A NOTE ON THE HILBERT ALGEBRAS WITH INFIMUM

Aldo V. Figallo **Guillermina Z. Ramón**
Susana Saad

Abstract

In this paper the research on Hilbert algebras with infimum is started. These algebras are presented as algebras $\langle A, \to, \wedge, 1\rangle$ of the type $(2, 2, 0)$, where the reduct $\langle A, \to, 1\rangle$ is a Hilbert algebra, and the identities $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \wedge x = x$, $x \wedge (x \to y) = x \wedge y$, $(x \to (y \wedge z)) \to ((x \to z) \wedge (x \to y)) = 1$ are verified. It is proved that the class of Hilbert algebras with infimum is a variety of algebras that strictly includes the variety of Hertz algebras. Besides, it is verified that the congruences of a Hilbert algebra with infimum can be obtained by means of absorbent deductive systems of A, that is to say those subsets of A which fulfil the properties: (D1) $1 \in D$, (D2) if $x, x \to y \in D$, then $y \in D$, (D3) if $d \in D$, then $x \to (x \wedge d) \in D$ for all $x \in A$. A subclass of subdirectly irreducible Hilbert algebras with infimum is determined as well.

1 Introduction and Preliminaries

The most important contribution of this work is having eliminated an "old confusion". This fact led us, in the first place, to find a new propositional calculus which could have some interesting application, and, in the second place, to give relevance within the classes of algebras which come from logic to an old class which have not reserved the attention it deserves on account of the confusion referred to above.

The Hertz algebras, as is well known, are the algebraic models for the fragment of intuitionistic propositional calculus, in which the implication ($\to$) and the conjunction ($\wedge$) appear as primitive connectors. In relation to this fact,

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several authors have stated, without any verification, that the class of Hertz algebras coincide with the class of Hilbert algebras whose underlying ordered structure is that of a semilattice with last element.(See [9, 1]). At one time we used to hold the same belief. However, that assertion proved to be untrue, as is shown in Example 1.1.

Thus, two problems, among others, arise. They are:

(i) Investigating the propositional calculus formulated in terms of the connectives $\to$ and $\wedge$, for which the algebraic models are precisely the Hilbert algebras with underlying ordered structure of meet-semilattice with last element (or Hilbert algebras with infimum).

(ii) Solving the problems that normally arise in the class of algebras mentioned in (i). For example, state whether the class is a variety, determine the congruences, determine the subdirectly irreducible algebras, etc.

We have positively solved the problem presented in (i) by determining scheme axioms and inference rules for a propositional calculus formulated in terms of the connectors $\to$, $\wedge$ which generalizes the corresponding fragment of intuitionistic propositional calculus. On account of shortness of room, these results will be presented elsewhere.

In this paper, we show the results obtained with relation to the problems mentioned in (ii). In Section 1 we recall some basic definitions. In Section 2 we introduce the variety of Hilbert algebras with infimum (or iH-algebras, to abbreviate), which are the Hilbert algebras with underlying structure of a meet-semilattice, but in which we have considered as primitive operations the semilattice operations yielded by the underlying order. Then, the iH-algebras constitute a generalization of the Hertz algebras. In Section 3 we obtain the congruences of any iH-algebra and, in Section 4, we include results on a class of subdirectly irreducible iH-algebras.

The results on the Hilbert algebras may be consulted in [1, 2, 7] . Throughout this paper we shall be including the definitions and properties necessary for the understanding of the remaining part.

Definition 1.1 ([2]) A Hilbert algebra is an algebra $\langle A, \rightarrow, 1\rangle$ of the type $(2,0)$ which satisfies these identities:

(H1) $x \rightarrow x = 1$,

(H2) $1 \rightarrow x = x$,

(H3) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(H4) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$.

We shall denote by **H** the variety of the Hilbert algebras. Then we state the following lemma.

Lemma 1.1 *For each $A \in \mathbf{H}$ the following properties are verified:*

(H5) $x \rightarrow 1 = 1$,

(H6) *if* $x \rightarrow y = y \rightarrow x = 1$, *then* $x = y$,

(H7) *the relation* $\leq$ *defined by* $x \leq y$ *if and only if* $x \rightarrow y = 1$ *is a partial order on* A,

(H8) $y \leq x \rightarrow y$,

(H9) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(H10) *if* $x \leq y \rightarrow z$, *then* $y \leq x \rightarrow z$,

(H11) *if* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$,

(H12) *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$.

In [1] and [11] a definition of the Hertz algebra equivalent to the following definition is used.

