MATHEMATICAL FUZZY LOGIC – STATE OF ART 2001

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Abstract

Present state of development of mathematical fuzzy logic is surveyed.

The aim of this paper is to survey the present state of development of mathematical fuzzy logic (or fuzzy logic in the narrow sense) based on the logical systems BL and BL∀ (basic fuzzy propositional and predicate logic) as introduced in my monograph [21]. Note that there was another survey [22] written in 1998. The present paper is based on my lectures held on WOLLIC'2001 in July 2001 in Brasília, Brazil and on Reason Park in August/September 2001 in Foligno, Italy. The paper cannot be any self-contained exposition; it should be understood as a guide for studying the book [21] and later results. (Needless to say, only a selection of results is presented.) The reader should be also informed on four recent monographs dealing with many-valued (fuzzy) logic, each from its specific point of view: Cignoli et al. [7], Gottwald [20] and Novák et. al. [39], and Turunen [40].

1 Propositional fuzzy logic.

Fuzzy logic is understood a logic with a *comparative notion of truth,* the standard set of truth values being the real interval $[0, 1]$ with its usual ordering. 1 is absolute truth, 0 absolute falsity. The logic is built as *truth-functional; continuous t-norms* (binary operations $*$ on $[0, 1]$ that are commutative, associative, non-decreasing and satisfy $1 * x = x$ for each x)[1] serve as truth-functions of conjunction &. Each such t-norm $*$ has its *residuum* $x \Rightarrow y = \max\{z | x * z \leq y\}$

[1] The monograph [36] is recommended as a modern monograph on t-norms, in general very good but unformulately its presentation of properties of our logic BL is incorrect.

serving as the corresponding truth function of implication $\rightarrow$. Note that $x*(x \Rightarrow y) = \min(x, y)$ and $\max(x, y) = \min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x)$. Formulas are built from propositional variables, connectives $\&, \rightarrow$ and truth constant $\bar{0}$ (denoting 0); one defines $\neg\varphi$ to be $\varphi \rightarrow \bar{0}$, $\varphi \wedge \psi$ to be $\varphi\&(\varphi \rightarrow \psi)$ and $\varphi \vee \psi$ to be $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$. Three most important continuous t-norms are Łukasiewicz $\max(0, x + y - 1)$, Gödel $\min(x, y)$ and product t-norm $x \cdot y$. Formulas for their residua are: $x \Rightarrow y = 1$ for $x \leq y$, otherwise $x \Rightarrow y = 1 - x + y$ for Łukasiewicz, $= y$ for Gödel, $= y/x$ for product. The residuum of product is called Goguen implication. Note also that the truth function$(-)x = x \Rightarrow 0$ for negation is $1 - x$ for Łukasiewicz, and Gödel negation($(-)0 = 1, (-)x = 0$ for $x > 0$) for Gödel and product. An evaluation e of propositional variables by truth values extends to an evaluation e_* of all formulas (depending of a chosen t-norm $*$); a formula φ is a $*$-*tautology* if $e_*(\varphi) = 1$ for all e; φ is a t-tautology if $e_*(\varphi) = 1$ for each e and each $*$.

1.1 The basic fuzzy logic and three stronger systems

The following t-tautologies are taken for *axioms* of the logic BL:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi\&\psi) \rightarrow \varphi$
(A3) $(\varphi\&\psi) \rightarrow (\psi\&\varphi)$
(A4) $(\varphi\&(\varphi \rightarrow \psi)) \rightarrow (\psi\&(\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi\&\psi) \rightarrow \chi)$
(A5b) $((\varphi\&\psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $\bar{0} \rightarrow \varphi$

Modus ponens is the deduction rule; this gives a usual notion of proof and provability.

Łukasiewicz logic Ł is BL plus the axiom of double negation $\neg\neg\varphi \rightarrow \varphi$; *Gödel logic* G is BL plus the axiom of idempotence of conjunction $\varphi \rightarrow (\varphi\&\varphi)$. *Product logic* Π, originally introduced in [30], is defined in [21] as the extension of BL by two axioms; Cintula [9] has shown that they can be replaced by the following axiom: $\neg\neg\varphi \rightarrow ((\varphi \rightarrow (\varphi\&\psi)) \rightarrow (\psi\&\neg\neg\psi))$.

General algebras of truth functions for BL are called *BL-algebras.* A BL-algebra is a structure $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1)$ where $(L, \cap, \cup)$ is a lattice with 0 and 1 as least and greatest elements, $(L, *, 1)$ is a commutative semigroup with identity 1 and the following holds for each $x, y, z \in L$:

$$z \leq x \Rightarrow y \text{ iff } x * z \leq y \text{ (residuation)}$$

$$x \cap y = x * (x \Rightarrow y) \text{ (divisibility)}$$

$$(x \Rightarrow y) \cup (y \Rightarrow x) = 1 \text{ (prelinearity)}$$

BL algebras from a variety, i.e. can be defined only by identities. Indeed, residuation may be replaced by identities and Cignoli showed [6] that the following two suffice:

$$x \Rightarrow (y \Rightarrow z) = (x * y) \Rightarrow z \quad \text{(cf. Axiom A5)}$$

$$(x \wedge y) * z = (x * z) \wedge (y * z) \text{ } (\wedge *\text{-distributivity}).$$

Agliano, Ferreirim and and Montagna [2, 3] related BL-algebras to hoops. A *hoop* is an algebra $\mathbf{L} = (L, *, \Rightarrow, 1)$ satisfying for all $x, y, z \in L$ the following:

$x * y = y * x$ (commutativity), $1 * x = x, \quad x \Rightarrow x = 1,$
$x * (x \Rightarrow y) = y * (y \Rightarrow x), \quad x \Rightarrow (y \Rightarrow z) = (x * y) \Rightarrow z.$

(Compare the last two axioms with (A4) and (A5).) One defines $x \leq y$ as $x \Rightarrow y = 1$. Particular hoops: satisfying

$(x \Rightarrow y) \Rightarrow z \leq ((y \Rightarrow x \Rightarrow z) \Rightarrow z$ (basic)
$(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$ (Wajsberg)

A hoop is *bounded* if it has a least element 0. It turns out that BL-algebras are precisely bounded basic hoops; more precisely, $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ is a BL-algebra iff its reduct $(L, *, \Rightarrow, 1)$ is a bounded basic hoop with least element 0.

