A NEW CHARACTERIZATION OF CLIQUE GRAPHS

Liliana Alcón **Marisa Gutierrez**

Abstract

The *clique graph*, $K(H)$, of a given graph H is the intersection graph of the family of maximal completes of H. A graph G is said to be a clique graph if there exists H such that $G = K(H)$. The only characterization of Clique Graphs known so far is due to Roberts and Spencer [2], however recognizing clique graphs through this characterization is in general difficult; the computational complexity of recognizing clique graphs is a long-standing open problem. In this paper we present a new characterization of clique graphs based on assignments of a vertex u_T to each triangle T of the graph.

1 Definitions and introduction

We consider finite, simple and undirected graphs. $V(G)$ and $E(G)$ denote the vertex set and the edge set of the graph G respectively. A *complete* of G is a subset of $V(G)$ inducing a complete subgraph. A *clique* is a maximal complete. We also use the terms *complete* and *clique* to refer to the corresponding subgraphs. A *triangle* is a complete with exactly three vertices, $T(G)$ denotes the set of triangles of G.

Given a *set family* $\mathcal{F} = (F_i)_{i \in I}$, the sets F_i are called *members* of the family. $F \in \mathcal{F}$ means that F is a member of $\mathcal{F}$. $\mathcal{F}$ is *pairwise intersecting* if the intersection of any two members is not the empty set. The *intersection* or *total intersection* of $\mathcal{F}$ is the set $\cap\mathcal{F} = \cap_{i \in I} F_i$. The family $\mathcal{F}$ has the *Helly property* or is a *Helly family*, if any pairwise intersecting subfamily has nonempty total intersection.

The *intersection operator*, L, maps a set family $\mathcal{F} = (F_i)_{i \in I}$ into the graph $L(\mathcal{F})$ satisfying

$$V(L(\mathcal{F})) = \{F_i, i \in I\} \text{ and } E(L(\mathcal{F})) = \{F_i F_{i'} / F_i \cap F_{i'} \neq \emptyset\}.$$

If G is a graph, $\mathcal{C}(G)$ denotes the *clique family* of G. *The clique graph* of G, denoted by $K(G)$, is the graph $L(\mathcal{C}(G))$. G is *a clique graph* if there exists another graph H such that $G = K(H)$. $K(Graph)$ is the set of all clique graphs.

The edge with end vertices u and v is represented by uv. We say that the complete C *covers* the edge uv when u and v belong to C. A *complete edge cover* of a graph G is a family of completes of G, covering the edges of G. A *Helly complete edge cover* of G is an complete edge cover of G satisfying the Helly property. Although clique graphs have been study widely, see for instance [3], the following is the only characterization of clique graphs known so far. This characterization has not led to an efficient algorithm for clique graphs: the computational complexity of the clique graph recognition problem reminds open.

Theorem 1 (Roberts and Spencer, [2]) *$G \in K(Graph)$ if and only if there exists a Helly complete edge cover of G.*

We say that the subsets V_1 and V_2 of $V(G)$ are *stuck one on the other* if $V_1 \cap V_2$ contains at least two vertices. We also say V_1 is stuck on V_2 or V_2 is stuck on V_1. If $\mathcal{F}$ is a complete edge cover of G and $V \subseteq V(G)$, $\mathcal{F}_V$ denotes the subfamily of $\mathcal{F}$ formed by all the members which are stuck on V, if any exists. It is easy to prove that a complete edge cover $\mathcal{F}$ of G, has the Helly property, if and only if,

$$T \in T(G) \Longrightarrow \cap \mathcal{F}_T \neq \emptyset. \tag{I}$$

Using this idea the characterization of clique graphs due to Roberts and Spencer is formulated in [1] as following.