Definition 1.2 *A Hertz algebra (or implicative semilattice, according to Nemitz [8]) is an algebra* $\langle A, \rightarrow, \wedge, 1\rangle$ *of the type* $(2,2,0)$ *which satisfies these identities:*

(He1) $x \rightarrow x = 1$,

(He2) $(x \to y) \wedge y = y$,

(He3) $x \wedge (x \to y) = x \wedge y$,

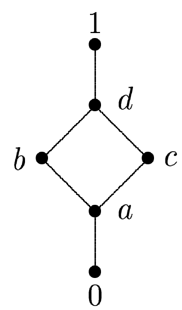
(He4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

In a subsequent step, Example 1.1, we shall show that the identities (He1) to (He4) do not characterize the Hilbert algebras in which the order given by (H7) is that of a meet-semilattice.

Example 1.1 It is well known (see [1, 2]) that, if in an ordered set $(A, \leq)$ with a last element 1 the operation $\to$ is defined by the prescription

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise,} \end{cases}$$

then $\langle A, \to, 1\rangle$ is a Hilbert algebra. Then, considering the set $A = \{0, a, b, c, d, 1\}$ with the order indicated in the following Hasse diagram,



$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	a	1	c	1	1
c	0	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

it results that every pair of elements has infimum, but it does not verify property (He4) since $b \to (b \wedge c) = b \to a = a$, $(b \to b) \wedge (b \to c) = 1 \wedge c = c$.

2 Hilbert Algebras with Infimum

Definition 2.1 *A Hilbert algebra $\langle A, \to, 1\rangle$ is called a Hilbert algebra with infimum when the underlying structure $\langle A, \leq\rangle$, with the order induced by $\to$, is a meet-semilattice.*

In the Theorem 2.1 we shall show the equivalence between Definition 2.1 and Definition 2.2, which is given below.

Definition 2.2 *An iH-algebra is an algebra* $\langle A, \rightarrow, \wedge, 1\rangle$ *of the type* $(2, 2, 0)$ *which satisfies the following conditions:*

(i) *The reduct* $\langle A, \rightarrow, 1\rangle$ *is a Hilbert algebra.*

(ii) *These identities are verified:*

(iH1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(iH2) $x \wedge x = x$,

(iH3) $x \wedge (x \rightarrow y) = x \wedge y$,

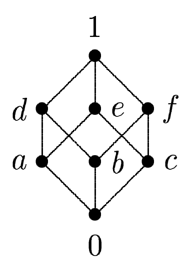
(iH4) $(x \rightarrow (y \wedge z)) \rightarrow ((x \rightarrow z) \wedge (x \rightarrow y)) = 1$.

We shall denote the variety of iH-algebras by **iH**. Following, we shall include some examples.

Example 2.1 If $\langle A, \wedge, 1\rangle$ is a meet-semilattice and $\rightarrow$ is defined as in Example 1.1, then $\langle A, \rightarrow, \wedge, 1\rangle \in$ **iH**.

Example 2.2 Every Heyting algebra is an iH-algebra.

Example 2.3 Let $\mathcal{C} = \langle C, \rightarrow, \wedge, 1\rangle$ be the algebra in which the Hasse diagram corresponds to that of a Boole algebra with three atoms and the operation $\rightarrow$ is the one indicated below:



$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	b	1	b	f	1	1	f	1
b	e	e	1	e	1	e	1	1
c	b	d	b	1	d	1	1	1
d	0	e	b	c	1	e	f	1
e	b	d	b	f	d	1	f	1
f	0	a	b	e	d	e	1	1
1	0	a	b	c	d	e	f	1

It is easy to verify that $\langle C, \rightarrow, \wedge, 1\rangle \in$ **iH**. In this case $a \rightarrow 0 = b \neq f = \text{máx}\{z : a \wedge z \leq 0\}$, which shows that $\mathcal{C}$ is not a Heyting algebra. Besides, as $a \rightarrow 0 \neq 0$, then the operation $\rightarrow$ defined on $\mathcal{C}$ does not coincide with the operation in Example 1.1.

Lemma 2.1 *In every iH-algebra A, the properties that follow are satisfied:*

(iH5) $x \wedge 1 = x$,

(iH6) $x \wedge y = y \wedge x$,

(iH7) *For every* $x, y \in A$, *the following properties are equivalent:* (a) $x \to y = 1$, (b) $x = x \wedge y$.

Proof.

(iH5) $x \wedge 1 = x \wedge (x \to x) = x \wedge x = x$.

(iH6) $x \wedge y = 1 \to (x \wedge y) \leq (1 \to y) \wedge (1 \to x) = y \wedge x$.