Each BL-algebra $\mathbf{L}$ can serve as algebra of truth functions for BL, $\Rightarrow, *, \cap, \cup$ being truth functions of $\rightarrow, \&, \wedge, \vee$ respectively. Our axioms A1–A7 are $\mathbf{L}$-tautologies for any BL-algebra $\mathbf{L}$. We have three varieties corresponding to our three stronger logic: A BL-algebra $\mathbf{L}$ is an *MV-algebra* if the axiom of double negation is an $\mathbf{L}$-tautology, i.e. $(-)(-)x = x$ is $\mathbf{L}$-valid ($(-)x$ being $x \Rightarrow 0$). Similarly $\mathbf{L}$ is a G-algebra if $x = x * x$ is $\mathbf{L}$-valid; $\mathbf{L}$ is a *product algebra* if the additional axiom of Π is an $\mathbf{L}$-tautology. As shown in [3], MV-algebras are precisely all bounded Wajsberg hoops. Relation of product algebras to hoops was studied in [1].

If $*$ is a continuous t-norm and $\Rightarrow$ its residuum then $([0,1], \min, \max, *, \Rightarrow, 0, 1)$ is a particular linearly ordered BL-algebra $[0,1]_*$ given by $*$. Such algebras are called t-algebras or standard BL-algebras. The standard MV-algebra (G-algebra, Π-algebra) is just $[0,1]_*$ where $*$ is the Łukasiewicz (Gödel, product) t-norm. Each BL-algebra $\mathbf{L}$ can be isomorphically embedded into a direct product of linearly ordered BL-algebras (subdirect representation).

Let now $\mathcal{C}$ stand for BL, Ł, G, Π, let $\mathcal{C}$-algebras be BL-algebras, MV-algebras, G-algebras and Π-algebras respectively.

General completeness theorem. For each formula φ, the following equivalent: (i) φ is provable in $\mathcal{C}$, (ii) for each $\mathcal{C}$-algebra $\mathbf{L}$, φ is an $\mathbf{L}$-tautology, (iii) for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$, φ is an $\mathbf{L}$-tautology, (iv) for each/the standard $\mathcal{C}$-algebra $\mathbf{L}\forall$ φ is an $\mathbf{L}$-tautology.

Note the the equivalence of (i)–(iii) generalizes to provability in a theory T over $\mathcal{C}$ and truth of φ in each $\mathbf{L}$-model of T; details are in [21]. Concerning the equivalence of (i) and (iv), called standard completeness, for $\mathcal{C}$ being Ł$, G, \Pi$ a proof is found in [21] (together with references to literature); for BL it is a result of Cignoli et. al. [8] (improving substantially my partial result from [23]). We shall discuss it in the next subsection.

1.2 Structure of linearly ordered BL-algebras

The proof of standard completeness for G is easy. For Ł and Π it uses the representation of linearly ordered MV-algebras (MV-chains) as intervals $[0, e]$ in a (linearly) ordered Abelian group (oag) with appropriately defined operations and the representation of Π-chains as intervals $[-\infty, 0]$ in an oag extended by $-\infty$. This is used together with theorem of Gurevich and Kokorin saying that for each oag G and each finite subset $X \subseteq G$ there is a subset Y of reals and one-one mapping of X onto Y preserving addition and ordering. Using this one can convert a counterexample showing that a φ is not an $\mathbf{L}$-tautology, $\mathbf{L}$ being an MV-chain (Π-chain) to a counterexample showing that φ is not a tautology over the standard MV-algebra (Π-algebra).

Showing standard completeness for BL is much more difficult and uses results on the structure of BL-chains that are of independent interest. (They are from [23] and [8].) To sketch them let us first recall the characterization (by Mostert and Shields) of continuous t-norms. Let $*$ be a continuous t-norm and let

$E = \{x | x * x = x\}$ be the (closed) set of its idempotents. For each $x \in E - \{1\}$, let x^+ be the least idempotent bigger than x if such element exists, otherwise $x^+ = x$. If $x \neq x^+$ than the restriction of $*$ to $[x, x^+]$ is isomorphic to Łukasiewicz or product t-norm; if for each $x \in E$, if u, v are not from the same interval $[x \subset x^+]$, then $u * v = \min(u, v)$.

This generalizes to BL-chains as below: each BL-chain can be embedded into another BL-chain which is saturated (roughly, a saturated BL-chain has "sufficiently many" idempotents).[2] Let $\mathbf{L}$ be a saturated BL-chain and let E be its set of idempotents; for $x \in E$ defined x^+ as above. Now if $x \neq x^+$ then the interval $[x, x^+]$ with the structure inherited from $\mathbf{L}$[3] is an MV-chain or a Π-chain. For u, v not belonging to any common $[x, x^+]$, $u * v = \min(u, v)$. Using this one can transfer a counterexample showing that φ is not an $\mathbf{L}$-tautology to a counterexample showing φ not to be a $[0,1]_*$-tautology for a suitable t-algebra.

