RIBBON BRAIDED MULTIPLICATIVE LINEAR LOGIC

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Abstract

We give here three different descriptions of ribbon braided multiplicative linear logic without units : In terms of sequent calculus, in terms of planar proof-net structures with explicit permutation and torsion operators and in term of semantical proof-nets which are isotopy classes of ribbon graphs embeded in $\mathbb{R}^3$. This allow us to decide the equality between proofs.

1 Introduction

In his sequent calculus Gentzen recognized three structural rules: contraction, exchange and weakening. This paper is dedicated to the study of the exchange rule. This rule allows us to permute formulae both to the left (left exchange rule) and to the right (right exchange rule) of the inference symbol of a sequent.

In linear logic the two other structural rules, contraction and weakening are forbidden except for the formulae starting with the "why not" modal operator (or more generally for negatively polarised formulae).

At this stage we can consider two variant of linear logic :

- Girard's commutative linear logic [6].

In this case (by far the most common one) exchange is free and left as an implicit rule. Indeed saying that a sequent is composed of two multi-sets of formulae means that it's invariant by permutation on the left and on the right of the inference symbol. This calculus is also often presented with an implicit negation: the calculus being symmetrical between left and right, we can apply De Morgan's symmetries to the formulae and write sequents with all the formulae on the right side of the inference symbol.

- Yetter's variant, circular linear logic [11].

Here we also use an implicit negation. This leads to a sequent system with all the formulae on the right where the exchange rule is restricted to circular permutation. Only the why-not-modalised formulae do commute with every formula. Why allowing circular permutation? Indeed there is also a strictly non commutative system by Abrusci [1] where one has to consider two negations. Circular permutation allows us to consider only one negation. More precisely, it amounts to say that between the two negations there is a natural equivalence restricting to equality on formulae. Usual models of non commutative (linear) logic enjoy this rule: phase spaces for example.

Moreover in presence of braiding the set of these natural equivalences is in bijection with the set of torsion operators. The proof of this assertions is not given here and left to a forthcoming paper. Similar ideas have been developed by A. Joyal and R. Street in their work on the Tensor Calculus [7]. Indeed an important motivation of our work lies in its possible applications to Category Theory: the main proof of this paper can be seen as the core of a coherence theorem for ribbon braided *-autonomous categories.

The proof-nets of multiplicative linear logic have nice geometrical properties. Between the planar proof-nets of non commutative linear logic and the abstract graphs of the commutative one, it was quite natural to look at what happens in the braided case. It turns out that a generalisation of Artin's treatment [2] of the braid group using algebraic topology works well and allows us to reach our result.

In chapter two we present the sequent calculus and define the equalities of proofs to be considered in presence of the exchange rules. In chapter three, we define planar proof-nets. Note that our system of proof-nets is without explicit axiom and cut links like Lamarche's essential nets [9] for intuitionistic linear logic and unlike the usual proof-nets for classical linear logic. In chapter four we show how to put exchange in a canonical form relatively to the commutative sequential proof underlying the braided one. In chapter five we exhibit a cut elimination procedure and an algorithm to compare proofs. In chapter six we introduce the three dimensional version of proof-nets. We show that two sequential proofs are equal if and only if their representations in $\mathbb{R}^3$ are ambient isotopic. This topological argument shows that our algorithm for comparing proofs is indeed complete.

2 Sequent calculus and structural group

2.1 Formulae, sequents, proofs

We have a denumerable set of *atomic formulae*, given together with a bijection on it which is involutive and without fixed point, denoted by $(.)^\perp$ and named *duality*, *negation* or *orthogonality*.

By structural induction a *formula* will be an atomic formula or an expression of one of the forms $A \parr B$ and $A \otimes B$ where A and B are formulae. Duality is extended to all formulae by the equalities $(A \parr B)^\perp = B^\perp \otimes A^\perp$ and $(A \otimes B)^\perp = B^\perp \parr A^\perp$. One can easily see that duality is still involutive and without fixed point.

Greek capitals $\Gamma, \Delta \ldots$ stand for sequences of formulae.

A sequent $\vdash \Gamma$ is a circularly ordered multi-set of formulae. This circular order is obtained by extending the successor relation of the sequent Γ in the obvious way.

Proofs are built according to the following set of rules:

$$\frac{}{\vdash A, A^\perp} : \mathit{axiom} \qquad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} : \mathit{cut}$$

$$\frac{\vdash A, B, \Gamma}{\vdash B, A, \Gamma} : p_{A,B} \qquad \frac{\vdash A, \Gamma}{\vdash A, \Gamma} : \theta_A$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma} : \mathit{par} \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} : \mathit{tensor}$$

The rule $p_{A,B}$ is called *permutation* and the rule θ_A is called *torsion*.

The formulae containing Latin capitals are called *principal formulae* of the corresponding rule. The principal formulae can be deduced from the pattern of each rule except for permutation and for torsion. In this case the principal formulae are signed at the level of the rule. Thus one can consider that the name of the rule is part of the rule itself.

As a consequence the ancestors of a(n occurence of a) formula in a sequent in a proof are uniquely determined. In fact we could say that this paper studies chains of ancestors, and the relations between these chains. They are represented as “ribbons” in a 3-dimensional space.

2.2 Commutations and simplification of proofs

In this sequent calculus, one can define permutations of rules as usual (Kleene 52)[8]: when a rule R_1 is followed by a rule R_2 and no principal conclusion of R_1 is principal premise of R_2, the rule R_2 can always be applied before R_1 except when R_1 is a *tensor* or a *cut*, R_2 is a *par* or a *permutation* and the two premises of R_2 come from different sequents in the premises of R_1.

We will not give a detailed treatment of this matter, as it follows from the representation of proof-nets as Bellin and Van De Wiele show in [4]. We will call these permutations of rules *trivial commutations.* Concerning structural rules we are interested here in a more general notion where a principal conclusion of the first one may be principal premise of the following one. Moreover we have a case of *simplification* of structural rules.

We have four cases of non trivial commutations:

$$\dfrac{\dfrac{\vdash A,B,\Gamma}{\vdash A,B,\Gamma}\,\theta_A}{\vdash B,A,\Gamma}\,p_{A,B} \quad\sim\quad \dfrac{\dfrac{\vdash A,B,\Gamma}{\vdash B,A,\Gamma}\,p_{A,B}}{\vdash B,A,\Gamma}\,\theta_A$$

$$\dfrac{\dfrac{\vdash A,B,\Gamma}{\vdash A,B,\Gamma}\,\theta_B}{\vdash B,A,\Gamma}\,p_{A,B} \quad\sim\quad \dfrac{\dfrac{\vdash A,B,\Gamma}{\vdash B,A,\Gamma}\,p_{A,B}}{\vdash B,A,\Gamma}\,\theta_B$$

$$\dfrac{\dfrac{\dfrac{\vdash A,B,C,\Gamma}{\vdash B,A,C,\Gamma}\,p_{A,B}}{\vdash B,C,A,\Gamma}\,p_{A,C}}{\vdash C,B,A,\Gamma}\,p_{B,C} \quad\sim\quad \dfrac{\dfrac{\dfrac{\vdash A,B,C,\Gamma}{\vdash A,C,B,\Gamma}\,p_{B,C}}{\vdash C,A,B,\Gamma}\,p_{A,C}}{\vdash C,B,A,\Gamma}\,p_{A,B}$$

$$\begin{array}{c}\dfrac{\dfrac{\vdash A, B_1,\dots B_n}{\vdash B_1, A,\dots B_n}\,p_{A,B_1}}{\vdots}\,p_{A,B_2}\\ \dfrac{\vdash B_1,\dots A, B_n}{\vdash B_1,\dots B_n, A}\,p_{A,B_n}\\ \Updownarrow\\ \vdash A, B_1,\dots B_n\end{array} \quad\sim\quad \begin{array}{c}\vdash A, B_1,\dots B_n\\ \Updownarrow\\ \dfrac{\dfrac{\vdash B_1,\dots B_n, A}{\vdash B_1,\dots A, B_n}\,p_{A,B_n}}{\vdots}\,p_{A,B_{n-1}}\\ \dfrac{\vdash B_1, A,\dots B_n}{\vdash A, B_1,\dots B_n}\,p_{A,B_1}\end{array}$$

…and the simplification:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash A, B_1, \dots B_n}{\vdash B_1, A, \dots B_n}\, p_{A,B_1}}{\vdots}\, p_{A,B_2}}{\cfrac{\vdash B_1, \dots A, B_n}{\vdash A, B_1, \dots B_n}\, p_{A,B_n}}}{\vdash A, B_1, \dots B_n}\, \theta_A \qquad \sim \qquad \vdash A, B_1, \dots B_n (Id)$$

2.3 The structural group

This set of equivalences of deductions turns the set of structural rules into a group. In order to give it explicitly we will mark a formula and situate the other relatively to this one (there is a little cheating here because we should have taken into account the change of the marked formula, but it doesn't have harmful consequence on our purpose). Let $\vdash \Gamma, A$ be a sequent with $n+1$ occurrences of formulae. The marked formula A is said of *index* 0, the other ones following the circular order are said of *index* $1 \dots n$. Let θ_i be the torsion on the formula of index i $(0 \leq i \leq n)$, let l (resp. r) be the permutation of the marked formula with the one immediately to its left (resp. to its right). For $1 \leq i \leq n-1$ let σ_i be the permutation between the formulae of indexes i and $i+1$.