(iH7) (a) $\Rightarrow$ (b) $x \wedge y = x \wedge (x \to y) = x \wedge 1 = x$,

(b) $\Rightarrow$ (a) $1 = (x \to (x \wedge y)) \to ((x \to y) \wedge (x \to x)) = 1 \to ((x \to y) \wedge 1) = x \to y$.

□

It is clear that from Lemma 2.1 it follows that in an iH-algebra the order determined by the structure of the meet-semilattice coincides with the order determined by the structure of a Hilbert algebra. On the other hand, from definitions 2.1, 2.2 and Lemma 2.1, the following theorem results.

Theorem 2.1 *Let* $\langle A, \to, \wedge, 1 \rangle$ *of the type* $(2, 2, 0)$. *Then the following conditions are equivalent:*

(i) *A is an iH-algebra,*

(ii) *A is a Hilbert algebra with infimum.*

Proof. It is routine. □

In Lemma 2.2 we shall show a condition that must be verified by an iH-algebra to be a Hertz algebra. In the proof we shall use the notion of (H)-Hilbert algebra introduced by M. Kondo [11].

Definition 2.3 ([11]) *A Hilbert algebra A is an (H)-Hilbert algebra when it satisfies the following condition:*

(H) *The set $A(a,b) = \{x \in A : a \to (b \to x) = 1\}$ has the least element for any $a, b \in A$.*

M. Kondo proves that the Hertz algebras are the (H)-Hilbert algebras ([11], Theorem 1, p. 197).

Lemma 2.2 *An iH-algebra in which this identity is verified:*

(I) $x \to (y \to (x \wedge y)) = 1$ *for every $x, y \in A$,*

is a Hertz algebra.

Proof. From (I), for every $a, b \in A$

(1) $a \wedge b \in B = \{z : a \to (b \to z) = 1\}$.

In addition, for every $z \in B$ the following properties are verified:

(2) $a \leq b \to z$,

(3) $a \wedge b \leq b \wedge (b \to z)$, [(2)]

$= b \wedge z$, [(iH3)]

$\leq z$,

(4) $a \wedge b = \min\ \{z : a \to (b \to z) = 1\}$. [(1),(3)]

Then, $\langle A, \to, 1\rangle$ is an (H)-Hilbert algebra; therefore, it is a Hertz algebra.

□

3 Congruences

Now we shall consider the way to determine the congruences of a iH-algebra. Let's recall the following definition.

Definition 3.1 ([2]) *Let $A \in \mathbf{H}$. $D \subseteq A$ is a deductive system if the following conditions are verified:*

(D1) $1 \in D$,

(D2) $x, x \to y \in D$ *imply* $y \in D$.

We shall denote by $\mathcal{D}(A)$ the family of the deductive systems of A .

Remark 3.1 *It is well known* [2] *that:*

(i) *If $A \in \mathbf{H}$, $D \in \mathcal{D}(A)$ and $R(D) = \{(x,y) \in A^2 : x \to y \in D, y \to x \in D\}$, then $R(D) \in Con_{\mathbf{H}}(A)$, where $Con_{\mathbf{H}}(A)$ is the set of all the $H-$congruences on A. If $R \in Con_{\mathbf{H}}(A)$ and x_R denotes the equivalence class of x, $x \in A$, then $1_{R(D)} = D$.*

(ii) *If $R \in Con_{\mathbf{H}}(A)$, then there exists a unique $D \in \mathcal{D}(A)$ such that $R = R(D)$ and $D = 1_R$.*

By means of Example 1.1 given above we shall show that the deductive systems do not determine the congruences of an iH-algebra. In fact, $D = \{b, c, d, 1\}$ is a deductive system of A. But the $H-$congruence $\theta(D)$ is not an iH-congruence because $(b,c), (c,c) \in \theta(D)$ but $(b \wedge c, c \wedge c) = (a, c) \notin \theta(D)$.

The previous result has led us to determine which are the deductive systems D such that $R(D)$ are iH-congruences.

Definition 3.2 *Let $A \in \mathbf{iH}$. We shall say that the deductive system $D \subseteq A$ is absorbent if it verifies this property:*

(D3) *If $x \in D$, then $z \to (z \wedge x) \in D$ for every $z \in A$.*

We shall denote by $\mathcal{D}_{ab}(A)$ the set of the absorbent deductive systems of A.

Lemma 3.1, which is given below, can be easily proved.