Agliano and Montagna [3] developed another representation of BL-chains. Let $(\mathbf{L}_i, i \in I)$ be a linearly ordered system of hoops having the same greatest element 1 and otherwise being pairwise disjoint, i.e. $L_i \cap L_j = \{1\}$ for $i \neq j$. Let $L = \bigcup_{i \in I} L_i$ and let $*$ be the operation on L that behaves on each L_i as $\mathbf{L}_i$ says and otherwise for $x \in L_i, y \in L_j, i < j$ let $x * y = x$. Define accordingly $\Rightarrow$ on L and you get a hoop $\mathbf{L}$ which can be called the AM-sum of $(\mathbf{L}_i, i)$. Agliano and Montagna show that each BL-chain is an AM-sum of a system of Wajsberg hoops, the system having a first element which is bounded (has a least element). Note that their paper contains deep results on varieties of BL-algebras, e.g. a characterization of BL-chains generating the whole variety (i.e. for such a BL-chain $\mathbf{L}$, $\mathbf{L}$-tautologies coincide with t-tautologies). Deep algebraic results on varieties of BL-algebras have been also obtained by diNola et al. [13, 14] and others. The paper [33] by Honzíková (=Haniková) on logics of particular continuous t-norms must also be mentioned.

1.3 Computational complexity

Let us close our survey on fuzzy propositional calculi by summarizing known result of computational complexity. $\mathcal{C}$ stands again for BL, Ł, G, Π; for each $\mathcal{C}$

[2] A pair $X, Y \subseteq L$ such that $(\forall x \in X)(\forall y \in Y)(x \leq y$ and $x * y = x)$ and Y is closed under $*$ is called a *cut*. $\mathbf{L}$ is saturated if for each cut there is an idempotent d such that $(\forall x \in X)(\forall y \in Y)(x \leq d \leq y)$.

[3] $*$ is restricted to $[x, x^+]$, also $v \Rightarrow u$ for $x \leq u < v \leq x^+$ is inherited; for $x \leq u \leq v \leq x^+$, $u \Rightarrow v$ becomes v.

we consider four sets of formulas denoted by $TAUT_1^{\mathcal{C}}, TAUT_{pos}^{\mathcal{C}}, SAT_1^{\mathcal{C}}, SAT_{pos}^{\mathcal{C}}$ and defined as follows:

$TAUT_1^{\mathcal{C}} = \{\varphi|$ for each $\mathcal{C}$-chain $\mathbf{L}$ and $\mathbf{L}$-evaluation e, $e_{\mathbf{L}}(\varphi) = 1\}$,
$TAUT_{pos}^{\mathcal{C}} = \{\varphi|$ for each $\mathcal{C}$-chain $\mathbf{L}$ and $\mathbf{L}$-evaluation e, $e_{\mathbf{L}}(\varphi) > 0\}$,
$SAT_1^{\mathcal{C}} = \{\varphi|$ for some $\mathcal{C}$-chain $\mathbf{L}$ and $\mathbf{L}$-evaluation e, $e_{\mathbf{L}}(\varphi) = 1\}$,
$SAT_{pos}^{\mathcal{C}} = \{\varphi|$ for some $\mathcal{C}$-chain $\mathbf{L}$ and $\mathbf{L}$-evaluation e, $e_{\mathbf{L}}(\varphi) > 0\}$,

Note that due to the results formulated above we may restrict our attention just to *standard* $\mathcal{C}$-chains keeping in mind that Ł, Π, G has just one standard $\mathcal{C}$-chain (t-norm) each and standard BL-chains are just t-algebras $[0,1]_*$ (all continuous t-norms). Elements of $TAUT_1^{\mathcal{C}}$ may be called 1-*tautologies* of $\mathcal{C}$ elements of $TAUT_{pos}^{\mathcal{C}}$ *positive tautologies* of $\mathcal{C}$, elements $SAT_1^{\mathcal{C}}, SAT_{pos}$ 1-*satisfiable* and *positively satisfiable* formulas of $\mathcal{C}$.

To put it briefly, everything is as expected: $SAT_1^{\mathcal{C}}$ and $SAT_{pos}^{\mathcal{C}}$ are NP-complete and $TAUT_1^{\mathcal{C}}, TAUT_{pos}^{\mathcal{C}}$ are co-NP-compete. The proofs of these facts are of varying degree of difficulty; for $\mathcal{C} =$ Ł$, G, \Pi$ all are in [21] (with references to original papers if there are any); $TAUT_1^{\mathrm{BL}}$ being co-NP-complete is proved in [5] and the remaining results for BL are (easy but) not yet published. There are several results of equality among these sets, namely $SAT_1^G = SAT_{pos}^G = SAT_1^{\Pi} = SAT_{pos}^{\Pi} = SAT^{Bool}$ (= formulas satisfiable in the classical Boolean logic), $TAUT_{pos}^G = TAUT_{pos}^{\Pi} = TAUT^{Bool}$ (see [21]), $SAT_1^{\mathrm{BL}} = SAT_1^{\text{Ł}}, SAT_{pos}^{\mathrm{BL}} = SAT_{pos}^{\text{Ł}}, TAUT_{pos}^{\mathrm{BL}} = TAUT_{pos}^{\text{Ł}}$ (unpublished). Evidently, $TAUT_1^{\mathcal{C}}$ for different $\mathcal{C}$ are pairwise distinct; also $SAT_{pos}^{\text{Ł}} \supset SAT_1^{\text{Ł}} \supset SAT^{Bool}$, $TAUT_1^{\text{Ł}} \subset TAUT_{pos}^{\text{Ł}} \subset TAUT^{Bool}$ and all these inclusions are strict (see [21]).