Commutations and simplifications give us the following equations:

trivial commutations between two permutations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad 1 \leq i, j \leq n-1 \quad |i-j| \geq 2 \tag{1}$$

$$l \sigma_{i+1} = \sigma_i l \quad 1 \leq i \leq n-2 \tag{2}$$

$$r \sigma_i = \sigma_{i+1} r \quad 1 \leq i \leq n-2 \tag{3}$$

between a permutation and a torsion:

$$\sigma_i \theta_j = \theta_j \sigma_i \quad \begin{cases} 1 \leq i \leq n-1 \\ 0 \leq j \leq n \end{cases} \quad j \neq i, i+1 \tag{4}$$

$$l \theta_{i+1} = \theta_i l \quad 1 \leq i \leq n-1 \tag{5}$$

$$r \theta_i = \theta_{i+1} r \quad 1 \leq i \leq n-1 \tag{6}$$

between two torsions:

$$\theta_i \theta_j = \theta_j \theta_i \quad 0 \leq i, j \leq n \tag{7}$$

1^{st} non trivial commutation:

$$\theta_i\sigma_i = \sigma_i\theta_{i+1} \quad 1 \le i \le n-1 \tag{8}$$

$$\theta_n l = l\theta_1 \tag{9}$$

$$\theta_0 r = r\theta_0 \tag{10}$$

2^{nd} non trivial commutation:

$$\theta_{i+1}\sigma_i = \sigma_i\theta_i \quad 1 \le i \le n-1 \tag{11}$$

$$\theta_0 l = l\theta_0 \tag{12}$$

$$\theta_1 r = r\theta_n \tag{13}$$

3^{rd} non trivial commutation:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad 1 \le i \le n-2 \tag{14}$$

$$\sigma_{n-1}l^2 = l^2\sigma_1 \tag{15}$$

$$l\sigma_1 r = r\sigma_{n-1}l \tag{16}$$

$$r^2\sigma_{n-1} = \sigma_1 r^2 \tag{17}$$

4^{th} non trivial commutation:

$$\sigma_i \ldots \sigma_{n-1} l \sigma_1 \ldots \sigma_{i-1} = \sigma_{i-1} \ldots \sigma_1 l \sigma_{n-1} \ldots \sigma_i \quad 1 \le i \le n \tag{18}$$

$$r^n = l^n \tag{19}$$

the simplification:

$$\sigma_i \ldots \sigma_{n-1} l \sigma_1 \ldots \sigma_{i-1}\theta_i = 1 \quad 1 \le i \le n \tag{20}$$

$$r^n\theta_0 = 1 \tag{21}$$

One can immediately check:

$\sigma_i^{-1} = \sigma_{i+1} \ldots \sigma_{n-1} l \sigma_1 \ldots \sigma_{i-1}\theta_i$
$\theta_i^{-1} = \sigma_i \ldots \sigma_{n-1} l \sigma_1 \ldots \sigma_{i-1}$
$l^{-1} = \sigma_1 \ldots \sigma_{n-1}\theta_n$
$r^{-1} = \sigma_{n-1} \ldots \sigma_1\theta_1$
$\theta_0^{-1} = r^n$

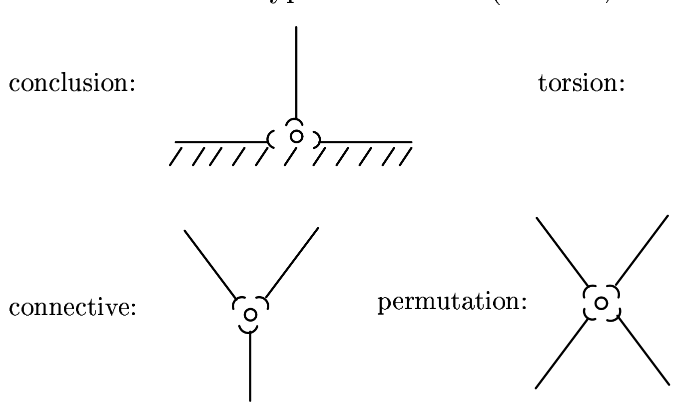
If we want to generate this set as a group (and not as a monoid) one can forget about l, r and θ_0 and keep only equations (1), (4), (7), (8), (11) and (14). This is the usual presentation of the ribbon braided group [2].

3 Proof-nets and sequentialisation

3.1 Definitions

A proof-structure $\mathcal{R}$ is a cellular complex of dimension 2(i.e. a disjoint union of 0-cells or vertices, of 1-cells or open lines and of 2-cells or open discs) isomorphic to the oriented closed disc D^2.

We will have four types of vertices (or links, or 0-cells):



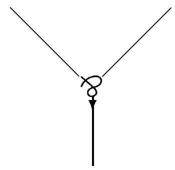
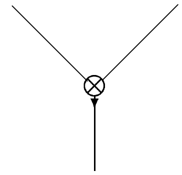
The lines on the border of D^2 are called *borders*, they inherit an orientation from D^2 (looking at the positive face you see them oriented positively i.e. counterclockwise). We have #borders=#conclusions,the set of borders and the set of conclusions are circularly ordered by the orientation of D^2.

A non-border line will be called an *edge*. A sequence of edges connected by torsion links and whose extremal vertices are not torsions will be called, by abuse of language, an edge too. The number of torsion on these so called edges is uniquely determined. We will say: this edge is carrying n ($n \in N$) torsion(s).

The set $\{\text{vertices}\} \cup \{\text{edges}\}$ is called the *graph* of the proof-structure.

The 2-cells will simply be called cells. We will say that a cell is *adjacent* to a vertex or to a line if this vertex or this line belong to the closure of the cell.

For each connective link, we need to know if the connective is a *par* link or a *tensor* link and wich one of the three incident edges is the *principal edge* of the link.

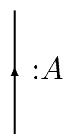
Graphically we will use the notation:  and 

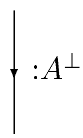
Turning around a connective link in the positive sens, after the principal edge or *concusion* of the link, the first edge we meet is the *right premise* of the link and the third one is the *left premise*.

The cell adjacent to the two premises is said to be *above* the connective..

Moreover, these proof-structures may be typed in the following way: to each edge with a *chosen orientation* one associate a formula. Typing obey to some coherence conditions:

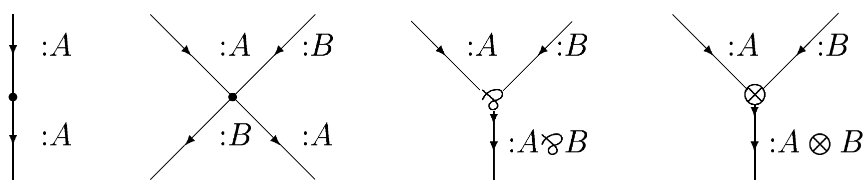
For each edge we have:



then


(same edge, opposite orientation)

For internal vertices we have:



Remarks: Unless the usual proof-net representation there is no canonical orientation of edges here. Indeed these orientatationless proof-nets can be seen as a quotient of oriented (and intuitionistic) ones in the style of Lamarche essential nets. The changing of orientation over edges beeing involutive and without fixpoint, the typing makes sens only because negation is itself involutive over the formulae.

