EVEN PAIRS IN SQUARE-FREE BERGE GRAPHS *

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Abstract

We consider the graphs that contain no odd chordless cycle on at least five vertices (an "odd hole"), no chordless cycle on exactly four vertices (a "square"), and no subgraph that consists of two triangles with three vertex-disjoint paths between them (a "stretcher"). We show that any such graph either is a complete graph or has two vertices that are not linked by an odd chordless path (an "even pair"). This is a partial answer, in the case of square-free graphs, to several conjectures concerning even pairs in Berge graphs.

1 Introduction

We consider only finite and undirected graphs. A graph G is *perfect* if for every induced subgraph H of G, the chromatic number $\chi(H)$ of H is equal to the maximum size of its cliques $\omega(H)$. An odd (even) *hole* is a chordless odd (even) cycle of G of length at least five. An odd (even) *anti-hole* is a complement of an odd (even) hole. We will follow the convention of calling *Berge graph* any graph that contains no odd hole and no odd antihole. The class of perfect graphs was defined in 1960 by Claude Berge who also made a famous conjecture (see [14] for a survey):

Conjecture 1 (Strong Perfect Graph Conjecture [2]) *Any graph that contains no odd hole and no odd anti-hole is perfect.*

A proof of Berge's conjecture has been announced recently [6]. The outline of the proof is that every Berge graph either is of some "basic" type or admits some property that cannot be satisfied by any minimally imperfect Berge graph. However, the details of this proof are not published and it is expected that the

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whole result will be very complex and lengthy. A proof of Berge's conjecture in the case of graphs not containing an induced subgraph isomorphic to a square (chordless cycle on four vertices) was given earlier in [7]. Here we want to consider some different questions concerning Berge graphs.

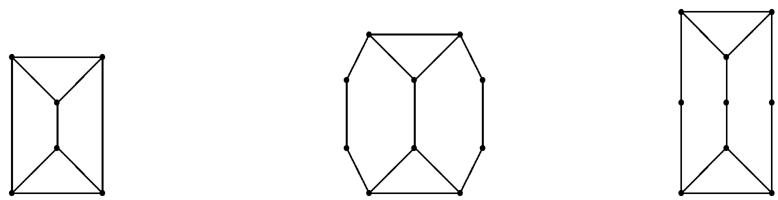
An *even pair* in a graph G is a pair of non-adjacent vertices of G such that the length (number of edges) of each chordless path between them is even. Meyniel [13] (see also Fonlupt and Uhry [9] and Bertschi and Reed [4]) proved that no minimally imperfect graph has an even pair, and called *strict quasi-parity* (SQP) the class of graphs where every induced subgraph that is not a clique has an even pair. So every strict quasi-parity graph is perfect. The converse is not true, as one can find infinitely many perfect graphs that are not strict quasi-parity [11] and are minimal with this property. In general, finding an even pair is co-NP-complete [5]. See [8] for a recent survey on even pairs.

Contracting two vertices x, y in a graph G means removing them and replacing them by a single vertex adjacent to every vertex of $G \setminus \{x, y\}$ that was adjacent to at least one of x, y. Bertschi [3] calls a graph G *even-contractile* if there is a sequence $G_0, \ldots, G_k$ of graphs such that $G = G_0$, each G_i is obtained from G_{i-1} by contracting an even pair of G_{i-1}, and G_k is a clique. Bertschi calls *perfectly contractile* any graph all induced subgraphs of which are even-contractile. Everett and Reed conjecture the following characterization of perfectly contractile graphs.

Conjecture 2 (Everett and Reed [15]) *A graph is perfectly contractile if and only if it contains no odd hole, no antihole and no odd stretcher.*

Here a *stretcher* is any graph that consists of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and three chordless paths P_1, P_2, P_3, such that P_i is from a_i to b_i, and there is no edge between these paths other than the two triangles' edges. A stretcher is odd (even) if all three chordless paths have odd (even) length. See Figure 1.