2 Predicate fuzzy logic

2.1 The basic predicate fuzzy logic and the three stronger systems

In [21] one works with an arbitrary predicate language given by its *predicates* (each having a given arity) and *constants;* formulas are built from them in the obvious way using object variables, logical connectives $\&, \rightarrow$, truth constant $\bar{0}$ and *quantifiers* $\forall, \exists$. *A standard interpretation* of the language is a structure $\mathbf{M} = (M, (r_P)_{P\,pred}, (m_c)_{c\,const}.)$ where each $m_c \in M$ and for each

n-ary predicate P, r_P is an n-ary fuzzy relation $r_P : M^n \to [0,1]$. To compute truth values one has to fix a continuous t-norm $*$, thus we may speak on a $[0,1]_*$-interpretation. A *general interpretation* over a BL-algebra $\mathbf{L}$ (or an $\mathbf{L}$-interpretation) is a structure as above but r_P is an $\mathbf{L}$-fuzzy relation, i.e. $r_P : M^n \to L$. The notion of free and bound variable is as usual. The *truth value* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of a formula φ, is given by a BL-algebra $\mathbf{L}$, an $\mathbf{L}$-interpretation $\mathbf{M}$ and an evaluation v assigning to each variable x an element $v(x) \in M$ (and for simplicality, assume $v(c) = m_c$). The definition is a la Tarski:

$$\|P(u_1, \ldots, u_n)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(v(u_1), \ldots, v(u_n)),$$

$$\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}, \text{ similarly } \&, *,$$

$$\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}, |v' \equiv_x v\}, \text{ similarly } \exists, \sup$$

(where $v' \equiv_x v$ means that $v'(y) = v(y)$ for all variables y except possibly x). This definition is total for $\mathbf{L} = [0,1]_*$ since the order of $[0,1]$ is complete; for a BL-algebra $\mathbf{L}$ some values may be undefined since the necessary inf/sup does not exist. The $\mathbf{L}$-interpretation $\mathbf{M}$ is *safe* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for all φ and v. (Caution: it is *not* demanded that $\mathbf{L}$ is completely ordered; just all sups/infs used in the definition exist.) A formula φ is an $\mathbf{L}$-*tautology* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 1$ for all safe $\mathbf{L}$-interpretations $\mathbf{M}$ and all corresponding evaluations v. We write $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ for $\inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}|v\}$ and say that φ is $\mathbf{L}$-true in $\mathbf{M}$ if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$. The following formulas are $\mathbf{L}$-tautologies for each linearly ordered BL-algebra $\mathbf{L}$ and are taken for *axioms on quantifiers:*

($\forall 1$) $(\forall x)\varphi(x) \to \varphi(y)$
($\exists 1$) $\varphi(y) \to (\forall x)\varphi(x)$
($\forall 2$) $(\forall x)(\chi \to \psi) \to (\chi \to (\forall x)\psi)$
($\exists 2$) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$
($\forall 3$) $(\forall x)(\varphi \vee \chi) \to ((\forall x)\varphi \vee \chi$

Deduction rules are modus ponens and generalization (from φ derive $(\forall x)\varphi$). Let $\mathcal{C}$ be BL, Ł, Π, G. The *predicate calculus* $\mathcal{C}\forall$ has axioms of $\mathcal{C}$ for connectives (in which φ, ψ, χ vary on formulas of predicate logic) and the above five axioms on quantifiers. A *theory* is a set of formulas; *provability* in a theory T over the logic $\mathcal{C}\forall$ is defined in the obvious way.[4] $T \vdash_{\mathcal{C}\forall} \varphi$ (or $T \vdash \varphi$ of $\mathcal{C}$ is clear from

[4]Deduction rules are modus ponens and generalization.

context) means that φ is provable in T (over $\mathcal{C}\forall$). An $\mathbf{L}$-model of T is a safe $\mathbf{L}$-interpretation $\mathbf{M}$ in which all $\alpha \in T$ are $\mathbf{L}$-true.

General completeness theorem (see [21]). For each theory T and formula φ, $T \vdash_{\mathcal{C}\forall} \varphi$ iff for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model $\mathbf{M}$ of T, φ is $\mathbf{L}$-true in $\mathbf{M}$. In particular, φ is provable in $\mathcal{C}\forall$ iff for each $\mathcal{C}$-algebra $\mathbf{L}$ φ is an $\mathbf{L}$-tautology.

Remarks. If the reader compares this completeness theorem with the corresponding completeness theorem for propositional logic, he/she may ask the following questions:

(1) Why only linearly ordered BL-algebras? What about BL-algebras that are not linearly ordered? The question is whether the axiom ($\forall$3) is $\mathbf{L}$-true in all safe interpretations over an arbitrary BL algebra (the other axioms are). Thus soundness of BL$\forall$ w.r.t. such interpretations had been an interesting open problem for long time; very recently, Esteva and Montagna have found counterexamples.

(2) What about standard completeness, i.e. completeness w.r.t. standard interpretations (over standard $\mathcal{C}$-algebras)? $G\forall$ has standard completeness but the other logics in question not, their standard tautologies are not recursively axiomatizable (see below).

(3) Couldn't we work with completely ordered $\mathcal{C}$-algebras (having all infima and suprema)? Again for $G\forall$ yes; but in general no. For example there are MV-chains that cannot be embedded into any completely ordered MV-chain. (More can be said.)

