple model where these issues can be explored in depth, and use it to show that the way a system responds to near–resonant forcing can be far from trivial. The model that we shall use is the forced inviscid Burgers equation:

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = f(x,\,t)\,, \tag{1}$$

where $f = f(x,\,t)$ is a smooth given function, periodic (of period 2π) in space and vanishing mean, and the solution $u = u(x,\,t)$ is also periodic and has zero mean.

Here the dependent variable $u(x, t)$ represents a mode (or set of modes) with linear frequency $\omega = 0$ (as follows from the zero mean condition.) On the other hand, the externally imposed force $f(x, t)$ represents other modes of the system, which (depending on the scale of their dependence on time) will be close or far from resonance with u.

The nonlinear term in (1) has two combined functions: to transfer energy among the various (Fourier) components of u, and to dissipate energy at shocks. Thus the "inertial cascade" of large nonlinear systems and their nonlinear dissipation are modeled by a single term. This not only implies a big gain in simplicity, but could also in fact be a rather realistic model for fluid systems, whose dissipation is almost invariably associated with some form of wave breaking.

The plan of this paper is the following. In section 2, we introduce the model, show its relation to real systems, and discuss some of its elementary properties. In section 3, we study the behavior of the solutions when the force

f is unimodal; in particular, we show that a sharp boundary divides resonant from non–resonant behavior, a boundary that can be characterized as a phase transition. Finally, in section 4, we study bimodal forcings; here the dominant feature is the appearance of intermittent events, that we denote "storms", where energy transfer is highly enhanced.

The tone of this paper is mostly descriptive, with the emphasis placed on the relevance of the striking behavior of our simple model to more general systems. For the proofs of many of its results, as well as for a more detailed account of the numerics, we refer the reader to [2].

2 The Model and its Elementary Properties.

When linearized, equation (1) becomes

$$u_t = f(x,\, t)\,. \tag{2}$$

Then the force f may be said to be resonant when it has a nonzero temporal mean, and hence makes u grow secularly. If, for concreteness, we consider forcings of the form

$$f(x,\, t) = A\,\cos(k\,x - \omega\,t)\,, \tag{3}$$

then f is resonant when $\omega = 0$, near–resonant when ω is small, and non–resonant when ω is large. We shall see below that the apparently tenuous distinction between near–resonant and far–from–resonant behavior can be made quite sharp for the model in (1).

It may appear that near–resonant behavior can be characterized at least asymptotically, through the introduction of a small parameter ϵ measuring the departure from resonance. The resulting equation should be

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \epsilon^2 f(x,\, \epsilon t)\,, \tag{4}$$

The reason for the factor ϵ^2 in front of the forcing term follows from considering a quasi–steady approximation to the solution to (4), namely:

$$u(x) \approx \epsilon\sqrt{2\int^x f(s,\, \epsilon t)\, ds}\,.$$

This indicates that a slow forcing of size $O(\epsilon^2)$ generally induces a response of amplitude ϵ in u. Hence any force stronger than ϵ^2 in (4) would render its own time modulation irrelevant, since the induced nonlinearity would act on a much faster time scale, thus effectively freezing f from a dynamical perspective.

However, the ϵ's in (4) can be scaled out by the simple transformation $\tilde{t} = \epsilon\, t$, $u = \epsilon \tilde{u}$. Hence near-resonances in (1) cannot be defined as an asymptotic limit involving a small parameter ϵ; if there is a distinction between near resonant and nonresonant forces, it will have to arise from a finite bifurcation in the behavior of the solutions to (1) — which in fact occurs, as we will show below.

Equation (1) develops shocks, which move at speed $s = \frac{1}{2}\,(u^+ + u^-)$ and satisfy the entropy condition $u^+ - u^- < 0$. Hence the energy

$$E(t) = \int_0^{2\pi} \frac{1}{2}\, u^2(x,\, t) dx$$

has typically a source given by the forcing and a sink at shocks; its dynamics is given by

$$\frac{dE}{dt} - \sum \left(\frac{1}{12} [u]^3 \right) = \int_0^{2\pi} u(x,t)\, f(x,t)\, dx\,, \tag{5}$$

where the sum on the left is over all the shocks in the solution. Here and throughout this paper, brackets stand for the jump across the shock of the enclosed expression.