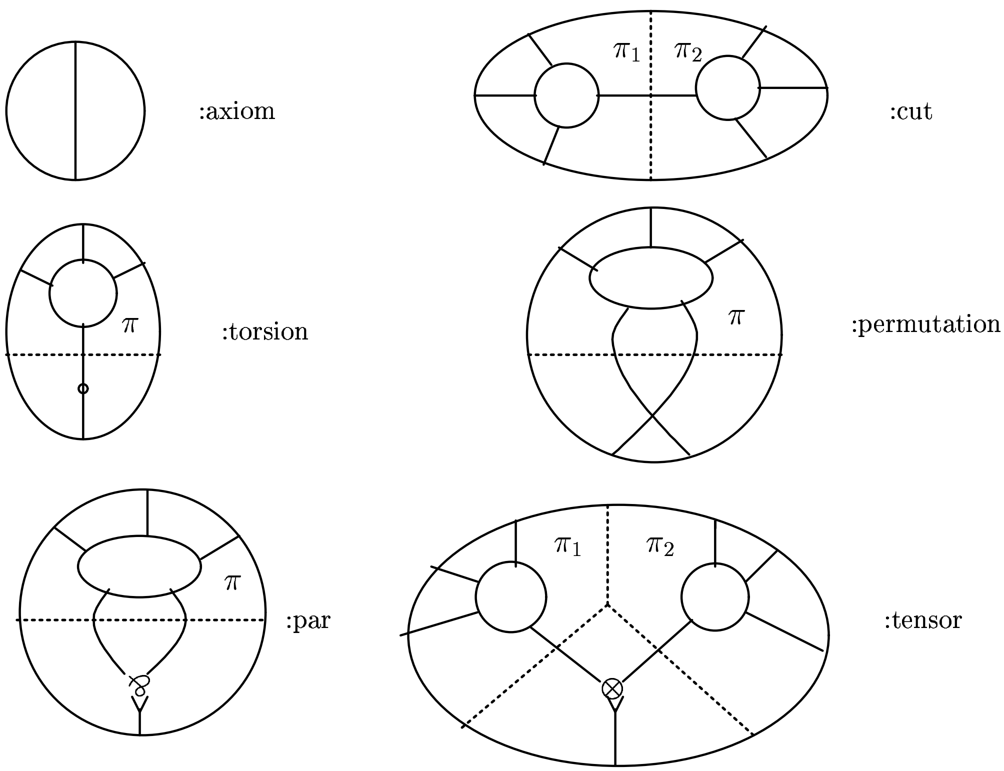
The type of a conclusion will be the one of the edge oriented to reach this link. The set of types of the conclusions inherits the circular order on these conclusions. In this way, it inherits a sequent structure. this sequent is called the *conclusion sequent* of the proof structure.

If the principal edge of a tensor reaches a conclusion and the three cells adjacent to this tensor are adjacent to a border, this tensor will be called *splitting*.

The embedding of a circle S^1 in the interior of the disk D^2 induces a partition of D^2 into a ring outside and a disk inside. In the case of a proof-structure if the image of S^1 doesn't contain any link and cuts the edges transversally, the embedding is said *regular*. The disk inside is then itself a proof-structure. We will say that it's a *sub-proof-structure* of the larger one.

3.2 Sequents and proof-nets

Translation of the sequent calculus into proof-structures:
the translation is made inductively starting with the axioms and for the other rules by gluing disks step by step:



Each sequential rule determines a sub-proof. We check that the translation of a sub-proof is a sub-proof-structure. this one will be said *associated* to the rule. Remark that the translation forgets about the position of the axioms and cuts and also forgets the orientation of the torsions.

Definition: Let $\mathcal{C}$ be the smallest subset of the set of 2-cells such that:

- If c is adjacent to a border b then $c \in \mathcal{C}$
- if the two cells adjacent to the principal edge of a par belong to $\mathcal{C}$ then the cell above this par also belongs to $\mathcal{C}$.
- If three of the four cells adjacent to a permutation belongs to $\mathcal{C}$ then the fourth one too.

We construct $\mathcal{C}$ by induction starting with $\emptyset$ and adding cells one by one. We describe this construction by the mean of a function $f : \mathcal{C} \to \{\text{borders}\} \cup \{\text{par}\} \cup \{\text{permutations}\}$ which to x associates the border , the par or the permutation used to show that $x \in \mathcal{C}$. The function f is injective, indeed:

Assume the work is already done for a subset $\mathcal{C}_0 \subset \mathcal{C}$ and that for all $⅋$ or perm. $\in f(\mathcal{C}_0)$ the cells adjacent to those links are already in $\mathcal{C}_0$. If $\mathcal{C}_0 = \mathcal{C}$ the proof is finished otherwise $\exists x \in \mathcal{C} \backslash \mathcal{C}_0$ in one of the three cases:

- x is adjacent to a border b. Since only one cell is adjacent to b, no cell of $\mathcal{C}_0$ is adjacent to it, so $b \notin f(\mathcal{C})$. We extend f to $\mathcal{C}_0 \cup \{x\}$ with $f(x) = b$
- x is above $⅋_0$, c and c' are two cells adjacent to the principal edge of $⅋_0$ belonging to $\mathcal{C}_0$. We set $f(x) = ⅋_0$

  The induction hypothesis still apply to $\mathcal{C}_0 \cup \{x\}$
- x adjacent to a permutation p_0 and c,c',c'' the three other cells adjacent to p_0 are in $\mathcal{C}_0$. We set $f(x) = p_0$

  The induction hypothesis still applies to $\mathcal{C}_0 \cup \{x\}$.

Definition: A *proof-net* is a proof-structure such that every cells is in $\mathcal{C}$ and such that #cells=#borders+#par+#permutations.

We say that a proof-net is a correct proof-structure. In this case the function f is bijective. We show now that it is unique :

Let f_1 and f_2 two such functions. Let $\mathcal{C}_1 = \{x / f_1(x) = f_2(x)\}$.

If $\mathcal{C}_1 \neq \mathcal{C}$ $\mathcal{C}_1$ is not close for our three case then it exist an x in one of the three following case:

- $f_1(x) = b$ border and $f_2(x) \neq b$ but b can only be the image of x. f_2 would not be bijective contradiction.

- $f_1(x) = ⅋_0$ and $f_2(x) \neq ⅋_0$ also here $⅋_0$ can only be the image of x contradiction.

- $f_1(x) = p_0$ $f_2(x) \neq p_0$ furthermore for c, c', c'' on a $f_1 = f_2$ and then $f_2(c) \neq p_0$, $f_2(c') \neq p_0$, $f_2(c'') \neq p_0$ p_0 is not in the image of f_2 contradiction.

For a proof-net there is a unique inductive way to check if cells belongs to $\mathcal{C}$. Starting by the border we move up between the premises of the par and the permutation. In the neighborhood of each non torsion link the status of germs of edges being premise or conclusion of this link is well defined: conclusion links have 1 premise and no conclusion, connective links have 2 premises and 1 conclusion(the principal edge), permutation links have 2 premises and 2 conclusions (the cell x such that $f(x) = p$ is the unique cell adjacent to the two premises of p). For an edge (in the abusive sense of an edge carrying some torsions) there are three possible status: on one end the edge is premise, on the other it is conclusion, the edge is called *ordinary edge*,the edge is premise on both ends, it's called *axiom edge*, the edge is conclusion on both ends, it's called *cut edge*.

Theorem: *A proof-structure is the translation of a sequential proof if and only if it is a proof-net.*

Proof: We show first that translations of sequential proofs are proof-nets: by induction

- The axiom: we have 2 cells on the border and the axiom has 2 conclusions. Remark that the convention on typing is coherent with the fact that the conclusions have dual types.

- The cut: We suppose that the result is true for the two sub-proof-structures π_1 et π_2. Gluing them together we obtain

  #conclusions(π)=#conclusions(π_1)+#conclusions(π_2)-2
  #cells(π)=#cells(π_1)+#cells(π_2)-2

  The fact that cells belong to $\mathcal{C}$ is checked independently in the two parts. Moreover this fits with the convention on typing.

- Torsion: trivial

- Permutation: one more cell, one more permutation. The cell above the permutation is immediately reached from the border and then the induction hypothesis allows us to conclude for every other cells of the proof-structure.

- Par: one mor par, one conclusion less. Same argument as for the permutation.

- Tensor: #conclusions(π)=#conclusions(π_1)+#conclusions(π_2)-1
  #cells(π)=#cells(π_1)+#cells(π_2)-1

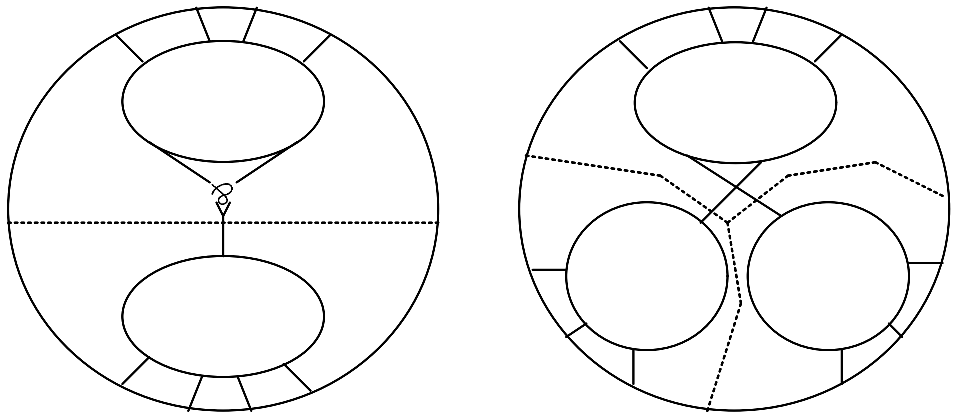
  Same argument as for the cut.

Remarks: A cut on an axiom doesn't change the translation. In the three last cases the new edges are ordinary (neither axiom nor cut).