Lemma 1 *[1] $G \in K(Graph)$ if and only if there exists a complete edge cover $\mathcal{F}$ of G such that, $T \in T(G)$ implies $\cap \mathcal{F}_T \neq \emptyset$.*

This Lemma says that G is a clique graph if and only if there exists a complete edge cover of G such that for every triangle T of G the intersection of all those members containing at least two vertices of T is not empty. It follows that if G is a clique graph every triangle of G can be related with a subset of vertices of G and so, in particular, with one vertex of G:

$$T \in T(G) \longrightarrow u_T \in \cap \mathcal{F}_T.$$

It is natural to ask if it is possible to develop a set $\mathcal{P}$ of properties independent of $\mathcal{F}$, such that if every triangle of G is related with a vertex of G, satisfying the properties in $\mathcal{P}$, then G is a clique graph. In section 2 we present different results about $\mathcal{P}$. In section 3 we show an affirmative answer to that question, and we obtain a new characterization of clique graphs. We think that this new characterization is a potential useful tool to be used to solve the clique graph recognition problem.

It is known that G is a clique graph if and only if the graph obtained from G by removing the edges which are not in a triangle, is a clique graph, thus, without loss of generality, we will only consider graphs whose edges belong to some triangle.

2 The main results

A *t-v assignment* of a graph G is a function $\mathbf{u} : T(G) \to V(G)$, ($\mathbf{u}(T)$ is denoted by u_T), such that for every triangle $T = \{x_1, x_2, x_3\}$ of G either

1. $u_T \in T$, or
2. u_T is adjacent to x_1, x_2 and x_3, and, in this case, $u_T = u_{T_1} = u_{T_2} = u_{T_3}$, where $T_1 = \{x_2, x_3, u_T\}$, $T_2 = \{x_1, x_3, u_T\}$ and $T_3 = \{x_1, x_2, u_T\}$.

Remark 1 *Notice that if* $\mathbf{u}$ *is a t-v assignment of* G *and* $uv \in E(G)$ *then there exists some triangle* T *of* G *covering* uv *and satisfying* $u_T \in T$*.*

In the following, given a t-v assignment $\mathbf{u}$ and a vertex subset V' of a graph G, a new vertex subset $A_{\mathbf{u}}(V')$ is obtained progressively by adding the vertices u_T assigned to the triangles which are stuck on. Let us put this idea into proper form: given $\mathbf{u}$, a t-v assignment of G, $V' \subseteq V(G)$, and s a positive integer, let $A^s_{\mathbf{u}}(V')$ be the vertex set:

$$\begin{aligned} A^1_{\mathbf{u}}(V') &= V' \cup \{u_T,\ T \in T(G),\ T \text{ stuck on } V'\}, \\ A^s_{\mathbf{u}}(V') &= A^1_{\mathbf{u}}(A^{s-1}_{\mathbf{u}}(V')). \end{aligned}$$

It is clear that for any $V' \subseteq V(G)$, there exists $s_{V'}$ such that

$$A^{s_{V'}}_{\mathbf{u}}(V') = A^{s_{V'}+1}_{\mathbf{u}}(V')$$

and then for every $k \geq s_{V'}$

$$A_{\mathbf{u}}^{s_{V'}}(V') = A_{\mathbf{u}}^{k}(V').$$

We define

$$A_{\mathbf{u}}(V') = A_{\mathbf{u}}^{s_{V'}}(V').$$

Remark 2 *Notice that $A_{\mathbf{u}}^1(A_{\mathbf{u}}(V')) = A_{\mathbf{u}}(V')$, thus, if T is stuck on $A_{\mathbf{u}}(V')$, then u_T belongs to $A_{\mathbf{u}}(V')$.*

The following lemma shows that a particular t-v assignment of a graph can be obtained from a given Helly complete edge cover.

Lemma 2 *Let $\mathcal{F}$ be a Helly complete edge cover of a graph G. Given $T \in T(G)$, choose $u_T \in V(G)$ such that $u_T \in \cap\mathcal{F}_T$ and, if it is possible, $u_T \in T$. The application $\mathbf{u} : T(G) \to V(G)$, defined by $\mathbf{u}(T) = u_T$, is a t-v assignment of G satisfying*