(4) Finally, discussing models of a formula φ, can we just assume that its value is defined, not bothering about safeness (all values of all formulas are defined)? No, we cannot, as proved in [31]. Call $\mathcal{C}\forall$ *supersound* is each (closed) formula φ provable in $\mathcal{C}\forall$ is $\mathbf{L}$-true in each $\mathbf{L}$-interpretation $\mathbf{M}$ ($\mathbf{L}$ any $\mathcal{C}$-chain) in which $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ is defined. One can show that $G\forall$ is supersound but BL, Ł, Π are not. (This is strengthened in [32].) These remarks seem to show that our choice of semantics of fuzzy predicate calculus is reasonable (adequate).

2.2 Equality and function symbols

Fuzzy equality is called *similarity*; a binary predicate $\approx$ is a similarity predicate in a theory T (over one of our logics $\mathcal{C}\forall$) if T proves axioms of reflexivity, symmetry and transitivity of $\approx$. What it means for models of T depends on the underlying logic; e.g for Ł$\forall$ similarities are closely related to metrics. (See [21] for details). If $\approx$ is a similarity in T then for each n, the predicate $\approx^n$ defined by $x \approx^k y \equiv x \approx y \& \ldots \& x \approx y$ (k conjuncts) is also a similarity for T. The *congruence axiom* of degree k for a predicate P and similarity $\approx^k$ reads $(x_1 \approx^k y_1 \& \ldots x_n \approx^k y_n) \rightarrow (P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n))$. Similarly for any formula $\varphi(x_1, \ldots)$ instead of the atomic formula $P(x_n, \ldots)$.

Axioms of a fuzzy function F w.r.t. $\approx$ (F being an $(n+1)$-ary predicate) are the congruence axiom for F and the functionality axiom:

$$(F(x, y_1) \& F(x, y_2)) \rightarrow y_1 \approx y_2.$$

This is elaborated in [21] and used for an analysis of "fuzzy IF-THEN rules", see also [25]. We shall not refer on this; but we shall present the result of [26] concerning the use of function symbols in fuzzy logic.[5] $\mathcal{C}\forall$ still varies over BL$\forall$, Ł$\forall$, $G\forall$, $\Pi\forall$.

Take a language consisting of some predicates $P_1, \ldots P_2$, (with arities, among them $\approx$ binary) function symbols $F_1, \ldots, F_n$ (with arities) and some constants. *Terms* are built from variables and constants using function symbols in the usual way. *Logical axioms* of $\mathcal{C}\forall F$ (i.e. $\mathcal{C}\forall$ with function symbols) will be those of $\mathcal{C}\forall$ (in substitution axioms $(\forall 1), (\exists 1)$ allowing y to be a substitutable term) plus similarity axioms for $\approx$, congruence axioms for predicates (each of a given degree) and the following congruence axioms for function symbols:

$$(x_1 \approx^k y_1 \& \ldots x_n \approx^k y_n) \rightarrow (F(X_1, \ldots, x_n) \approx F(y_1, \ldots, y_n)).$$

A **L**-*interpretation* of this language is a structure

$$\mathbf{M} = (M, (r_P)_{P\, predicate}, (f_F)_{F\, funct.\, symb.}, (m_c)_{c\, const.})$$

where r_P, m_c are as above and for each function symbol F, f_F is a (crisp) mapping of M^n into M ((n being the arity of F). The definition of the value $\|t\|_{\mathbf{M},v}$

[5] Let us mention that the fuzzy logic developed by Novák, Perfilieva and Močkoř in [39] is a logic with function symbols.

of a term is obvious. The truth value of an atomic formula is defined as follows: $\|P(t_1,\ldots,t_n)\|_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v})$; the rest is as before. **M** is **L**-*admissible* if it is a safe **L**-interpretation making all the logical axioms (of similarity and congruence) **L**-true. For these notions we get the following.

Strong completeness theorem Let T be a theory over $\mathcal{C}\forall F$, let φ be a formula. $T \vdash_{\mathcal{C}\forall F} \varphi$ iff for each linearly ordered $\mathcal{C}$-algebra **L**, and each admissible **L**-model of T, φ is **L**-true in **M**.

For more results see [26].

2.3 Arithmetical complexity

Here I assume that the reader knows the notion of a recursive set of natural numbers (words, formulas etc.) and the corresponding arithmetical hierarchy of Σ_n sets and Π_n sets. A set is *arithmetical* if it belongs to some Σ_n or Π_n. A set X is Σ_n-*complete* of X is Σ_n and each Σ_n set is recursively reducible to X.

We are interested in arithmetical complexity of sets of predicate tautologies and sets of satisfiable formulas. In contradiction to the discussion in propositional calculus we restrict ourselves to the value 1 (absolute truth) and do not discus positive tautologicity/satisfiability. Another difference is that we have to distinguish standard tautologies from general tautologies and similarly for satisfiability. Let $\mathcal{C}\forall$ be as above.

$TAUT^{\mathcal{C}} = \{\varphi|$ for all standard $\mathcal{C}$-chains **L** and each **L**-safe **M**, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1\}$
$genTAUT^{\mathcal{C}} = \{\varphi|$ for all $\mathcal{C}$-chains **L** and each **L**-safe **M**, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1\}$
$SAT^{\mathcal{C}} = \{\varphi|$ for some standard $\mathcal{C}$-chain **L** and some **L**-safe **M**, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1\}$
$genSAT^{\mathcal{C}} = \{\varphi|$ for some $\mathcal{C}$-chain **L** and some **L**-safe **M**, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1\}$

(Caution: recall that for Ł, Π, G there is a unique standard **L** chain.) The results are summarized in the following table:

	PROVABLE = gen.TAUT	stand. TAUT	gen. SAT = consistent	stand. SAT
BL	Σ_1-compl.	NOT AR.	Π_1-compl.	NOT AR.
Ł	Σ_1-compl	Π_2-compl	Π_1-compl	Π_1-compl
G	Σ_1-compl	Σ_1-compl	Π_1-compl	Π_1-compl
Π	Σ_1-compl	NOT AR.	Π_1-compl	NOT AR.