A perfectly contractile graph contains no odd hole and no anti-hole since such graphs have no even pair. Moreover it can be proved (see [12]) that any sequence of even-pair contractions in an odd stretcher leads to the anti-hole with six vertices, which has no even pair. Thus no perfectly contractile graph may contain an odd stretcher. So the "only if" part of Everett and Reed's conjecture holds.



An odd stretcher: $\overline{C}_6$ An odd stretcher   An even stretcher

Figure 1: Some stretchers

Our purpose here is to examine the existence of even pairs in square-free Berge graphs in the direction suggested by the above concepts and conjectures. Our main result is:

Theorem 1 *Let G be a square-free Berge graph that does not contain a stretcher. Then G either is a clique or contains an even pair.*

The proof of Theorem 1 is given in Section 3, using results from Section 2. This theorem is only a partial answer to Conjecture 2 in the case of square-free graphs; indeed, the Theorem does not cover the more general case when the graph may contain even stretchers. Some comments on this question are proposed in the conclusion.

2 Tools

For two vertices x, y in a graph G, we frequently say 'x sees y' instead of 'x is adjacent to y' and 'x misses y' instead of 'x is not adjacent to y'.

In a graph G, given a vertex x, we call *x-edge* any edge uv whose two endvertices see x (so uvx is a triangle). A ΔP *(configuration)* is a graph that consists of a triangle abc, a vertex x, three chordless paths $a \cdots x$, $b \cdots x$, $c \cdots x$ such that at most one of the three paths has length 1, any two of these paths have only x as a common vertex, and the graph has no other edge. We may also say that we have a $\Delta P(abc, x)$.

The following three lemmas are classical and easy.

Lemma 1 *Any ΔP configuration contains an odd hole.*

Proof: Consider a ΔP with the notation above. Clearly, two of the three defining paths have the same parity. Thus their union induces an odd hole.

□

Lemma 2 *In a graph G with no odd hole, let P be a chordless path and x be a vertex that sees both endvertices of P. If P has odd (resp. even) length then the number of x-edges in P is odd (resp. even).*

Proof: We write $P = p_0 \cdots p_k$ and prove the lemma by induction on k. If $k = 1$ the lemma is trivial. Suppose $k \geq 2$. Let j be the smallest index such that x sees p_j and $j > 0$. If $j = 1$ then p_0p_1 is an x-edge and the desired result follows by induction on $P \setminus p_0$. If $j \geq 2$ then none of $p_0p_1, \ldots, p_{j-2}p_{j-1}$ is an x-edge, the vertices $x, p_0, \ldots, p_j$ induce a hole, so j is even, and the result follows by induction on $P \setminus \{p_0, \ldots, p_{j-1}\}$.

□

Lemma 3 *In a graph G with no odd hole, let H be a hole and x be any vertex that sees two consecutive vertices of H. Then either x has no other neighbour on H, or the number of x-edges in H is even.*

Proof: The lemma holds if x sees no other vertex of H and also if x sees all other vertices of H. If x sees some but not all of the other vertices, then the cycle H can be labelled $h_0, \ldots, h_{k-1}$ (k even) such that x sees h_0, h_1 and h_j for some j with $2 \leq j \leq k-2$. One of the two paths $h_1 \cdots h_j$ and $h_j \cdots h_0$ has odd length and the other has even length. Applying Lemma 2 to them yields the desired result.

□

Lemma 4 *Let G be a square-free graph that contains a hole, and let H be a shortest hole of G. Then every vertex of $G \setminus H$ sees either zero, one, two consecutive, three consecutive, or all vertices of H.*

Proof: A routine examination shows that if a vertex x of $G \setminus H$ violates the conclusion of the lemma then $H \cup \{x\}$ induces a subgraph that contains a square or a hole shorter than H.

□

The preceding lemma will frequently be used in the following form: if $x \in G \setminus H$ sees two non-consecutive vertices of H but not all of H, then x sees exactly three consecutive vertices of H and no other.

Given a path P and a set X of vertices that induces a connected subgraph of G of size at least two, we define the *X-segments* of P as follows: mark every vertex of P that has a neighbour in X; an X-segment is then any subpath of P of length at least one whose endvertices are marked and whose interior vertices are not marked. Note that it is not assumed that P and X do not intersect; actually, every vertex of $P \cap X$ is marked because every vertex of X has a neighbour in X. When each endvertex of P itself is marked, the path P is (edge-wise) partitioned into its X-segments.