$$T \subseteq F \in \mathcal{F} \Longrightarrow A_{\mathbf{u}}(T) \subseteq F.$$

Proof: Since $\mathcal{F}$ is a Helly complete edge cover of G, then for every triangle T of G, $\cap\mathcal{F}_T \neq \emptyset$, thus for every triangle T of G, it is possible to choose a vertex $u_T \in \cap\mathcal{F}_T$; we also ask the simple condition that, if it is possible, $u_T \in T$.
Let us see that the application defined by $\mathbf{u}(T) = u_T$ is a t-v assignment of G: let $T = \{x_1, x_2, x_3\}$ be a triangle of G and assume that $u_T \notin T$, then $\cap\mathcal{F}_T \cap T = \emptyset$. It follows that $x_1, x_2, x_3 \notin \cap\mathcal{F}_T$, thus (since every edge of T is covered by some member of $\mathcal{F}$) there exist members of $\mathcal{F}$ satisfying: $x_1 \notin F_1 \supseteq \{x_2, x_3, u_T\}$, $x_2 \notin F_2 \supseteq \{x_1, x_3, u_T\}$ and $x_3 \notin F_3 \supseteq \{x_1, x_2, u_T\}$ (See Figure 1).

Clearly, u_T must be adjacent to x_1, x_2 and x_3. Consider the triangles $T_1 = \{x_2, x_3, u_T\}$, $T_2 = \{x_1, x_3, u_T\}$ and $T_3 = \{x_1, x_2, u_T\}$, we have to prove that $u_{T_1} = u_{T_2} = u_{T_3} = u_T$. By symmetry it is enough to prove that $u_{T_1} = u_T$.
First notice that, since F_2 is stuck on T_1 and $x_2 \notin F_2$, and since F_3 is stuck on T_1 and $x_3 \notin F_3$, then

$$u_{T_1} \neq x_2 \ , \ u_{T_1} \neq x_3.$$

On the other hand, if $\{x_2, x_3\} \subseteq F \in \mathcal{F}$ then $u_T \in F$, thus u_T belongs to any member of $\mathcal{F}$ stuck on T_1, i.e. $u_T \in \cap\mathcal{F}_{T_1} \cap T_1$. It follow from how u_{T_1} was chose, that $u_{T_1} = u_T$.
We have proved that $\mathbf{u}$ is a t-v assignment. Now, let $T \in T(G)$ and $F \in \mathcal{F}$

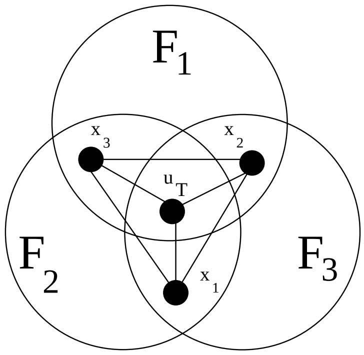

Figure 1: The members F_i, $1 \leq i \leq 3$.

such that $T \subseteq F$. To show that $A_{\mathbf{u}}(T) \subseteq F$ we will prove by induction over s that $A^s_{\mathbf{u}}(T) \subseteq F$.
Let s=1,

$$A^1_{\mathbf{u}}(T) = T \cup \{u_{T'}, T' \text{ stuck on } T\}.$$

If T' is a triangle which is stuck on T, since $T \subseteq F$, then T' and F are stuck, i.e. $F \in \mathcal{F}_{T'}$ thus $u_{T'} \in F$.
The inductive hypothesis says for $s = k$ that

$$A^k_{\mathbf{u}}(T) \subseteq F.$$

Let $s = k + 1$,

$$A^{k+1}_{\mathbf{u}}(T) = A^1_{\mathbf{u}}(A^k_{\mathbf{u}}(T)) = A^k_{\mathbf{u}}(T) \cup \{u_{T'}, T' \text{ stuck on } A^k_{\mathbf{u}}(T)\}.$$

By inductive hypothesis $A^k_{\mathbf{u}}(T) \subseteq F$, thus if T' is stuck on $A^k_{\mathbf{u}}(T)$ then T' is stuck on F, i.e. $F \in \mathcal{F}_{T'}$. It follows $u_{T'} \in F$.

□

The following lemma shows that it is possible to get a Helly complete edge cover of a given graph from a t-v assignment $\mathbf{u}$ satisfying a particular condition. The set of triangles of G holding $u_T \in T$ will be denoted by $T(G)_{\mathbf{u}}$.

Lemma 3 *Let* $\mathbf{u}$ *be a t-v assignment of a graph* G *such that, for every* $T \in T(G)_{\mathbf{u}}$, $A_{\mathbf{u}}(T)$ *is a complete of* G. *Then* $(A_{\mathbf{u}}(T))_{T \in T(G)_{\mathbf{u}}}$ *is a Helly complete edge cover of* G.