Since by the completeness theorem general tautologies coincide with provable formulas, $genTAUT^{\mathcal{C}}$ is evidently Σ_1; Σ_1-completeness is proved in [27]. Similarly, general satisfiability coincide with consistence; thus $genSAT^{\mathcal{C}}$ is Π_1 and completeness needs a proof. $TAUT^{G\forall} = genTAUT^{G\forall}$ by the standard completeness of $G\forall$. $TAUT^{Ł\forall}$ being Π_2-complete is a classical result of Ragaz. The fact that $SAT^{\Pi\forall}$ is not arithmetical is proved in my [27]; the three other results, of $TAUT^{\Pi\forall}, TAUT^{\mathrm{BL}\forall}, SAT^{\mathrm{BL}\forall}$ not being arithmetical are due by Montagna [38] by a very tricky improvement of my proof just mentioned.

The questions of positive tautologies and satisfiable formulas have to be studied; I only mention that the set $TAUT^{Ł\forall}_{pos}$ of standard positive tautologies of Ł is Σ_2-complete and the set $SAT^{Ł\forall}_{pos}$ is Σ_1-complete. (This follows from the preceding thanks to the properties of Łukasiewicz negation.) Also it is easy to show that $SAT^{G\forall}_{pos} = SAT^{G\forall}$. (If $\|\varphi\|^G_{\mathbf{M}} = a > 0$ use the function $h(x) = x \cdot a^{-1}$ for $x \leq a, h(x) = 1$ for $a \leq x \leq 1$ to produce an $\mathbf{M}'$ with $\|\varphi\|^G_{\mathbf{M}'} = 1$.) Similarly for $genSAT^{G\forall}_{pos}$. (Not much more seems to be known.)

2.4 Monadic fuzzy predicate logics

A logic is *monadic* if all its predicates are unary. Classical monadic logic is very simple: it is decidable, has finite model property and each closed formula can be equivalently expressed as a propositional combination of closed formulas of the form $(\forall x)\varphi(x)$; thus one object variable suffices. Fuzzy monadic logics are investigated in my [24]; I present the main results. We shall deal with our logics BL$\forall$, Ł$\forall$, $G\forall$, $\Pi\forall$, (monadic) and their *standard* semantics. $TAUT^{\mathcal{C}}$ and $SAT^{\mathcal{C}}$ has the same meaning as above (but for monadic languages); $fTAUT^{\mathcal{C}}$ stands for the sets of all *finite* tautologies, i.e. sentences $[0,1]_*$-true in all $[0,1]_*$-structures $\mathbf{M} = (M, \ldots)$ with a finite M (and $*$ any/the $\mathcal{C}$-t-norm); similarly $fSAT^{\mathcal{C}}$ for the set of all φ such that for a finite $\mathbf{M}$ (and some/the $\mathcal{C}$-t-norm) φ is $[0,1]_*$-true in $\mathbf{M}$.

We have two different notions of finite model property: FMP_1 stands for $TAUT = fTAUT$, whereas FMP_2 stands for $SAT = fSAT$. Furthermore, *mon*-$\mathcal{C}\forall$ stands for the monadic predicate logic $\mathcal{C}\forall$ and *mon*$_1$-$\mathcal{C}\forall$ for its sublogic with just one object variable x. We present the known results in a table.

	$TAUT$	$fTAUT$	SAT	$fSAT$	FMP_1	FMP_2
mon-Ł∀	Π_1	$=\Pi_1$	Π_1-comp	Σ_1	yes	no
mon_1-Ł∀	Δ_1	$=\Delta_1$	Δ_1	$=\Delta_1$	yes	yes
mon-$G\forall$	Σ_1	Π_1	Π_1	Δ_1	no	no
mon_1-$G\forall$	Σ_1	Π_1	Π_1	Δ_1	no	no
mon-$\Pi\forall$	?	Π_1	?	Δ_1	no	no
mon_1-$\Pi\forall$	?	Π_1	?	Δ_1	no	no
mon-BL	?	Π_1	?	Σ_1	no	no
mon_1-BL	?	Π_1	?	Σ_1	no	no

Here Δ_1 stands for recursive (both Σ_1 and Π_1). Note that the full (non-monadic) Ł∀ has neither FMP_1 non FMP_2; for other logic the same follows from the fact that the corresponding monadic logic lacks the property.[6] The equality sign means that the two sets in question are the same. As you can see there are many open problems concerning the arithmetical complexity and the corresponding Σ_1-completeness (Π_1-completeness). Thus monadic fuzzy logic (which can e.g. express simple facts and notions on "fuzzy IF-THEN rules") is by far not an uninteresting branch of fuzzy logic.