Given an induced subgraph H of a graph G and a vertex x of H, it will be convenient to use the notation

$$J_H(x) = \{v \in G \setminus (H \setminus x) \mid N(v) \cap (H \setminus x) = N(x) \cap (H \setminus x)\}.$$

Obviously $x \in J_H(x)$. Note that, if $v \in J_H(x)$, the subgraph $(H \setminus x) \cup \{v\}$ is isomorphic to H, and $J_{(H\setminus x)\cup\{v\}}(v) = J_H(x)$.

We observe that if G is square-free, H is a shortest hole in G and x is any vertex of H, then, by Lemma 4, a vertex is in $J_H(x)$ if and only if it sees the two neighbours of x in H and misses at least one vertex of H.

3 Proof of Theorem 1

Let G be a graph satisfying all the hypotheses of Theorem 1, and let us prove that G is a clique or has an even pair.

Recall that a graph G is *chordal* if every cycle of G of length at least four has a chord. It is well known that every chordal graph is perfect (see [1, 2] or [14, Chap. 1]), and it is known that every chordal graph that is not a clique contains an even pair (see [8, 10]; actually, it is not hard to see that any two non-adjacent vertices x, y that maximize the size of $N(x) \cap N(y)$ form an even pair). So, in proving the theorem, we may assume that G contains a hole.

Let H be a shortest hole of G (H has length at least six). Call α, u, β in this order three consecutive vertices of H, and let $Q = v_0 \cdots v_q$ (q even, $q \geq 2$) be the chordless path formed by $H \setminus \{\alpha, u, \beta\}$, where v_0 sees α and v_q sees β. For simplicity write

$$A = J_H(\alpha) \text{ and } B = J_H(\beta).$$

Note that A is non-empty ($\alpha \in A$) and is a clique (for otherwise two non-adjacent vertices from A plus u and v_0 would induce a square). Likewise B is a non-empty clique. Moreover, there is no edge ab with $a \in A$, $b \in B$, for otherwise $Q \cup \{a, b\}$ would be an odd hole. For any $a \in A$ and $b \in B$, denote by $H^{a,b}$ the hole formed by $Q \cup \{u, a, b\}$.

Our aim is to show that a well-chosen pair of vertices $a \in A$, $b \in B$ forms an even pair of G. This will be established using several lemmas regarding paths between A and B, as follows.

Lemma 5 *Suppose that there exists a chordless odd path $P = x_0 x_1 \cdots x_{p-1} x_p$ (p odd, $p \geq 1$) with $x_0 \in A$ and $x_p \in B$ (it is not assumed that P and H^{x_0,x_p} have no other vertices in common). Then we have $p \geq 3$, and either $x_1 \in A$ or $x_{p-1} \in B$.*

Proof: We have $p \geq 3$ because there is no edge between A and B, as observed above. Suppose now that none of $x_1 \in A$, $x_{p-1} \in B$ holds. We note that no x_i with $0 < i < p$ sees all of H^{x_0,x_p}, because x_i misses at least one of x_0, x_p as p is odd and $p \geq 3$. Moreover,

$$\text{No } x_i \text{ with } 0 < i < p \text{ sees both } u \text{ and a vertex of } Q. \tag{1}$$

Indeed, suppose that (1) fails for some vertex x_i. If $2 \leq i \leq p-2$, vertex x_i would miss x_0 and x_p, thus Lemma 4 would be violated along the hole H^{x_0,x_p}. If $i = 1$, the only possibility allowed by Lemma 4 would be $N(x_1) = \{u, x_0, v_0\}$, that is, $x_1 \in A$, which we have excluded. Likewise $i = p-1$ is excluded. So (1) holds.

Consider the u-edges of P. Since u sees both endvertices x_0, x_p of P, Lemma 2 implies that P has an odd number of u-edges.