Proof: Let $\mathcal{F} = (A_{\mathbf{u}}(T))_{T \in T(G)_{\mathbf{u}}}$. By Remark 1, the members of $\mathcal{F}$ cover the edges of G, and by hypothesis, they are completes; thus $\mathcal{F}$ is a complete edge cover of G. To show that $\mathcal{F}$ has the Helly property, it is enough, by implication I, to prove that $T \in T(G)$ implies $\cap \mathcal{F}_T \neq \emptyset$. We claim that u_T belongs to that intersection, so it is not empty. Indeed, let $T \in T(G)$ and $A_{\mathbf{u}}(T')$ be a member of $\mathcal{F}$ belonging to $\mathcal{F}_T$, then T is stuck on $A_{\mathbf{u}}(T')$. It follows from Remark 2 that $u_T \in A_{\mathbf{u}}(T')$, as we wanted to show.

□

3 A new characterization of Clique Graphs

The following theorems are characterizations of clique graphs. The proof of the theorems are obtained from the lemmas of the previous section immediately.

Theorem 2 *A graph G is a clique graph if and only if there exists* $\mathbf{u}$*, a t-v assignment of G, such that $A_{\mathbf{u}}(T)$ is a complete of G, for every triangle T of G satisfying $u_T \in T$.*

Proof: Let G be a clique graph. By Theorem 1, there exists $\mathcal{F}$, a Helly complete edge cover of G. Let $\mathbf{u}$ be a t-v assignment of G obtained from $\mathcal{F}$ as described in Lemma 2. We have to prove that if $u_T \in T$ then $A_{\mathbf{u}}(T)$ is a complete of G. If $u_T \in T$, it is clear that there exists $F \in \mathcal{F}$ such that $T \subseteq F$, then, since Lemma 2 is satisfied, $A_{\mathbf{u}}(T) \subseteq F$, thus, as F is a complete, $A_{\mathbf{u}}(T)$ is a complete of G. The converse of the present theorem arises from Lemma 3 and Theorem 1.

□

Theorem 3 *A graph G is a clique graph if and only if there exists* $\mathbf{u}$*, a t-v assignment of G, such that $A_{\mathbf{u}}(e)$ is a complete of G for every $e \in E(G)$.*

Proof: Let G be a clique graph and $\mathbf{u}$ a t-v assignment of G satisfying Theorem 2. Let e be an edge of G. If every triangle T, which is stuck on e, is such that $u_T \in e$, then $A_{\mathbf{u}}(e) = e$, thus it is a complete. If there exists a triangle T which is stuck on e and $u_T \notin e$, then there exists a triangle T' that is stuck on e (T' can be the same T), such that $u_{T'} \in T'$ and $u_{T'} \notin e$. It is easy to see that $A_{\mathbf{u}}(e) = A_{\mathbf{u}}(T')$. By Theorem 2, $A_{\mathbf{u}}(T')$ is a complete of G, then the proof follows.

Conversely, let $\mathbf{u}$ be a t-v assignment of G satisfying the hypothesis. Let T be any triangle of G such that $u_T \in T$. Let e be the edge of T satisfying u_T is not an end vertex of e. It is clear that $A_{\mathbf{u}}(e) = A_{\mathbf{u}}(T)$, then $A_{\mathbf{u}}(T)$ is a complete; it follows from the previous theorem that G is a clique graph.

□

References

[1] Alcón, L.; Gutierrez, M., *Cliques and Extended Triangles. A necessary condition to be Clique Planar Graph*, accepted by Discrete Applied Mathematic.

[2] Roberts, F. S.; Spencer, J. H., *A characterizations of clique graphs*, Journal of Combinatorial Theory B. 10 (1971), 102-108.

[3] Szwarcfiter, J. L., *A survey on Clique Graphs*, In : Recent Advances in Algorithms and Combinatorics, C. Linhares and B. Reed, eds.,Springer-Verlag, to appear.

Departamento de Matemática
Universidad Nacional de La Plata
C. C. 172, (1900) La Plata, Argentina
E-mail: {liliana,marisa}@mate.unlp.edu.ar