Throughout this paper, we shall only consider forces of zero mean; for these, equation (1) preserves the mean of u, that we take always zero (since a non–vanishing mean can be absorbed by a Galilean change of coordinates).

3 A Single Forcing Mode.

In this section, we study the effect of unimodal forcings in (1) For concreteness, we shall consider forces of the form

$$f(x,t) = \sin(x - \omega t) \tag{6}$$

(More general unimodal forcings have been considered in [2].) Interestingly, equation (1) then admits traveling wave solutions of the form

$$u(x,t) = \omega \pm \sqrt{2\,(D - \cos(x - \omega t))}\,, \tag{7}$$

where $D(\omega) > 1$ and the sign of the square root follow from the condition on the mean:

$$\int_0^{2\pi} u(x,t)\, dx = 0\,.$$

Instead of writing a closed expression for $D(\omega)$, it is actually easier to write ω as a function of D:

$$\omega = \omega(D) = \pm\frac{1}{2\pi}\int_0^{2\pi}\sqrt{2(D-\cos z)}\,dz\,, \quad \text{where} \quad D \geq 1\,. \tag{8}$$

Notice that $\omega(D)$ is a growing function of D, which takes its minimum value when $D = 1$. This minimum value ω_{cr} can be calculated in closed form; it is given by

$$\omega_{cr} = \frac{4}{\pi}\,.$$

For values of ω smaller than ω_{cr}, the traveling wave solution develops a shock, (7) freezes at $D = 1$, and becomes

$$u(x,t) = \omega \pm 2\left|\sin(\frac{x-\omega t}{2})\right|\,. \tag{9}$$

In each period (say $0 \leq x < 2\pi$) there is a continuous switch from the minus to the plus sign as $z = x - \omega t$ crosses $z = 0$, and a discontinuous switch from the plus to the minus sign across a shock at some position $z = s$. The position of this shock follows from the zero mean condition

$$0 = \int_0^s G^+(z)\,dz + \int_s^{2\pi} G^-(z)\,dz = 2\pi\omega - 8\,\cos(s/2)\,, \tag{10}$$

where $G^+ = \omega + 2\,|\sin(z/2)|$, and $G^- = \omega - 2\,|\sin(z/2)|$. Thus

$$s = 2\arccos\left(\frac{\pi}{4}\omega\right)\,, \quad \text{with} \quad 0 \leq s < 2\pi\,. \tag{11}$$

Notice that, at the critical value $\omega = \omega_{cr}$, the solution has no shock, but a corner instead. Similar solutions with corners at the threshold between dissipative and non–dissipative behavior seem to be a common feature of forced systems that dissipate energy through shocks [3],[1], [5], [4], [6]. Figure 1 displays the three kinds of traveling wave solutions above: smooth, critical and discontinuous.

The work per unit time W_f done by the external force f on the exact solution above must agree with the energy E_d dissipated at the shock (since the traveling wave has constant energy), so we have:

$$W_f = \int_0^{2\pi} f\,u\,dx = E_d = -\frac{1}{12}\,[u]^3 = \frac{16}{3}\left\{1-\left(\frac{\pi\omega}{4}\right)^2\right\}^{3/2}\,. \tag{12}$$

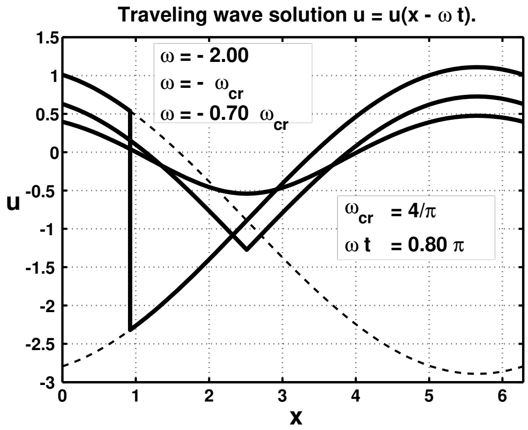

Figure 1: Examples of traveling waves for the equation $u_t + (0.5\, u^2)_x = \sin(x - \omega\, t)$.