Consider the Q-segments of P. Note that x_0 and x_p have a neighbour in Q (they are "marked"), so P is (edge-wise) partitioned into its Q-segments. Moreover, at least one interior vertex of P has a neighbour in Q (it is marked), for otherwise $P \cup Q$ would induce an odd hole. Thus P has at least two Q-segments.

It follows from the previous two paragraphs that there exists a Q-segment S of P that contains an odd number of u-edges, and that S does not contain both x_0, x_p; by symmetry we may assume that S does not contain x_p. By (1), at most one u-edge of S contains a vertex that has a neighbour in Q, and if there is such an edge it must be x_0x_1 (and x_0 is the first vertex of S). Write $S = x_h \cdots x_j$ with $0 \le h < j < p$. If $j = h+1$ then x_hx_{h+1} is the u-edge of S, so x_j is adjacent to both u and Q, a contradiction to (1). So $j \ge h+2$.

If x_h is in Q, then x_{h+1} has a neighbour in Q (it is marked), thus $j = h+1$, which we have just excluded. So $x_h \notin Q$, and similarly $x_j \notin Q$. Also no interior vertex x_i of S is in Q (else three vertices x_{i-1}, x_i, x_{i+1} of S would be marked, contradicting the definition of a Q-segment). In summary, we have $S \cap Q = \emptyset$ and no interior vertex of S has any neighbour along Q.

By the definition of a Q-segment, each of x_h, x_j has a neighbour in Q. So there exists a subpath Q' of Q such that one endvertex of Q' is adjacent to x_h, the other is adjacent to x_j, and Q' is as short as possible with these properties (Q' may have length 0). It follows that $S \cup Q'$ induces a chordless cycle in G, of length at least $(j-h)+2$; since $j \ge h+2$, this cycle is not a triangle, thus it is a hole. Since there are an odd number of u-edges along this hole, Lemma 3 implies that u has exactly two neighbours x_i, x_{i+1} along S. We have $h \le i < i+1 < j < p$; $h = i$ is possible only if $h = 0$, else x_h would violate (1).

Let k be the smallest integer such that x_jv_k is an edge, and let l be the largest integer such that x_hv_l is an edge. Each of $k < l$, $k = l$, $k > l$ is possible. For the sake of convenience we write $v_{q+1} = x_p$ and $v_{q+2} = u$. Recall from Lemma 4 that each of x_h, x_j has one, two or three consecutive neighbours along H; more precisely, $N(x_h) \cap H = \{v_l\}$ or $\{v_{l-1}, v_l\}$ or $\{v_{l-2}, v_{l-1}, v_l\}$, and $N(x_j) \cap H = \{v_k\}$ or $\{v_k, v_{k+1}\}$ or $\{v_k, v_{k+1}, v_{k+2}\}$. We can now prove that $H \cup P$ contains an induced ΔP or a stretcher (a contradiction). This is done formally by distinguishing between the following two cases.

Case 1: $l \le k$.

Here $S \cup Q[v_l, v_k]$ is a chordless cycle.

If x_j misses v_{k+1} then x_j has no neighbour on $Q[v_{k+1}, v_q]$, and so the triangle ux_ix_{i+1} with the three chordless paths $S[x_i, x_h] \cup Q[v_l, v_k]$, $S[x_{i+1}, x_j] \cup v_k$, $Q[v_k, v_q] \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, v_k)$, a contradiction.

If x_j sees v_{k+1} and not v_{k+2} (possibly $k = q$) then the triangles ux_ix_{i+1} and $x_jv_kv_{k+1}$ with the three chordless paths $S[x_i, x_h] \cup Q[v_l, v_k]$, $S[x_{i+1}, x_j]$, $Q[v_{k+1}, v_q] \cup x_p \cup u$ form a stretcher, a contradiction.

If x_j sees v_{k+1} and v_{k+2} (possibly $k = q-1$), then the triangle ux_ix_{i+1} with the three chordless paths $S[x_i, x_h] \cup Q[v_l, v_k] \cup x_j$, $S[x_{i+1}, x_j]$, $x_j \cup Q[v_{k+2}, v_q] \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, x_j)$, a contradiction.