Lemma 3.1 *Let $A \in \mathbf{iH}$ and $D \in \mathcal{D}_{ab}(A)$, then D is a filter of A.*

Remark 3.2 *Let A be the algebra introduced in Example 1.1. Then $D = F(b) = \{b, d, 1\}$ is a filter which is not an absorbent deductive system. In effect, $b \in D$, and $c \to (c \wedge b) = c \to a \notin D$.*

Next we shall determine the iH-congruences of A and establish an isomorphism between the lattices $Con_{\mathbf{iH}}(A)$ and $\mathcal{D}_{ab}(A)$.

Lemma 3.2 *Let $A \in \mathbf{iH}$ and $D \in \mathcal{D}_{ab}(A)$. Then the following properties are verified:*

(i) $R(D) \in Con_{\mathbf{iH}}(A)$.

(ii) $1_{R(D)} = D$.

Proof. We shall only prove that, if $(x, y) \in R(D)$, then $(x \wedge z, y \wedge z) \in R(D)$ for every $z \in A$.

(1) Let $(x, y) \in R(D)$ and $z \in A$,

then, these properties are verified:

(2) $x \to y \in D$,

(3) $y \to x \in D$,

(4) $(z \wedge x) \to ((z \wedge x) \wedge (x \to y)) \in D$, [(2), (D3)]

(5) $(z \wedge x) \to (z \wedge (x \wedge y)) \in D$, [(4), (iH3)]

(6) $z \wedge (x \wedge y) \leq z \wedge y$,

(7) $(z \wedge x) \to (z \wedge (x \wedge y)) \leq (z \wedge x) \to (z \wedge y)$, [(6)]

(9) $(z \wedge x) \to (z \wedge y) \in D$. [(5), (7)]

From (3), we can prove:

(10) $(z \wedge y) \to (z \wedge x) \in D$,

then

(11) $(z \wedge x, z \wedge y) \in R(D)$. [(9), (10)]

$\square$

Lemma 3.3 *Let* $A \in \mathbf{iH}$, $R \in Con_{\mathbf{iH}}(A)$ *y* $D = 1_R$. *Then the following assertions are verified:*

(i) $D \in \mathcal{D}_{ab}(A)$,

(ii) $R(D) = R$,

(iii) *The lattices* $Con_{\mathbf{iH}}(A)$ *and* $\mathcal{D}_{ab}(A)$ *are isomorphic if we consider the correspondences*
$R \longmapsto 1_R$ *and* $D \longmapsto R(D)$, *one being the inverse of the other.*

As a direct consequence of lemmas 3.2 and 3.3 we can formulate the following theorem:

Theorem 3.1 $Con_{\mathbf{iH}}(A) = \{R(D) : D \in \mathcal{D}_{ab}(A)\}$.

4 Subdirectly Irreducible Algebras

In this section we shall show some results on the subdirectly irreducible iH-algebras.

Lemma 4.1 *Let* $A \in \mathbf{iH}$, $D = F(a)$ *be a principal filter of* A *and* $S = \{z \in A : z \to (x \to (x \wedge z)) = 1, \text{for all } x \in A\}$. *Then, the following assertions are equivalent:*

(i) $F(a) \in \mathcal{D}_{ab}(A)$,

(ii) $a \in S$.

Proof. (i)⇒(ii):

(1) $D = F(a) \in \mathcal{D}_{ab}(A)$, [(i)]

(2) $x \to (x \wedge a) \in D$, for every $x \in A$, [(1), D3]

(3) $a \leq x \to (x \wedge a)$, [(2), (1)]

(4) $1 = a \to (x \to (x \wedge a))$, [(3)]

(5) $a \in S$. [(4)]

(ii)⇒(i):

(1) $a \in S$ y $D = F(a)$, [(ii)hyp.]

(2) $a \to (x \to (x \wedge a)) = 1$ for all $x \in A$, [(1)]

(3) $a \leq x \to (x \wedge a)$, [(2)]

(4) $x \to (x \wedge a) \in D$, for every $x \in A$, [(3), (1)]

(5) Let $y \in D$,

(6) $a \leq y$, [(5),(1)]

(7) $x \wedge a \leq x \wedge y$, [(6)]

(8) $x \to (x \wedge a) \leq x \to (x \wedge y)$, [(7)]

(9) $x \to (x \wedge y) \in D$, for every $x \in A$, [(8),(4)]

(10) $D \in \mathcal{D}_{ab}(A)$. [(9),(5)]

□

Lemma 4.2 *For every $x, y, z \in A$, the following properties are equivalent:*

(i) $z \to (x \to (x \wedge z)) = 1$,

(ii) $z \to (x \wedge z) = z \to x$,

(iii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Proof. (i)⇒(iii)

(1) $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$, [iH4]

(2) $z \leq (y \to (y \wedge z))$, [(i)]

(3) $x \to z \leq (x \to y) \to (x \to (y \wedge z))$, [(2)]

(4) $(x \to y) \wedge (x \to z) \leq (x \to y) \wedge ((x \to y) \to (x \to (y \wedge z)))$, [(3)]

$= (x \to y) \wedge (x \to (y \wedge z))$, [iH3]

$\leq x \to (y \wedge z)$.