For the smooth solutions (7) corresponding to $\omega > \omega_{cr}$, on the other hand, there are no shocks, hence no dissipation, and therefore no net work by the forcing. In other words, there is a sharp transition at $\omega = \omega_{cr}$ between near–resonant behavior, corresponding to energy input from the force into the system, to non–resonant behavior, with the solution and forcing in quadrature, and no energy exchange between them. This corresponds to a third order phase transition, with the solution switching from a dissipative (with shocks) to a non-dissipative (smooth) configuration. The work done by the forcing as a function of ω is shown in figure 2.

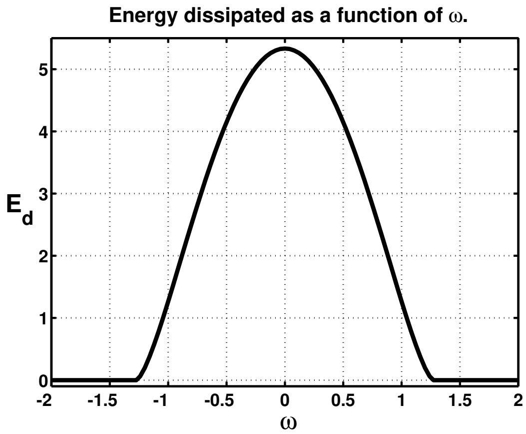

Figure 2: Energy dissipated (as a function of ω) by the traveling waves.

One may wonder whether the behavior of this family of exact traveling wave solutions is exceptional, or it represents the general dynamics of the model (1) subject to unimodal forcing. The latter possibility turns out to hold. In fact, it can be proved that all solutions to (1) with forcing given by (6) converge to the traveling wave solutions above. The proof, for which we refer the interested reader to [4], involves a rather unusual combination of a Hamiltonian formulation for the model written in characteristic form, a Hamiltonian and a pseudo-Hamiltonian formulation for the model in its original PDE form, and the dissipative dynamics of shocks. Figure 3 displays a numerical experiment showing convergence of an arbitrarily chosen initial data to the exact solution with shocks in (9).

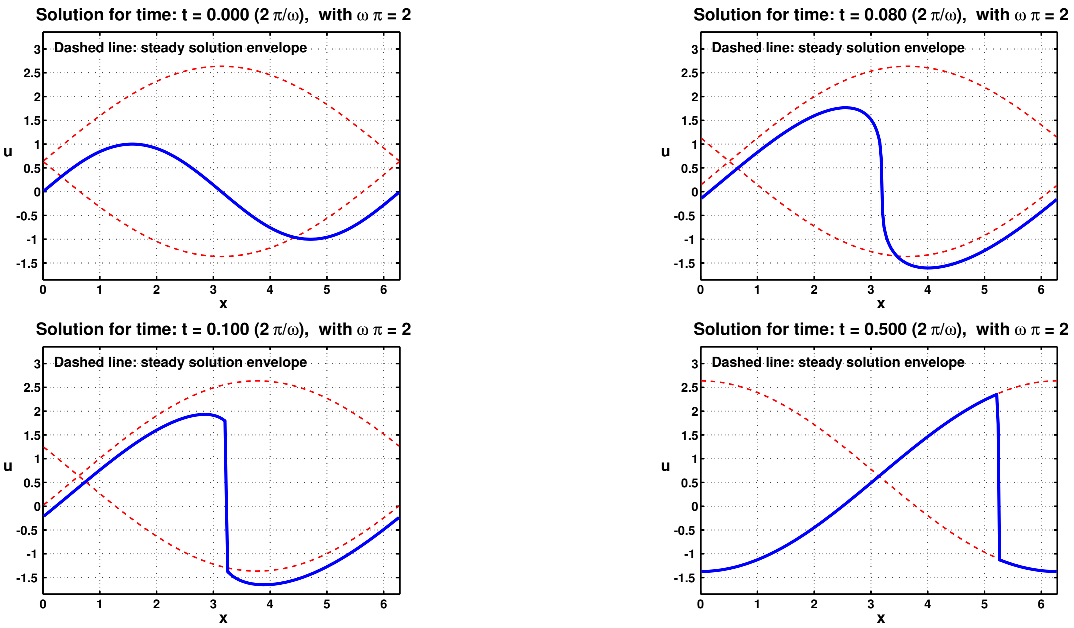

Figure 3: Forcing $f = \sin z$ in the equation, with $z = x - \omega t$ and $\omega = \omega_{cr}/2 = 2/\pi$.