Reciprocally proof-nets are translation of sequential proofs by induction on #par+#permutation

In the case there is neither par nor permutations, by correctness, each cell is adjacent to exactly one border: the graph of the proof-net is then a tree. If there is no tensor link, it's an axiom (which is perhaps carrying some torsions) and one check trivially that the proof-net is indeed the translation of a sequential proof. If there is a tensor we go down along his principal edge reaching either another tensor by one of his premise in which case we repeat the process either to a conclusion (the typing forbids us to reach another tensor by it's principal edge) and the last visited tensor is then splitting. It's a good candidate to be the translation of a last non torsion rule of a sequential proof (indeed if the complete proof structure is correct then the two sub-proof-structures π_1 and π_2 are correct to).

#⅋+#perm.$\geq$ 1: The set of cells adjacent to the border is not closed, then we have a par or a permutation in a configuration like this:



1^{st} case: Each of the two edges conclusions of the permutation (resp. the principal edge of the par) either reach a conclusion of the proof-net either are cut edges. We apply the cut if it's the case, some possible torsion and at last the permutation (resp. the par). By induction hypothesis the proof-net is sequentialisable.

2^{nd} case: At least one of the conclusion edge of the permutation or the principal edge of the par are neither cut edge neither conclusion of the proof-net. In this case it's not necessary to make a cut (it's indeed possible, this will be a cut on an axiom and then apply the 1^{st} case). But in the sub-proof-structure , this edge is an axiom edge and this proof-structure is a proof-net which isn't reduced to an axiom. By induction hypothesis it's sequentialisable (it contain at least one par or one permutation less). The last non torsion rule associated to a possible sequentialisation of this proof-net is still a good candidate as the last non torsion rule of the whole proof-net.

□

This second case has been considered to show that a proof-net without cut edge sequentialize in a cut free sequential proof . For the converse we must consider that if one premise of a cut rule is an axiom followed by torsions, this cut will disappear in the translation process.

3.3 Contexts and sub-proof-nets

A*context* is a proof-net with “holes”; This mean that we have the same cells, the same typing rules but we are no more necessarily isotope to D^2: A context is a zero gender orientated surface with a distinguished connected component of the border. Such a surface is represented as embedded in the plan with the distinguished border outside.

Internal borders are sequent-hypothesis. The types are read starting from the conclusion link (wich in this case should be called *hypothesis links*. Germ of edges are conclusion in the neighborhood of such a link.

A context with only one internal border and only ordinary edge connecting the internal and the external border is said *trivial*.

The sub-contexts are defined as the sub-proof-structures but relatively to a context instead of a proof-structure.

The correctness is defined exactly as for the proof-structures but only the external border is taken into account.

We call *sequential deduction* a sequential proof where some leaves are hypotheses (it can be an arbitrary sequent but marked as an hypothesis) instead of logical axioms.

Proposition: *The translation of a sequential deduction is a correct context, furthermore sequential deductions and context can be composed and the translation of the composite is the composite of the translations.*

The proof is easy and it is left to the reader. Reciprocally we have:

Proposition: *Let $\mathcal{C}$ a correct context $\mathcal{C}_1 \ldots \mathcal{C}_n$ sub-contexts of $\mathcal{C}$ disjoint and correct then $\mathcal{C} \backslash \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$ and $\mathcal{C}_1 \ldots \mathcal{C}_n$ are translations of sequential deductions.*

Proof: We only need to make the proof for $\mathcal{C} \backslash \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$ furthermore, we can suppose that all the internal borders of $\mathcal{C}$ are in $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$. Elsewhere, for these borders we take a trivial context and if $\mathcal{C}_1 \ldots \mathcal{C}_n$ are all trivial it simply remain to prove the sequentialisation of a correct context (this also allow us to conclude for $\mathcal{C}_1 \ldots \mathcal{C}_n$).
The proof follows the same pattern as the one for proof-nets.

- There is neither par, tensor nor permutation in $\mathcal{C} \backslash \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$. To see what is happening we can consider $\mathcal{C}_1 \ldots \mathcal{C}_n$ as internal n-ary vertices, By the correctness criterion the graph is a tree and then:

  If there is no internal borders it's an axiom (with some possible torsions).

  If there is exactly one internal border it's a trivial context (with some possible torsions).

If there are severals internals borders, there are some edges between these internal borders. These edges are cut edges and we apply the cut rule. In each part we have now less internal borders.

- there is no par no permutation but at least one tensor:

  The graph of the context still being without cycle, we follow the principal edges until we reach a border. If this border is an internal one, the principal edge of the last visited tensor is a cut edge. Applying the cut this edge now reach the external border and is splitting.

- There is at least one par or one permutation: We apply the same argument that in the proof-net case.

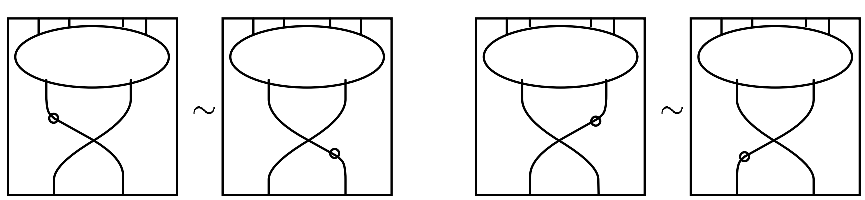
□

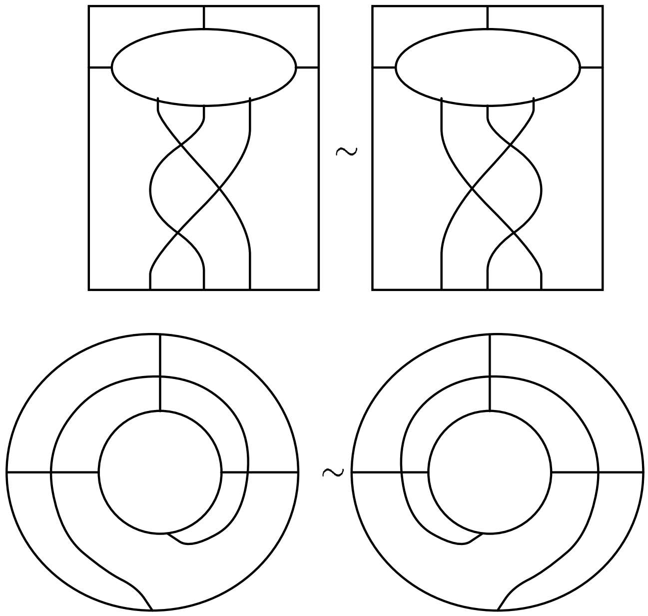
Suppose we have regularly embedded a disjoint union of circle S^1 in a proof-net such that the part inside each circle is correct. The nesting of the circles give us a tree structure of sub-proof-nets and sub-sub-proof-nets. By an inductive application of the last proposition, we see that it's possible to sequentialize keeping the trace of this tree structure.

3.4 Equivalences of proof-nets

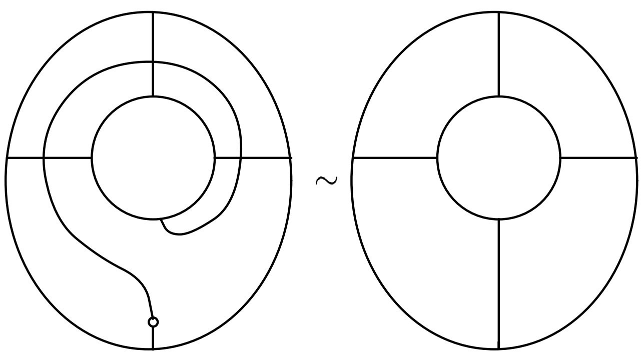
The translation of sequential proofs into proof-nets is quotienting the structure up to trivial commutations. But it's quotienting more. There is an apparent ambiguity related to the lack of orientation of torsions. We will then translate now the equivalences of proofs and show that this identification is harmless.

Boxes stand here for arbitrary sub-proof-nets. The four non trivial commutations give us:

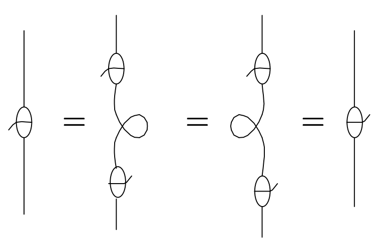




And the simplification rule:



About the torsions, using equivalence 2 and 3 we find:



The fact that axiom-cut are not translated is normal. This is because the proof-nets are a kind of natural deduction system: Substituting an axiom in an hypothesis doesn't change the proof.

4 Commutation of the exchanges

4.1 Commutation of the exchanges with fixed logical skeleton

We will call *logical skeleton* of a sequential proof with explicit exchange, the proof without explicit exchange where we have simply forgotten about the rules of permutation and of torsion. This is indeed meaningful because in the case without explicit exchange, sequents are multi-sets of formulae (this is the ordinary sequent calculus for commutative linear logic).