Case 2: $k \le l-1$.

This case is slightly different from Case 1 as the cycle $S \cup Q[v_l, v_k]$ is not necessarily chorldess.

First assume that $N(x_h) \cap N(x_j) \cap Q = \emptyset$. Let k' be the largest integer such that $x_j v_{k'}$ is an edge. Let l' be the smallest integer such that $x_h v_{l'}$ is an edge. By Lemma 4, we have $k \leq k' \leq k+2$ and $l-2 \leq l' \leq l$. By the assumption, we have $k' < l'$. Since $k' < q$, we have $j \leq p-2$. Since $l' > 0$, we have $h \geq 2$, hence $i > h$.

If $l' = l$, then the triangle ux_ix_{i+1} with the three paths $S[x_i, x_h] \cup v_l$, $S[x_{i+1}, x_j] \cup Q[v_{k'}, v_l]$, $Q[v_l, v_q] \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, v_l)$, a contradiction.

If $l' = l-1$, then the triangles ux_ix_{i+1} and $x_hv_lv_{l-1}$ with the three paths $S[x_i, x_h]$, $S[x_{i+1}, x_j] \cup Q[v_{k'}, v_{l-1}]$, $Q[v_l, v_q] \cup x_p \cup u$ form a stretcher.

If $l' = l-2$, then the triangle ux_ix_{i+1} and the three paths $S[x_i, x_h]$, $S[x_{i+1}, x_j] \cup Q[v_{k'}, v_{l-2}] \cup x_h$, $x_h \cup Q[v_l, v_q] \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, x_h)$, a contradiction.

Now assume that $N(x_h) \cap N(x_j) \cap Q \neq \emptyset$, and let t be the largest integer such that v_t sees both x_h, x_j $(t \leq q)$.

Suppose v_{t+1} misses both x_h, x_j. So $j \leq p-2$. Then the triangle ux_ix_{i+1} with the three paths $S[x_i, x_h] \cup v_t$, $S[x_{i+1}, x_j] \cup v_t$, $Q[v_t, v_q] \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, v_t)$, a contradiction.

Suppose v_{t+1} sees x_h. Thus v_{t+1} misses x_j (so $l = t$) and $t \leq q-1$ (so $j \leq p-2$). If v_{t+2} misses x_h, the triangles ux_ix_{i+1} and $x_hv_tv_{t+1}$ with the three paths $S[x_i, x_h]$, $S[x_{i+1}, x_j] \cup v_t$, $Q[v_{t+1}, v_q] \cup x_p \cup u$ form a stretcher, a contradiction. If v_{t+2} sees x_h (so $t \leq q-2$), then the triangle ux_ix_{i+1} with the three paths $S[x_i, x_h]$, $S[x_{i+1}, x_j] \cup v_t \cup x_h$, $x_h \cup Q[v_{t+2}, v_q] \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, x_h)$, a contradiction.

Suppose v_{t+1} sees x_j (and thus misses x_h). If v_{t+2} misses x_j, we have either $t \leq q-1$ and $j \leq p-2$ or $t = q$ and $j = p-1$. Accordingly, write $R = Q[v_{t+1}, v_q]$ if $t \leq q-1$ and $R = \emptyset$ if $t = q$. In either case the triangles ux_ix_{i+1} and $x_jv_tv_{t+1}$ with the three paths $S[x_i, x_h] \cup v_t$, $S[x_{i+1}, x_j]$, $R \cup x_p \cup u$ form a stretcher, a contradiction. If v_{t+2} sees x_j, then we have either $t \leq q-2$ and $j \leq p-2$ or $t = q-1$ and $j = p-1$. Accordingly, write $R' = Q[v_{t+2}, v_q]$ if $t \leq q-2$ and $R' = \emptyset$ if $t = q-1$. In either case the triangle ux_ix_{i+1} with the three paths $S[x_i, x_h] \cup v_t \cup x_j$, $S[x_{i+1}, x_j]$, $x_j \cup R' \cup x_p \cup u$ form a $\Delta P(ux_ix_{i+1}, x_j)$, a contradiction. This completes the proof of the lemma.