Then, $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$. [(1),(4)]

(iii)⇒(ii)

(1) $z \to (x \wedge z) = (z \to x) \wedge (z \to z)$, [(iii)].

$= z \to x$.

(ii)⇒(i)

(1) $z \to (x \wedge z) = z \to x$, [(ii)]

(2) $z \to (x \to (x \wedge z)) = (z \to x) \to (z \to (x \wedge z))$,

$= 1$. [(1)]

□

Lemma 4.3 *If $s \in S$, then $x \to s \in S$, for every $x \in A$.*

Proof.

(1) $s \in S$, [hyp.]

(2) $x \to s = x \to (x \wedge s)$, [(1),Lemma 4.2]

(3) $1 = (x \to s) \to (x \to s)$,

$= (x \to s) \to (x \to (x \wedge s))$, [(2)]

$= (x \to s) \to (x \to (x \wedge (x \to s)))$. [iH3]

Then $x \to s \in S$. [(3)]

□

Corolary 4.1 *For every $s, t \in S$ the following conditions are verified:*

(i) $s \to t \in S$,

(ii) $s \wedge t \in S$.

Proof.

(i) It is a direct consequence of Lemma 4.3.

(ii) (1) $s \wedge t \leq (x \to s) \wedge (x \to t)$,

$= (x \to (x \wedge s)) \wedge (x \to t)$, [Lemma 4.2 (ii)]

$= x \to ((x \wedge s) \wedge t)$. [Lemma 4.2 (iii)]

(2) $(s \wedge t) \to (x \to (x \wedge (s \wedge t))) = 1$. [(1)]

Then, $s \wedge t \in S$.

□

Corolary 4.2 $\langle S, \wedge, \to, 1\rangle$ *is a subalgebra of* $\langle A, \wedge, \to, 1\rangle$.

Lemma 4.4 $\langle S, \to, \wedge\rangle$ *is a Hertz algebra.*

Proof. It is enough to prove that $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, for every $x, y, z \in S$.

(1) $x, y, z \in S$,

(2) $(x \to (y \wedge z)) \to ((x \to y) \wedge (x \to z)) = 1$, [iH4]

(3) $z \to (y \to (y \wedge z)) = 1$, [(1)]

(4) $z \leq y \to (y \wedge z)$, [(3)]

(5) $x \to z \leq x \to (y \to (y \wedge z))$, [(4)]

$= (x \to y) \to (x \to (y \wedge z))$,

(6) $(x \to y) \wedge (x \to z) \leq (x \to y) \wedge ((x \to y) \to (x \to (y \wedge z)))$, [(5)]

$$= (x \to y) \wedge (x \to (y \wedge z)), \qquad \text{[iH3]}$$

$$\leq x \to (y \wedge z).$$

□

Definition 4.1 *Let $\langle A, \leq, 1\rangle$ be a poset with a last element 1. $p \in A$, is the penultimate element if $p \neq 1$ and $a \leq p$, for every $a \in A$.*

Lemma 4.5 *Let A be an iH-algebra for which every absorbent deductive system is a principal filter. The conditions that follow are equivalent:*

(1) *A is subdirectly irreducible,*

(2) *The lattice $\mathcal{D}_{ab}(A)$, of the absorbent deductive systems of A possesses exactly one atom,*

(3) *$S = \{z \in A : z \to (x \to (x \wedge z)) = 1, \text{for every } x \in A\} = \{z \in A : z \to (x \wedge z) = z \to x, \text{for every } x \in A\}$, has a penultimate element.*

Proof. It follows immediately. □

Lemma 4.6 *Let A be an iH-algebra for which every absorbent deductive system is a principal filter. The following conditions are equivalent:*

(1) *A is simple,*

(2) *$S = \{0, 1\}$.*

Proof. It follows immediately. □

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Aldo V. Figallo
Departamento de Matemática
Univ. Nac. del Sur
800 Bahía Blanca, and
Instituto de Ciências Básicas
Univ. Nac. de San Juan
5400 San Juan, Argentina
e-mail: matfiga@criba.edu.ar

Guillermina Ramón
Susana Saad
Instituto de Ciencias Básicas
Univ. Nac. de San Juan
5400 San Juan, Argentina
e-mail: gramon@unsj.edu.ar