4 Two Forcing Modes.

In this section, we study the solutions to equation (1) when the forcing term is the sum of two traveling waves of different speeds. For concreteness, we shall specifically look at a typical example, where the force f has the form

$$f(x, t) = \sin(x) + 2\,\sin(2(x - \Omega\, t))\,. \tag{13}$$

We are particularly interested in the behavior of $u(x,t)$ when the frequency shift Ω is small. The motivation for this choice is two–fold: on the one hand, it addresses the general question of the effects of the superposition of more than one near–resonant interaction acting on the same mode of a general system. On the other, it is an example of unimodal forcings similar to the one studied in the previous section, but modulated over a long time–scale. In real systems, such modulations are typically brought about by nonlinear effects, or by conditions external to the system. As we shall see, such slow modulations may have highly nontrivial consequences in systems where the mechanisms for energy dissipation are nonlinear.

Since Ω is small, one can in principle think of the response $u(x,t)$ to the forcing in (13) as frozen in time near each value $t = t_0$. In order to implement this idea, we propose the following simple asymptotic expansion in the small parameter Ω:

$$u(x,\, t) = u_0(x,\, \tau) + \Omega\, u_1(x,\, \tau) + O(\Omega^2)\,, \tag{14}$$

where $\tau = \Omega\, t$ is a slow time variable. Then, at leading order, (1) becomes

$$\left(\frac{1}{2}\, u_0^2\right)_x = \sin(x) + 2\, \sin(2(x-\tau)\,. \tag{15}$$

Thus

$$u_0(x,\, \tau) = \pm\sqrt{2\, G(x,\, \tau)}\,, \tag{16}$$

where $G = G(x,\, \tau)$ is defined (for each τ) by

$$G = C(\tau) - \left(\cos(x) + \cos(2(x-\tau)\right)\,, \tag{17}$$

where $C = \max\limits_x(\cos(x) + \cos(2(x-\tau))$. In each period $0 \le x < 2\pi$ the solution crosses (continuously) from the negative to the positive root at the point $x = x_m(\tau)$ where $G = 0$, and has a shock (jumping from the positive to the negative root) at a position $x = s(\tau)$, chosen so that the mean of u_0 vanishes. Notice that this quasi–steady solution is a very mild generalization of the steady one found in section (3) when $\omega = 0$, with the only difference that the force is no longer sinusoidal. In fact, the theorem of convergence of the solutions to traveling waves proved in [2] applies to forcing terms with nearly arbitrary shape.

The solution (16) above works as long as G has a single minimum per period, in which case $x_m = x_m(\tau)$ and $s = s(\tau)$ are well defined and depend smoothly on τ. However, there are some special times,

$$t_n = \frac{(2n+1)\,\pi}{2\,\Omega}\,, \tag{18}$$

at which this fails. At these times

$$G(x,\,\Omega\,t_n) \;=\; \frac{1}{8}\,(1-4\cos(x))^2\,, \tag{19}$$

and

$$u_0(x,\,\Omega\,t_n) \;=\; \pm\frac{1}{2}\,|1-4\cos(x)|\,, \tag{20}$$

with two candidate crossings of zero.

At the critical times t_n, there are two solutions $u_0(x)$ of the form (20), in which u_0 switches from negative to positive at one of the zeros, has a corner at the other, and switches once from positive to negative through a shock, at a position determined by the condition that u_0 has a vanishing average. The quasi–steady solution given by the asymptotic expansion in (14) approaches one (or the other) of these two solutions as $t \to t_n$ from below (or above.) The reason is that at almost all times G has several local minima, evolving in time, with one of them smaller than all the others. The critical times occur when two local minima exchange the property of being the global minimum. At these times x_m ceases to be smooth, jumps discontinuously from one position to another, and the expansion in (14) becomes inconsistent and fails. We call the fast transitions occurring at the critical times "storms"

Figure 4 displays the numerical solution to equation (13), starting from the asymptotic solution shortly before the critical time t_1, for a value of the frequency $\Omega = 0.01$. The dotted line gives the envelope for the asymptotic, quasi-steady solution $u_0(x,\,\tau)$ (i.e.: the curves $u = \pm\sqrt{2\,G}$.) In this figure we can see the actual solution $u = u(x,\,t)$ switching its upward crossing point from one zero of $G(x,\,\Omega\,t_1)$ to the other, through a relatively fast transition, involving the development, growth, travel and eventual disappearance of a second shock. During this transition, the solution sticks very closely to the envelope of the quasi-steady solution.