□

The proof of Theorem 1 continues as follows. Define a binary relation $<_A$

on A as follows: for $a, a' \in A$, write $a <_A a'$ if there exists a chordless odd path from a to a vertex of B such that the second vertex of this path is a'. We will prove:

Lemma 6 *The relation $<_A$ is antisymmetric on A.*

Lemma 7 *The relation $<_A$ is transitive on A.*

Clearly, the preceding two lemmas imply:

Lemma 8 *The relation $<_A$ is a strict partial order on A.*

Proof of Lemma 6 (antisymmetry of $<_A$). Suppose that the lemma is false: there exist two vertices $x, y \in A$ with $x < y$ and $y < x$. Thus, there exists a chordless odd path $P_x = x_0x_1 \cdots x_r$ with $x_0 = x$, $x_1 = y$, and $x_r \in B$ (with r odd, $r \geq 3$), and there exists a chordless odd path $P_y = y_0y_1 \cdots y_s$ with $y_0 = y$, $y_1 = x$, and $y_s \in B$ (with s odd, $s \geq 3$). We choose the paths P_x, P_y such that the number of vertices in their union is minimized. Possibly $x_r = y_s$. If $x_r \neq y_s$ then x_ry_s is an edge as B is a clique. We claim that:

$$\{x_2, \ldots, x_r\} \cap A = \emptyset \quad \text{and} \quad \{y_2, \ldots, y_s\} \cap A = \emptyset \tag{2}$$

$$\{x_2, \ldots, x_{r-1}\} \cap B = \emptyset \quad \text{and} \quad \{y_2, \ldots, y_{s-1}\} \cap B = \emptyset \tag{3}$$

Indeed (2) holds because A is a clique containing x_0 and y_0. To see that (3) holds, suppose on the contrary that some x_i is in B with $i < r$. Since B is a clique we have $i = r - 1$. Thus we have a chordless odd path $x_1 \cdots x_i$ with $x_1 \in A$, $x_i \in B$, and $i \geq 4$ (because there is no edge between A and B); applying Lemma 5 to this path, we should have either $x_2 \in A$, contradicting that A is a clique, or $x_{r-2} \in B$, contradicting that B is a clique.

Next, we claim that:

> There exist integers i, j (with $2 \leq i \leq r$, $2 \leq j \leq s$) such that x_iy_j is an edge, $P_x[x_1, x_i]$ and $P_y[y_1, y_j]$ are vertex-disjoint and $P_x[x_1, x_i] \cup P_y[y_1, y_j]$ is a hole, and either (a) $i = r$ and $j = s$ or (b) $i < r$, $j < s$ and $P_x[x_{i+1}, x_r] = P_y[y_{j+1}, y_s]$. (4)

To prove this, let i be the smallest integer ($i \leq r$) such that x_i sees a vertex of $P_y \setminus \{y_0, y_1\}$, and let j be the smallest integer with $0 < j \leq s$ such that there is an edge x_iy_j. Note that $i \geq 2$ because $x_1 = y_0$; likewise $j \geq 2$ because

$y_1 = x_0$. The definition of i, j implies that the paths $P_x[x_1, x_i]$, $P_y[y_1, y_j]$ are vertex-disjoint. Since none of x_0x_2, y_0y_2 are edges, $P_x[x_1, x_i] \cup P_y[y_1, y_j]$ is a hole of length at least four, thus it must be an even hole, and i, j have the same parity. Let k be the largest integer (with $j \le k \le s$) such that x_iy_k is an edge.