3 Extended and combined systems

3.1 Adding connectives

First we discuss the unary connective Δ whose truth function on $[0,1]$ (denoted also Δ) is as follows: $\Delta(1) = 1, \Delta(x) = 0$ for $x < 1$. Δ is called "Baaz's delta" [4] and $\Delta\varphi$ may be read "φ is absolutely true". BL_Δ is the extension of BL by the connective Δ; new axioms are $\Delta\varphi \vee \neg\Delta\varphi$, $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$, $\Delta\varphi \to \varphi$, $\Delta\varphi \to \Delta\Delta\varphi$, $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$ (we may call them $\Delta 1$–$\Delta 5$); there is a new deduction rule of Δ-generalization: *from* φ *infer* $\Delta\varphi$. For a natural notion of BL_Δ-algebras[7]one gets the usual completeness theorem (see [21]). The predicate version works also smoothly, as it is easy to check. Note that Δ is a particular *truth-stressing* connective (saying in general, "φ is *very* true"). This is analyzed and axiomatized in [29].

[6]Also mention that for the full (non-monadic) $\mathcal{C}\forall$ ($\mathcal{C}$ being BL, Ł, G, Π), $fTAUT$ is Π_1-complete and $fSAT$ is Σ_1-complete (i.e. Trakhtenbrot theorem holds for fuzzy logic). For $\mathcal{C} = L, G, \Pi$ it is proved in [28], for BL in [24]. The same can be proved for $fSAT_{pos}, fTAUT_{pos}$.

[7]One translates the axioms $\Delta 1$–$\Delta 5$ into corresponding identities and adds the identity $\Delta(1) = 1$ (guaranteeing soundness of Δ-generalization).

Now let us discuss negation. The negation of Łukasiewicz in *involutive,* i.e. $\neg\neg\varphi$ is equivalent to φ. G and Π have Gödel negation but sometimes one would be happy to have there also an involutive negation. To be satisfactorily general let us first introduce a theory SBL strict basic logic) weaker than both G and Π and enforcing the negation to be Gödel (to be made precise).[8]

SBL is the extension of BL by the axiom $(\varphi \wedge \neg\varphi) \to 0$ (caution: this is the min-conjunction; recall that BL proves $(\varphi \& \neg\varphi) \to 0$). Equivalently, this axiom may be replaced by $((\varphi \& \psi) \to 0) \to ((\varphi \to 0) \vee (\psi \to 0))$, i.e. $\neg(\varphi \& \psi) \to (\neg\varphi \vee \neg\psi)$. This axiom is a $*$-tautology for each strict continuous t-norm (having no non-trivial zero divisors, i.e. $x, y > 0$ with $x * y = 0$). They all have Gödel negation; moreover, if $\mathbf{L}$ is an arbitrary SBL-chain (BL-chain for which the additional axiom is a tautology) then $\mathbf{L}$ has Gödel negation: $(-)0 = 1, (-)x = 0$ for $x > 0$.

Let $\mathbf{L}$ be a SBL-chain and let n be an *involutive negation* for $\mathbf{L}$, i.e. a decreasing mapping of L onto itself satisfying $n(n(x)) = x$ for $x \in L$. If $(-)$ is the Gödel negation then obviously $\Delta x = (-)n(x)$, i.e. Δ is definable from the two negations. Thus let SBL$_\sim$ be the extension of SBL by a new unary connective $\sim$ (called also *involutive negation*) and a defined connective Δ ($\Delta\varphi$ is $\neg \sim \varphi$). The axioms are those of SBL plus

$$\sim\sim \varphi \equiv \varphi,\ \neg\varphi \to \sim \varphi,\ \Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi);$$

deduction rules are modus ponens and Δ-generalization.

There is an obvious notion of SBL$_\sim$-algebras with general completeness.

For $G_\sim$ and $\Pi_\sim$ we can say more: The *standard* $G_\sim$-algebra is the structure $([0,1]_{*,n})$ where $*$ is Gödel t-norm (minimum) and $n(x) = 1 - x$ for all $x \in [0,1]$.

Standard completeness theorem states that $G_\sim \vdash \varphi$ iff φ is a tautology of the standard $G_\sim$-algebra. For $\Pi_\sim$ the analogous claim is false but we have completeness w.r.t. *semistandard* $\Pi_\sim$-algebras of the form $([0,1]_*, n)$ where $*$ is now the product t-norm and n is an arbitrary involutive negation: $\Pi_\sim \vdash \varphi$ iff φ is a tautology over each semistandard $\Pi_\sim$-algebra.

We have the corresponding predicate calculi with general completeness.

[8] The presented material is from [17] and from [11].

3.2 Putting Łukasiewicz and product logic together

We present results of [15], [18], [10, 11]. One develops a logic in which we have both connectives of Ł and those of Π (and we get also connectives of G, Baaz's Δ and several other things). The language has connectives $\rightarrow_{\text{Ł}}, \rightarrow_{\Pi}$ (for Łukasiewicz and Goguen implication) and $\odot$ (product conjunction). We use $\neg$ for Gödel negation ($\neg\varphi$ is $\varphi \rightarrow_{\Pi} \bar{0}$) and $\sim$ for Łukasiewicz negation ($\sim \varphi$ is $\varphi \rightarrow_{\text{Ł}} \bar{0}$). & is defined as $\varphi \& \psi$ being $\sim (\varphi \rightarrow_{\text{Ł}} \sim \psi)$. Needles to say, the standard ŁΠ-algebra is $([0,1], \min, \max, *_{\Pi}, \rightarrow_{\text{Ł}}, \rightarrow_{\Pi}, 0, 1)$ where $*_{\Pi}$ is product t-norm and $\rightarrow_{\text{Ł}}, \rightarrow_{\Pi}$ are residua of Łukasiewicz and product t-norm respectively. Then clearly, $\neg$ and $\sim$ get their standard semantics and thus Δ can be defined in the obvious way ($\Delta\varphi$ is $\neg \sim \varphi$). Further define

$\varphi \vee \psi \equiv (\varphi \rightarrow_{\text{Ł}} \psi) \rightarrow_{\text{Ł}} \psi$,
$(\varphi \rightarrow_G \psi) \equiv \Delta(\varphi \rightarrow_{\text{Ł}} \psi) \vee \psi$,
$\varphi \ominus \psi \equiv \varphi \& \sim \psi$.