Figures 5 and 6 display the dissipation at the shocks and the work done by the forcing respectively. Both show a marked spike during the storm, approximately duplicating the regular amount of work and dissipation. Such doubling of the energy dissipation rate is due to the appearance of an extra shock during a storm, of a size comparable to the regular one. The close agreement between the energy dissipated and the work performed by the forcing, on the other hand, has its origin in the relatively slow evolution of storms, faster than the regular $O(\Omega\,t)$ rate, but clearly slower than a $O(t)$ rate. It follows that, at any particular

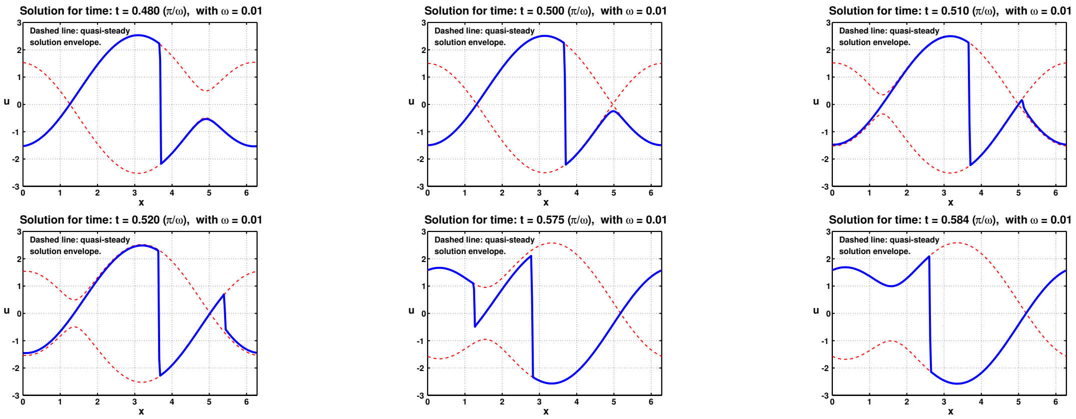

Figure 4: Asymptotic $t \to \infty$ solution to the equation $u_t + (\frac{1}{2}\,u^2)_x = \sin(x) + 2\sin(2\,(x - \Omega\,t))$, with $\Omega = 0.01$.