Now we want to commute permutations and torsions in order to do them as late as possible, the logical skeleton staying unchanged. We will see that the only rule blocking this process is the par rule. This will allow us to internalize permutation and torsion in the par rule. The remaining uncaught ones reach the end of the proof. But remember that the structural group is a group. Thus these rules are reversible (until something has been caught by a par) so we will just forget about them. An other possibility could have been to consider this last sequent as an implicit par, for example, by marking one particular formula of the sequent (as the left-most formula in a $n - ary$ par).

Lemma: *Let $g' = \sigma_{n-1} g$ and $d' = d\sigma_{n-1}$ $(n \geq 2)$ then for all word m in the structural monoid, there exist l and n such that $m = ln$, l being built from g, d and θ_0 and n being build from g', d', $\sigma_1 \dots \sigma_{n-2}$, $\theta_1 \dots \theta_{n-1}$ (if $n = 1$ we have $m = l$).*

Proof: If $n = 1$ $\quad \theta_1 = \theta_0$ else we have:

$\sigma_{n-1} = d^{n-1}\theta_0 d'$ and $\theta_n = d^{n-1}g^{n-1}\theta_0 \dots \theta_{n-1}(\sigma_1 \dots \sigma_{n-2})^{n-1}$

so the monoid is generated by g, d, θ_0, g', d', $\sigma_1 \dots \sigma_{n-2}$ and $\theta_1 \dots \theta_{n-1}$.

Since θ_0 commutes with everything it's enough to show that the generators of the 2^{nd} group can commute to the right of g and d:

$\sigma_i d = d\sigma_{i-1} \qquad 2 \leq i \leq n-2$

$\sigma_i g = g\sigma_{i+1} \qquad 1 \leq i \leq n-3$

$\theta_i d = d\theta_{i-1} \qquad 2 \leq i \leq n-1$

$\theta_i g = g\theta_{i+1} \qquad 1 \leq i \leq n-2$

$\sigma_1 d = g^n\theta_0\sigma_1 d = g^{n-1}\theta_0 g\sigma_1 d = g^{n-1}\theta_0 d\sigma_{n-1} g = g^{n-1}\theta_0 dg'$

$\sigma_{n-2} g = g\sigma_{n-1} = gd^{n-1}\theta_0 d'$

$\theta_1 d = d\theta_n = g^{n-1}\theta_1 \dots \theta_{n-1}(\sigma_1 \dots \sigma_{n-2})^{n-1}$

$$\theta_{n-1}g = g\theta_n = gd^{n-1}g^{n-1}\theta_0 \ldots \theta - n - 1(\sigma_1 \ldots \sigma_{n-2})^{n-1}$$

$$d'd = d\sigma_{n-1}d = d^2\sigma_{n-2} \qquad n \neq 2$$

$$d'd = d\sigma_1 d = dg\sigma_1 = dgd\theta_0 d' \qquad n = 2$$

$$d'g = d\sigma_{n-1}g = dg'$$

$$g'd = \sigma_{n-1}gd = d^n\theta_0\sigma_{n-1}gd = d^{n-1}\theta_0 d\sigma_{n-1}gd$$
$$= d^{n-1}\theta_0 g\sigma_1 d^2 = d^{n-1}\theta_0 gd^2\sigma n - 1 = d^{n-1}\theta_0 gdd'$$

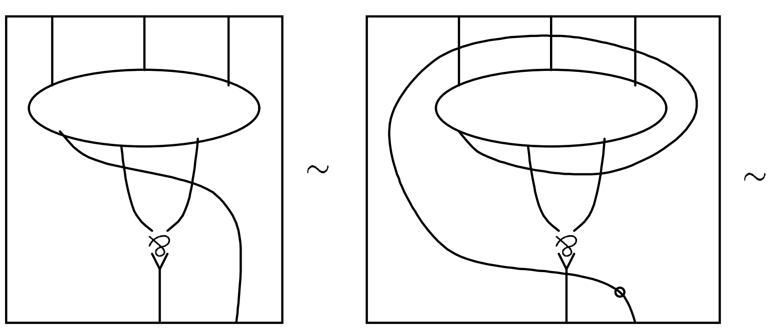
$$g'g = \sigma_{n-1}g^2 = g^2\sigma_1 \qquad n \neq 2$$

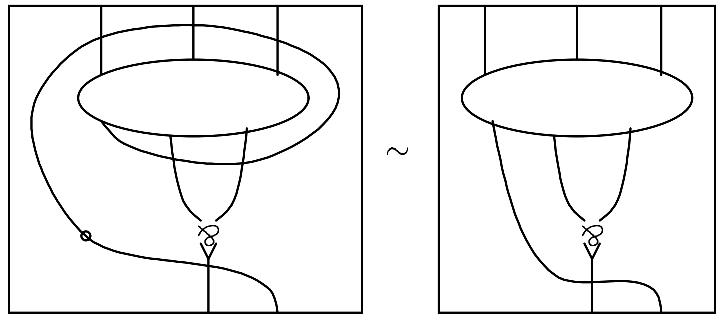
$$g'g = \sigma_1 g^2 = g^2\sigma_1 = g^2 d\theta_0 d' \qquad n = 2$$

Now take the leftmost d or g which is on the right of some generator of the second group and apply the commutation. if there is more than one generator of the second group in this position then their number has diminished. If you had only one, then the maximum number of d and g on the right of some generator of the second group has diminished. So the process of reaching the desired form is terminating and the lemma is true. Graphically, the distinguished formula is in the first time part of every exchange and then it "moves parallely" to the formula which is just on his left.

We will now make the exchanges commute below the connectives:

Case of the par: Suppose that we have a sequence of permutations and torsions followed by a par. We distinguish the right premise of the par and we put the exchanges under the form described in the preceding lemma. $\sigma_1 \ldots \sigma_{n-2}$ and $\theta_1 \ldots \theta_{n-1}$ commute trivially with the par. Let us show graphically what is happening in the case of g':

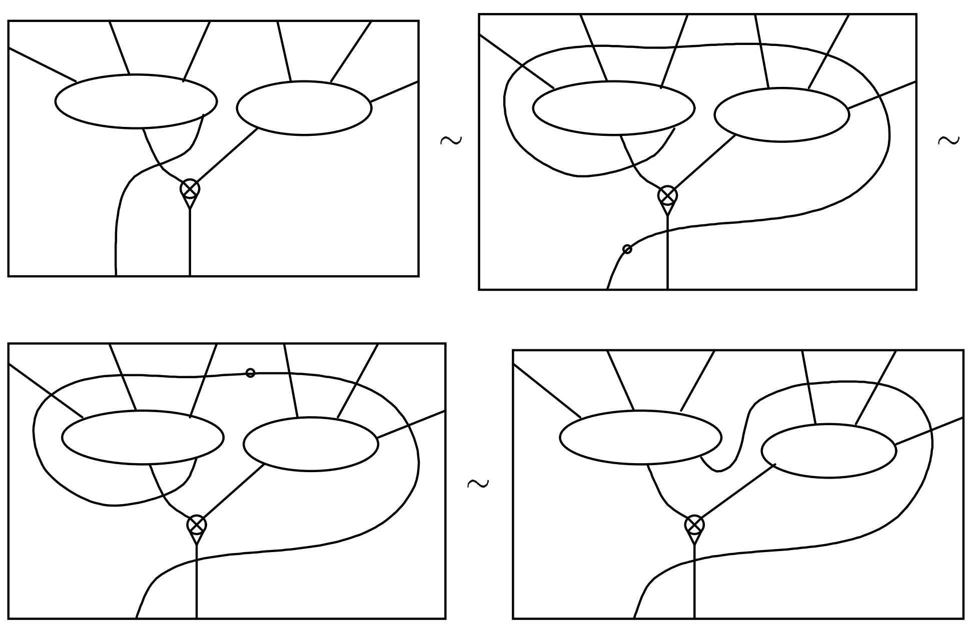




The exchanges have passed from the premises to the conclusion of the par. The case of d' is symmetrical to this one. The g, d and θ_0 stay there.

Case of the tensor: We have some exchange rule and then a tensor. Let's distinguish one of the premise of the tensor (say the left one). The only exchanges which do not commute trivially with the tensor are g, d and θ_0.

Let's show graphically what is happening in the case of d:



The permutations are now after the tensor.

For the case of θ_0 consider the following calculation:

Let's do $20(1) \times \ldots \times 20(n)$ and by (2), (5), (9) and (15) we obtain: $(\sigma_1 \ldots \sigma_{n-1}\theta_n)^n g^n = 1$ then by (19) we find: $\theta_0 = (\sigma_1 \ldots \sigma_{n-1}\theta_n)^n$ and the work is done.

The other cases (g for the left premise and g, d and θ_0 for the right premise) are similar and also the commutations with a cut instead of a tensor.