Axioms: axioms of Ł for $\rightarrow_L, \&, \bar{0}$
axioms of Π for $\rightarrow_{\Pi}, \odot, \bar{0}$,
$\neg\varphi \rightarrow \sim \varphi, \quad \Delta(\varphi \rightarrow_{\text{Ł}} \psi) \equiv \Delta(\varphi \rightarrow_{\Pi} \psi)$,
$\varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$.

Deduction rules are modus ponens and Δ-generalization. For a natural notion of an ŁΠ-algebra we get completeness. Note that $L\Pi$ proves the axioms $\Delta 1$–$\Delta 5$ for Δ (see [11]).

Moreover, ŁΠ enjoys standard completeness: ŁΠ $\vdash \varphi$ iff φ is a tautology over the standard ŁΠ-algebra. This is proved in [17] by characterizing ŁΠ-algebras using linearly ordered fields and using known results on (real closed) fields.

Recently Cintula has shown [11] that ŁΠ is equivalent to $\Pi_{\sim}$ with $\varphi \rightarrow_{\text{Ł}} \psi$ defined as $\sim (\varphi \odot \sim (\varphi \rightarrow_{\Pi} \psi))$ and only one axiom added, namely the transitivity of $\rightarrow_{\text{Ł}}$ (i.e. A1 for $\rightarrow_{\text{Ł}}$).

3.3 A logic of left-continuous t-norms

A t-norm has the residuum iff it is left-continuous (for each increasing sequence $\{x_n | n \in \omega\}$ of elements of $[0,1]$ and each $y \in [0,1]$, $lim_n(x_n * y) = (lim_n x_n) * y)$]. Fodor's nilpotent minimum [19], defined as $x * y = 0$ for $x + y \leq 1$, $x * y = \min(x, y)$ otherwise is the classical example of a non-continuous left-continuous t-norm; surprisingly, its residuum is $1 - x$. There are very many left-continuous t-norms, see e.g. [34]. If a left continuous t-norm is not continuous we loose divisibility (the identity $\min(x,y) = x * (x \Rightarrow y)$ is no more valid).

Esteva and Godo in their pioneering paper [16] develop a logic MTL (monoidal t-norm logic, logic of left-continuous t-norms) in a very smooth way. Just delete (A4) from the axioms of BL, add $\wedge$ as a new primitive connective and add the axioms

$$(\varphi \wedge \psi) \to \varphi,\ (\varphi \wedge \psi) \to (\psi \wedge \varphi),\ (\varphi \& (\varphi \to \psi)) \to \varphi \wedge \psi.$$

(Keep modus ponens as the only deduction rule.) This logic MTL proves surprizingly many of consequences of BL. Define MTL algebras by deleting the divisibility axiom from the definition of a BL-algebra; you can prove general completeness (MTL $\vdash \varphi$ iff φ is true over all (linearly ordered) MTL-algebras) by usual methods and also standard completeness (MTL$\vdash \varphi$ iff φ is true over all $[0,1_*$-algebras, $*$ being a left-continuous t-norm) by some more tricky means. (Standard completeness was proved by Jenei and Montagna [35].) General completeness of the corresponding predicate calculus MTL$\forall$ is also easily obtained. Various extensions of MTL have been studied, from which we mention IMTL (involutive MTL, i.e. MTL plus the axiom of double negation $\neg\neg\varphi \to \varphi$, sound e.g. for Fodor's t-norm, hence weaker that Łukasiewicz logic Ł) and ΠMTL (MTL plus the additional axiom of product logic, weaker that product logic Π). Note that MTL plus idempotence of conjunction $(\varphi \to (\varphi\&\varphi))$ is equivalent to G. NMTL is the logic of Fodor's t-norm – the extension of IMTL by the axiom $((\varphi\&\psi) \to 0) \vee ((\varphi \wedge \psi) \to (\varphi\&\psi))$. Remarkably, this logic is equivalent to $G_\sim$ (Gödel logic with involutive negation added) in the sense that the connectives of $G_\sim$ are definable in NMTL and vice versa (see [16]).

The following is a very surprizing and beautiful result of Montagna and Ono [37] on the predicate logic MTL$\forall$,:

Theorem. *Standard completeness of MTL$\forall$.* MTL$\forall$ proves φ iff for each left-continuous t-norm $*$ and each $[0,1]_*$-interpretation $\mathbf{M}$, φ is $[0,1]_*$-true in $\mathbf{M}$, i.e. φ is a standard tautology of MTL$\forall$.

This is in sharp contrast with BL$\forall$ since as we have seen the set of standard tautologies of BL$\forall$ is not arithmetical.

4 Conclusion

There are several results and papers that I could not present; I apologize to their authors. In particular, let me call the reader's attention to the long-term and continuing work of Vilém Novák and his group on a fuzzy logic which is the extension of Łukasiewicz logic by truth constants (known as Pavelka logic or logic with evaluated syntax). See [39] for detailed exposition.

I also mention a generalization of BL-algebras different from MTL-algebras, so-called pseudo-BL-algebras; see e.g. [12].

Nevertheless, I hope that this survey will be useful for people interested in mathematical fuzzy logic and shall be grateful for comments.

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