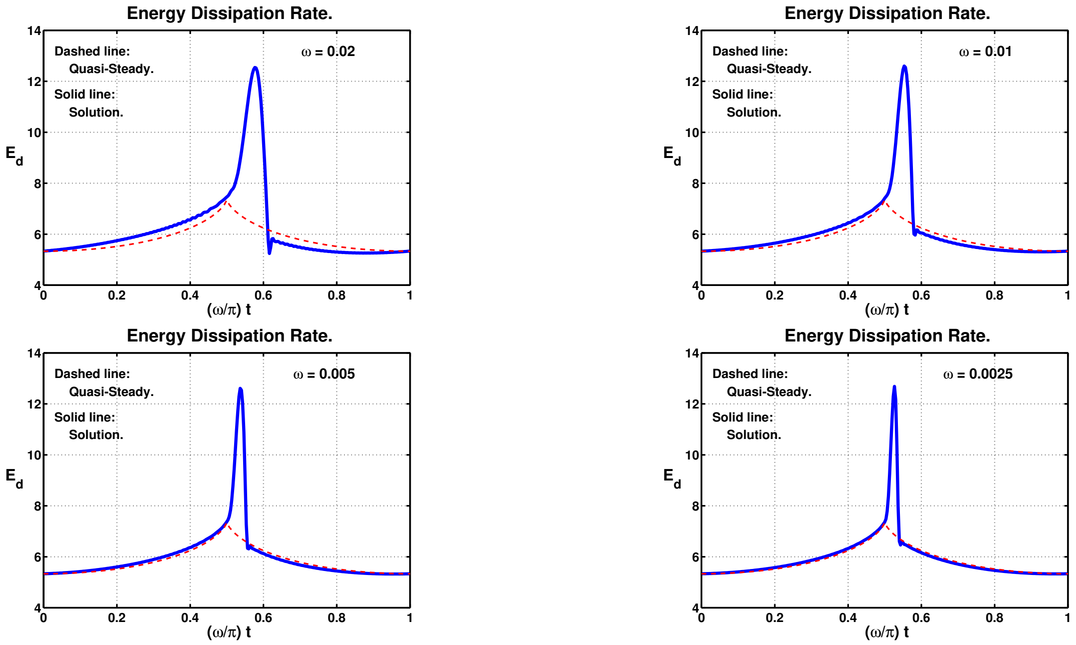

Figure 5: Energy dissipation rate $E_d = E_d(t)$ for the $t \to \infty$ asymptotic solution for the equation with double slow forcing.

time, the energy input and output need to be in balance to leading order, since even during a storm the solution remains quasi–steady. In fact, the duration of storms can be estimated quite precisely; it scales with the square–root of the frequency Ω [2].

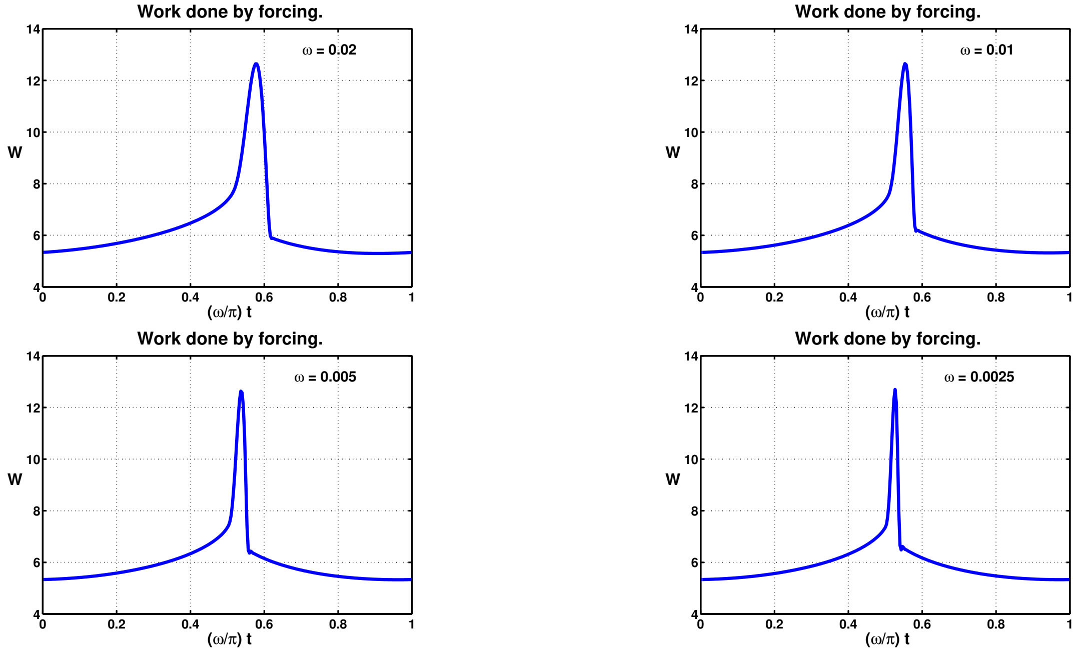

Figure 6: Work done by the forcing $W_f = W_f(t)$ for the $t \to \infty$ asymptotic solution for the equation with double slow forcing.

The fact that the effects caused by a storm scale with $\sqrt{\Omega}$, not Ω, may have important consequences when considering the effects of a complex set of near-resonances. The enhanced rate of energy exchange during storms hints at the possibility that nonlinear systems may have regimes where the energy exchange among modes is dominated by fast, intermittent events, involving coherent phase and amplitude adjustments of the full spectrum, rather than by the slow evolution of individual resonant sets. For this to be the case, the storms need to be strong and frequent enough to overcome the regular means of energy flux. We speculate that such strong intermittent events are likely occurrences in complex systems, particularly those whose behavior has stochastic or chaotic components.

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