□

We actually reach what we claimed by the same kind of induction as for the preceding lemma (taking the deepest par, tensor or cut which is behind a non canonical exchange rule). Except in the cases of g, d and θ_0 on the right premise of a par, the permutations and the torsions reach the end of the proof where we forget about them.

4.2 Exchange in canonical form

Let a proof in the usual MLL sequent calculus, the implicit use of exchange rules can be made explicit in the following way: We never make any permutation nor torsion except perhaps just before a par. Indeed the par can be done only if it's left premise is immediately to the left of it's right premise. In this case we just do it. otherwise we permute the right premise with the formula immediately on it's left and so on until we reach the left premise. Then we do the par. In our braided case, if the only remaining exchanges are on the right premise of some par, they can be encoded in a word in g, d and θ_0. Now multiply this word on the left by $g^n\theta_0$. The right premise of the par is then coming immediately to the right of the left premise and then do some exchanges and then the par. These exchanges are represented by a word in g, d, θ_0 such that $\#g \equiv \#d\,[n]$.

Lemma: *The subset of the structural group generated by g, d and θ_0 such that $\#g \equiv \#d\,[n]$ is isomorphic to $\mathbb{Z} \times FG(n-1)$ where $FG(n)$ is a free group on n generators.*

We have $g^n = d^n = \theta_0^{-1}$ and θ_0 do commutes with all the elements of the group, It will be the generator of $\mathbb{Z}$. For $0 < i < n$ let $u_i = d^i g^i$ and $\overline{u}_i = g^{n-i}d^{n-i}\theta_0^2$. The u_i will form the bases of our free group. We first easily verify that the application from $\mathbb{Z} \times FG(n-1)$ to our subset of the structural group is a surjective morphism of groups. For the injectivity we will use a topological argument (cf next proposition).

We call *canonical form of the exchange* the ordinary commutative sequent calculus where a comment is attached to every par rule, this comment being an element of $\mathbb{Z} \times FG(\#\Gamma)$, Γ being the context of the par (the relative number represents the torsions and the element of the free group represents the permutations).

Proposition: *Two proofs with the same logical skeleton are equal if and only if the comments on each par rule are equal (taking the reduced representation of the element of the free group).*

This proposition will be proved in paragraph 5.3. It implies the injectivity of the morphism of the precedent lemma.

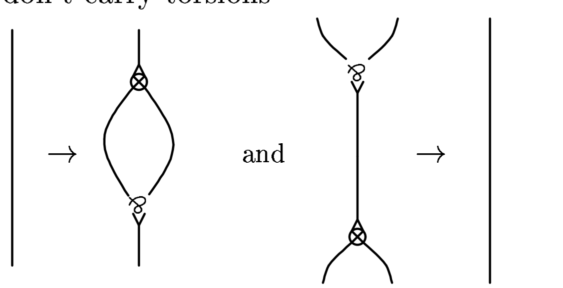
5 Cut eliminatition and equality of proofs

5.1 Cut elimination and η-rule

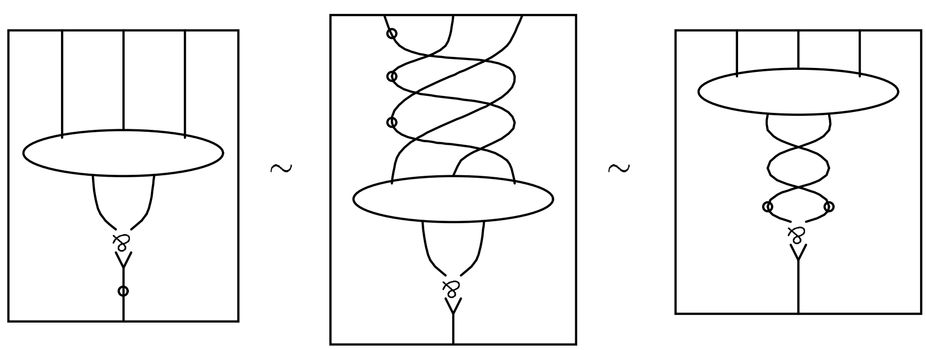
In a proof-net, When the two ends of a cut edge are connective links, we will call it a *multiplicative* cut. All the other cases will be called *exchange* cuts (one at least of the premise of the cut is a permutation link).

Let's show first how to η-expand the axioms and eliminating multiplicative cuts. If they don't carry torsions

we have:



If there are some torsions we can "push them over the par" and we can conclude. Indeed $gd\theta_0\theta_n = (\sigma_1 \ldots \sigma_{n-2}\theta_{n-1})^{n-1}$ and then:



In the translation of a sequential proof in canonical form the cuts are multiplicative cuts and don't carry torsions. There elimination is then easy. It enough to put the proof again in canonical form to iterate the process (which strictly shrinking) or to notice that the cut have already been eliminated. Even if this proof is constructive it cannot be taken as a serious normalization algorithm. In particular it is not confluent, this is why we need the next chapter.

5.2 Commutations of multiplicatives rules

We are able to decide the equality of two proofs with the same logical skeleton and also those without explicit exchange (usual multi-set calculus) using the trivial commutations. But with the comments on the par, these commutations are a bit less trivial (or even forbidden). When a par is before another rule the techniques of the preceding chapter are enough to conclude. When a tensor is before a par the situation is not so simple: We cannot always apply the commutation we would have without the comment.

We produce here a semantical argument which allow us to say when the commutation can be done.

The phases

One associate to each cell an element in a free group whose number of generators is equal to to #border+#par. This function will be called *phase* and denotated ϕ. The computation of the phases will be done following the correctness checking pattern.(the function f is the one describe in paragraph..)..

- if $f(c) = b$ is a border then $\phi(c) = x_b$
- if $f(c) = \wp_0$ is a par then $\phi(c) = x_{\wp_0}$
- if $f(c) = p_0$ is a permutation, let c_1, c_2, c_3 the three other cells adjacent to p_0, respectively on the left of the left conclusion, between the two conclusions and on the right of the right conclusion of the permutation. By the hypothesis of correctness of the proof-net, the values $\phi(c_1)$, $\phi(c_2)$ and $\phi(c_3)$ have already been calculated. We pose: $\phi(c) = \phi(c_1)\phi(c_2)^{-1}\phi(c_3)$

In the case of a context there is of course no generators associated to the internal borders. Let R a proof-net, R' a sub-proof-net, $R\backslash R'$ being a correct context. If we have calculated the phases for the proof-net R' and the context $R\backslash R'$, outside of R' the phases for R are the same as for $R\backslash R'$ and inside the sub-proof-net they are obtained by substituting to the generators of the border of R' the values obtained on the internal border of the context $R\backslash R'$ (easy checking).

Definition: Let a an edge with a chosen orientation let l and r the cells respectively on the left and on the right of the edge (a oriented upward). We call *phase* of a and we denotate $\phi(a) = \phi(g)\phi(d)^{-1}$.

Proposition: *If a tensor is followed by a par, the commutation is possible if and only if each of the premises of the tensor is conjugated to a term who doesn't contain the generator associated to this par.*

Proof: We will show, first that either one knows how to make the commutation, or all the conjugates of the phase of at least one of the premise of the tensor contain $x_\otimes$ (the reciprocal will be proved in the next chapter).

We are reasoning in the sub-proof-net associated to the par.

We start by computing the phases of a context with $n+1$ conclusions where the marked formula is permutated with k ($k \leq n$) formulae to it's left (resp. to it's right). We denotate x_0, $x_1 \ldots x_n$(resp. x'_0, $x'_1 \ldots x'_n$) the terms associated to the cells of the external (resp. internal)border starting with the marked formula and turning in the positive direction.

$$\text{We find: } \begin{cases} x'_0 = x_k \\ \vdots \\ x'_{n-k} = x_n \\ x'_{n-k+1} = x_n x_0^{-1} x_1 \\ \vdots \\ x'_n = x_n x_0^{-1} x_k \end{cases} \text{ for } g^k \text{ and: } \begin{cases} x'_0 = x_{n-k} x_n^{-1} x_0 \\ \vdots \\ x'_{k-1} = x_{n-1} x_n^{-1} x_0 \\ x'_k = x_0 \\ \vdots \\ x'_n = x_{n-k} \end{cases} \text{ for } d^k.$$

Let p (resp. q) the number of formulae in the context of the left premise A (resp. right premise B) of the tensor. Suppose that just after having introduced the tensor this one is of index t relatively to the marked formula (future right premise of the par). To the cells corresponding to this border will be associated the terms x'_0, $x'_1 \ldots x'_{p+q}$. The cells respectively to the left, to the right and above the tensor will then be of the following indices:

$t-1$, t and $t+q$ ($t-1$ and $t+q$ are taken modulo $p+q+1$) and then $\phi(A) = x'_{t+q} x'^{-1}_{t-1}$ $\phi(B) = x'_t x'^{-1}_{t+q}$

After having introduced the tensor we bring the marked formula just to the right of the future left premise of the par (by g^k $k < p+q$). If we denote $x_0 \ldots x_{p+q}$ the image by ϕ of the cells of the border of this sub-proof-net, we will find for $\phi(A)$ and $\phi(B)$ the following values:

- $t = 0$
  - $k = 0$

    $\phi(A) = x_q x_{p+q}^{-1}$

    $\phi(B) = x_0 x_q^{-1}$
  - $0 < k \leq p$

    $\phi(A) = x_{q+k} x_k^{-1} x_0 x_{p+q}^{-1}$

    $\phi(B) = x_k x_{q+k}^{-1}$
  - $p < k < p + q$

    $\phi(A) = x_{p+q} x_0^{-1} x_k x_{k-p}^{-1} x_0 x_{p+q}^{-1}$

    $\phi(B) = x_k x_{k-p}^{-1} x_0 x_{p+q}^{-1}$
- $1 \leq t \leq p$
  - $t + q \leq p + q - k$

    $\phi(A) = x_{t+q+k} x_{t-1+k}^{-1}$

    $\phi(B) = x_{t+k} x_{t+q+k}^{-1}$
  - $t \leq p + q - k < t + q$

    $\phi(A) = x_{p+q} x_0^{-1} x_{t-p+k} x_{t-1+k}^{-1}$

    $\phi(B) = x_{t+k} x_{t+k-p}^{-1} x_0 x_{p+q}^{-1}$
  - $t - 1 = p + q - k$

    $\phi(A) = x_{p+q} x_0^{-1} x_{q+1} x_{p+q}^{-1}$

    $\phi(B) = x_{p+q} x_0^{-1} x_{q+1} x_0 x_{p+q}^{-1}$
  - $p + q - k < t - 1$

    $\phi(A) = x_{p+q} x_0^{-1} x_{t-p+k} x_{t-1-(p+q-k)}^{-1} x_0 x_{p+q}^{-1}$

    $\phi(B) = x_{p+q} x_0^{-1} x_{t-(p+q-k)} x_{t-p+k}^{-1} x_0 x_{p+q}^{-1}$
- $t > p$
  - $t \leq p + q - k$

    $\phi(A) = x_{t-p-1+k} x_{t-1+k}^{-1}$

$\phi(B) = x_{t+k}x^{-1}_{t-p-1+k}$

- $p + q - k = t - 1$

  $\phi(A) = x_q x^{-1}_{p+q}$

  $\phi(B) = x_{p+q}x_0^{-1}x_{p+q+1-k}x_q^{-1}$

- $t - p - 1 \leq p + q - k < t - 1$

  $\phi(A) = x_{t-p-1+k}x^{-1}_{t-1-(p+q-k)}x_0 x^{-1}_{p+q}$

  $\phi(B) = x_{p+q}x_0^{-1}x_{t-(p+q-k)}x^{-1}_{t-p-1+k}$

- $p + q - k < t - p - 1$

  $\phi(A) = x_{p+q}x_0^{-1}x_{t-p-1-(p+q-k)}x^{-1}_{t-1-(p+q-k)}x_0 x^{-1}_{p+q}$

  $\phi(B) = x_{p+q}x_0^{-1}x_{t-(p+q-k)}x^{-1}_{t-p-1-(p+q-k)}x_0 x^{-1}_{p+q}$

Suppose that we do the par right now (this correspond to a $e \in Z \times FG(p+q-1)$). We have $x_\wp = x_{p+q}$. The two phases admit a conjugate which is not containing x_{p+q} exactly in the cases: 2.1($t + k \neq p$), 2.3, 2.4, 3.1($t + k \neq p + q$), 3.3($t+k = 2p+q+1$) et 3.4. These are precisely those for which the commutation arise in the case without explicit exchange. For the case 2.1 ($t + k \neq p$) and 3.1 ($t + k \neq p + q$) commutation is trivial. If t' is the new position of the tensor we have, modulo $p + q + 1$, $\phi(A) = x_{t'+q}x^{-1}_{t'-1}$ and $\phi(B) = x_{t'}x^{-1}_{t'+q}$. In the other cases we use the commutation between the tensor and the permutation of paragraph ... in these cases we have $\phi(A) = x_{p+q}x_0^{-1}x_{t'+q}x^{-1}_{t'-1}x_0 x^{-1}_{p+q}$ and $\phi(B) = x_{p+q}x_0^{-1}x_{t'}x^{-1}_{t'+q}x_0 x^{-1}_{p+q}$. In the other cases we have $\phi(A)$ or $\phi(B) = x_{p+q}v$ or vx^{-1}_{p+q} (v doesn't contain x_{p+q})

We will now reasoning by induction on the length of the comments $\theta_0^n.t$ or $\overline{\theta}_0^n.t$ with $n \in N$ and $t \in FG(p+q-1)$ (t is in reduce form) to show that, if for a comment a all the conjugates of A and B contain $x_\otimes$ then for the comment $a.u$ (u generator) it's also the case. The phases will always be alternated products of x and of x^{-1}. In the cases where one know how to do the commutation for a (then up to conjugation $\phi(A) = x_{t'+q}x^{-1}_{t'-1}$ et $\phi(B) = x_{t'}x^{-1}_{t'+q}$), some u will block it. The will then have the expected property.

- $u = \theta_0$ trivial

- $u = \overline{\theta}_0$ we compose the sub-net for which the calculus is already done with the context g^n ($n = p + q$). Here we have to substitute $x_{p+q}x_0^{-1}x_i$ to

x_i. One obtains $\phi_{a\overline{\theta}_0}(A) = x_{p+q}x_0^{-1}\phi_a(A)x_0x_{p+q}^{-1}$ and the hypotheses are satisfied.

- $u = u_k = d^k g^k$ we check first the effect produced on a context. For $0 \le i \le k$ we obtain: $x'_i = x_{p+q}x_0^{-1}x_ix_k^{-1}x_0x_{p+q}^{-1}x_k$ and for $k \le i \le p+q$: $x'_i = x_i$

  Substituting x'_i to x_i all the generators and conserved and in the same order, then $\phi_a \ni x_{p+q} \Rightarrow \phi_{a.u_k} \ni x_{p+q}$. Suppose now that up to conjugation $\phi_a(A) = x_{t'+q}x_{t'-1}^{-1}$ and $\phi_a(B) = x_{t'}x_{t'+q}^{-1}$ then for $a.u_k$ one finds:

  If $1 \le t' < p$

  $t'+q \le k$: $\phi_{a.u_k}(A) = x_{p+q}x_0^{-1}\phi_a(A)x_0x_{p+q}^{-1}$ and similarly for B and we know how to perform the commutation (should have been a good idea to put some more picture).

  $t'-1 < k < t'+q$: $\phi_{a.u_k}(A) = x_{t'+q}x_k^{-1}x_{p+q}x_0^{-1}x_kx_{t'-1}^{-1}x_0x_{p+q}^{-1}$ this phase and all its conjugates contain x_{p+q}. We will see that the commutation doesn't hold.

  $k \le t'-1$: $\phi_{a.u_k}(A) = \phi_a(A)$ similarly for B. The commutation is trivial.

  If $p < t' < p+q$

  $t' \le k$: $\phi_{a.u_k}(A) = x_{p+q}x_0^{-1}\phi_a(A)x_0x_{p+q}^{-1}$ similarly for B and we know how to perform the commutation.

  $t'-p-1 < k < t'$: $\phi_{a.u_k}(B) = x_{t'}x_k^{-1}x_{p+q}x_0^{-1}x_kx_{t'-p-1}^{-1}x_0x_{p+q}^{-1}$ this phase and all its conjugates contain x_{p+q}. We will see that the commutation doesn't hold.

  $k \le t'-p-1$: $\phi_{a.u_k}(A) = \phi_a(A)$ similarly for B. The commutation is trivial.

- $u = \overline{u}_k = g^{p+q-k}d^{p+q-k}\theta_0^2$ Over a context for $0 \le i \le k$ we obtain: $x'_i = x_i$, for $k \le i \le p+q$: $x'_i = x_kx_0^{-1}x_{p+q}x_k^{-1}x_ix_{p+q}^{-1}x_0$

  Similarly for u_k one check that: $\phi_a \ni x_{p+q} \Rightarrow \phi_{a.\overline{u}_k} \ni x_{p+q}$ and for the cases where $\phi_a(A) = x_{t'+q}x_{t'-1}^{-1}$ and $\phi_a(B) = x_{t'}x_{t'+q}^{-1}$ up to conjugation we find

  If $1 \le t' < p$

$t' + q \leq k$: $\phi_{a.\overline{u}_k}(A) = \phi_a(A)$ similarly for B. The commutation is trivial.

$t' - 1 < k < t' + q$: $\phi_{a.\overline{u}_k}(A) = x_k x_0^{-1} x_{p+q} x_k^{-1} x_{t'+q} x_{p+q}^{-1} x_0 x_{t'-1}^{-1}$ this phase and all its conjugates contain x_{p+q}. We will see that the commutation doesn't hold.

$k \leq t'-1$: $\phi_{a.\overline{u}_k}(A) = x_k x_0^{-1} x_{p+q} x_k^{-1} \phi_a(A) x_k xp + q^{-1} x_0 x_k^{-1}$ similarly for B and we know how to perform the commutation. If $p < t' < p + q$

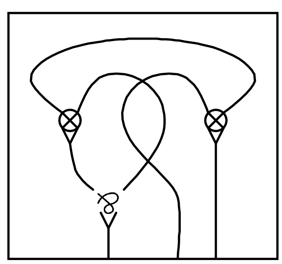
$t' \leq k$: $\phi_{a.\overline{u}_k}(A) = \phi_a(A)$ similarly for B. The commutation is trivial.

$t' - p - 1 < k < t'$: $\phi_{a.\overline{u}_k}(B) = x_k x_0^{-1} x_{p+q} x_k^{-1} x_{t'} x_{p+q}^{-1} x_0 x_{t'-p-1}^{-1}$ this phase and all its conjugates contain x_{p+q}. We will see that the commutation doesn't hold.

$k \leq t'-p-1$: $\phi_{a.\overline{u}_k}(A) = x_k x_0^{-1} x_{p+q} x_k^{-1} \phi_a(A) x_k xp + q^{-1} x_0 x_k^{-1}$ similarly for B and we know how to perform the commutation.

□

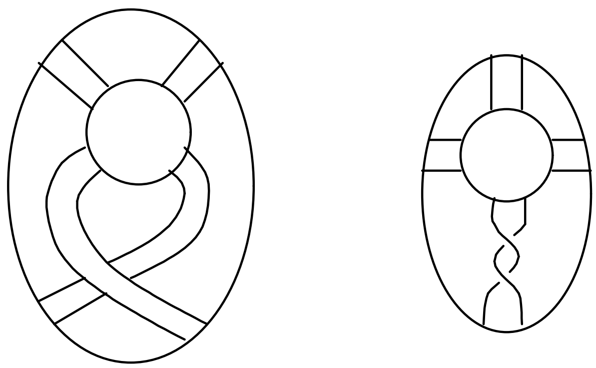
Let's now give an example where it's the exchange who blocks the commutation:



Without explicit exchange, the tensor on the right would have been splitting. Here it doesn't commute with the par and can't be the last rule. Intuitively, in order to have the commutation, the two premises of the par must be in the context of the same premise of the tensor and for each generator of the free group if we have crossed one formula in the context of the other premise of the tensor or the conclusion of the tensor, we must have crossed every one of them.

6 The proof-nets in $\mathbb{R}^3$

We give here a topological interpretation of proof-nets which will allow us to prove that the method we used to compare proofs is indeed complete. Take one more time the translation of the sequent calculus but giving a thickness to the planar proof-structures. The edges will have a certain width becoming like ribbons. The border of planar proof-structure being the equator of a sphere border of a ball B^3 where the "graph" (a surface) will be embedded (this one "touches" δB^3 only on this equator). We show graphically how permutation and torsion are interpreted:



In the case of the permutation, the left premise passes over the right premise. In the case of the torsion, the ribbon corresponding to the principal edge does a complete turn on itself in the clockwise sense (to my knowledge M.C.Shum [10] introduce this kind of structure). For the other two rules take the evident translation where nothing is going out of the equatorial plane. We can check that the equivalences of rules preserve ambient isotopy class. This means that we can pass from a form to the other by a continuous deformation of the ball B^3 (and then of the included ribbon). But we can do better than this:

Theorem: *Two cut-free proof-nets with only atomic axioms, are equal if and only if they are isotope as proof-nets in B^3. Furthermore, the method described in the former paragraph allows to decide about this equality.*

Let us now describe the decision algorithm. Given two proofs of the same multi-set of formulae, first put them in canonical form, cut-free and η-expanded. If they have the same logical skeleton, we compare the comments, otherwise by applying commutation rules, we try to impose on the second proof the same logical skeleton as the first one. More precisely: if the last rule of the first proof is a par then it's evidently possible to make it commute in order to reach the lowermost position and we can pass to the preceding rule. We can then suppose by induction that the last rule of the first proof is a tensor. Let's compute the phases of the premises of the corresponding tensor for the second proof. If the conjugation classes of these phases both contain an element of the group of the border, we know how to make the tensor commute and reach the lowermost position. It remains to prove that in case of failure of this algorithm the two proofs are not isotope.

Lemma: *Let $\pi_1(\Pi, p_0)$ be the fundamental group of the complement of the graph of a proof Π in B^3 relatively to a point $p_0 \in b_0$ a border. We have a surjective morphism of phase group over $\pi_1(\Pi, p_0)$ whose kernel is generated by x_{b_0}. This application sends the group generated by the phases of the border onto the fundamental group of the border of Π: $\delta B_3 \backslash \{conclusions\}$. In particular, this fundamental group is a free group with $\#conclusions + \#par - 1$ generators, the one of the border has $\#conclusions - 1$ generators.*

Proof: Let c be a cell, the image of $\phi(c)$ is the path which, starting from $p_0 \in b_0$, passes under the graph of the proof, rises through the cell c and goes back to p_0 passing over the graph. We proceed by induction on the sequential proof:

- In the case of the axiom, as expected we have a fundamental group isomorph to $\mathbb{Z}$ generated by the path which is turning around the ribbon.
- In the case of the cut or of the tensor, the result follows from an elementary application of the Seifert-Van Kampen theorem (saying that, if A, B and $A \cap B$ are connected by arcs, then the fundamental group of $A \cup B$ is the amalgamated sum of those of A and B over the one of $A \cap B$).
- Adding a par induce a retraction of the complementary of the graph and then the fundamental group is unchanged. The cell who was between the premises of the par is now above the par.
- The case of the torsion is trivial.
- In the permutation case, it's enough to check that the equations on the phases do pass to the fundamental group. □

Now let Π_1 and Π_2 be two proofs of the same multi-set of formulae, where Π_1 is in canonical form, cut-free and with only atomic axioms. We suppose that only the isotopy classes of Π_2 is known. To eliminate the cuts and η-expand the axioms, we just have to cut the ribbons in the sense of the length. Let's now test the equality of the proofs.

If the last rule of Π_1 is a par, it does exist in Π_2 a unique path starting from the left premise of this par following the graph but never an edge which is the right premise of another par and reaching the right premise of our par. The number of time the ribbon we are following turns on itself gives us the torsion associated to this par. This definition depends only on the isotopy class (in particular it is invariant by commutations) and we check on Π_1 that it is indeed the torsion of the last rule. Let's now cut the right premise of every par, the complementary of the graph retracts now on its border. Start from a point p_0 of the border in a neiborhood of $A ⅋ B$, following the path we just described, we come back to p_0. By the retraction we inherit an element of $\pi_1(\delta B_3 \backslash \Gamma \cup \{A ⅋ B\}, p_0)$. The circular order of the formulae in the root sequent depends only of the graph of the proof (and not of the particular embedding in B^3). A base can then be chosen knowing only the isotopy class of the proof. As a base, we choose the paths passing above the equator, making one turn around the formulae of Γ and back to the starting point by the same way. In the case of Π_1 one checks that the word produced in this way does correspond to the free group part of the comment. If the comments are different then the proofs are not isotope. If the last rule of Π_1 is a tensor, it's enough to check that this tensor is splitting in Π_2. The phase of an edge is associated to a path, passing under the graph turning around the edge in the positive sense and returning by the same way. If we are reasoning up to isotopy, we can take a path $c(A)$ turning around the edge A, but, in order to have an element of the fundamental group, we still have to choose a path to the base point. We will take all the possible path, this will give us a conjugation class $C(A) \subset \pi_1(\Pi, p_0)$. The condition $A \otimes B$ is splitting if and only if $C(A) \cap \delta B_3 \backslash \{conclusions\} \neq \emptyset$ et $C(B) \cap \delta B_3 \backslash \{conclusions\} \neq \emptyset$ does depend only on the isotopy class of Π. If this condition is not true for Π_2, this last one is then not isotope to Π_1.

7 Conclusion

Despite of the strong geometrical intuition about 3D-proof-nets, this work sticks to the syntactical aspects. Several other aspects have to be considered, first of all the categorical semantics which will give this work its natural audience. Rick Blute [5] has already given a definition of braided *-autonomous categories. The typical example of such a category is the category of modules over a quantum group. We can even hope a completeness result in this area. On the other hand, a proof-object is an ambient isotopy class of ribbon graph. So the correctness criterion should be itself invariant by isotopy. It's relatively easy to find a property which looks characteristic to our objects : If we generalize the notion of causal chain defined by Asperti [3], we find that these causal chains are unknotted. I conjecture that any proof-structure whose all the causal chains are unknotted is correct.

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