Suppose that $k - j$ is even (so i, j, k have the same parity; possibly $k = j$). The path $x_1x_2 \cdots x_iy_k \cdots y_s$, with $x_1 \in A$ and $y_s \in B$, is chordless by the choice of i and k, and its length is $(i - 1) + 1 + (s - k)$ which is odd. Call γ the neighbour of y_s on this path ($\gamma = y_{s-1}$ if $k \le s-1$; $\gamma = x_i$ if $k = s$). By Lemma 5 applied to this path, we should have either $x_2 \in A$ or $\gamma \in B$. The former is precluded by (2), so we have $\gamma \in B$. By (3), this is possible only if $\gamma = x_i$ and $i = r$ (so i, j, k are odd). If $j = s$ we have conclusion (a) of (4). If $j < s$ we have $j \le s - 2$ since j is odd. Now the path $y_1y_2 \cdots y_jx_i$ is a chordless odd path with $y_1 \in A$ and $x_i \in B$; applying Lemma 5 to this path, we should have $y_2 \in A$ or $y_j \in B$, which are both impossible by (2) and (3). So we may assume that $k - j$ is odd. In particular, $k > j$.

If $k \ge j + 3$, then $y_1y_2 \cdots y_jx_iy_k \cdots y_s$ is a chordless odd path from A to B. We call γ the neigbhour of y_s along this path ($\gamma = y_{s-1}$ if $k \le s - 1$; $\gamma = x_i$ if $k = s$). Applying Lemma 5 to this path, we must have either $y_2 \in A$ or $\gamma \in B$. By (2) this implies $\gamma \in B$, and by (3) this is possible only with $\gamma = x_r$ ($i = r$) and $k = s$; but these contradict the fact that $k - j$ is odd while i, j have the same parity.

So we must have $k = j + 1$. Observe that $P'_x = x_0x_1 \cdots x_iy_{j+1} \cdots y_s$ is a chordless odd path, and that $P'_x \cup P_y \subseteq P_x \cup P_y$. The choice of P_x, P_y (minimizing the size of their union) implies $P_x \cup P_y = P'_x \cup P_y$, which is possible only if $P_x = P'_x$. Thus we have $x_{i+1} \cdots x_r = y_{j+1} \cdots y_s$, and we have conclusion (b) of (4). Thus (4) is proved.

We now claim that:

On every u-segment of $x_1 \cdots x_r$ the number of v_0-edges is even. (5)

On every u-segment of $y_1 \cdots y_s$ the number of v_0-edges is even. (6)

To see that (5), holds, let us suppose on the contrary that there exists a u-segment $S = x_g \cdots x_h$ of $x_1 \cdots x_r$ that contains an odd number of v_0-edges. We have $1 \le g < h \le r$. Note that, since v_0, x_0, u, x_i cannot induce a square ($g \le i \le h$), vertices u and v_0 have no common neighbour along $x_g \cdots x_h$, except if $x_g = x_1$ and in this case x_1 is their only common neighbour on S. In either case $h \ge g + 2$, and so $S \cup \{u\}$ is an even hole. Thus there is an odd number of v_0-edges on $S \cup \{u\}$, and Lemma 3 applied to $S \cup \{u\}$ and v_0 implies that v_0 has only two (consecutive) neighbours on S, say x_l, x_{l+1} for some integer l with $g \le l < h$. If $l > 1$ then also $g > 1$ (else v_0 would have three neighbours x_1, x_l, x_{l+1} on $S \cup \{u\}$), and the triangle $v_0 x_l x_{l+1}$ together with the three paths $v_0 x_0 u$, $u \cup S[x_g, x_l]$ and $S[x_{l+1}, x_h] \cup u$ form a $\Delta P(v_0 x_l x_{l+1}, u)$, a contradiction. If $l = 1$, we have also $g = 1$, and since $S \cup \{u\}$ is an even hole, h is odd; but then $u \cup x_0 \cup v_0 \cup S[x_2, x_h]$ induces an odd hole, a contradiction. Thus (5) holds. Similarly (6) holds.

In view of (4), we have $r - i = s - j$ and we call this value d. In case (a) of (4) we have $d = 0$ and u sees both x_i, y_j. In case (b) of (4) we have $d \ge 1$, and it will be convenient to denote by $v_{q+d}, \ldots, v_{q+1}$ the vertices of $P_x[x_{i+1}, x_r] = P_y[y_{j+1}, y_s]$ in that order ($v_{q+d} = x_{i+1} = y_{j+1}, \ldots, v_{q+1} = x_r = y_s$), and u sees v_{q+1}. Note that $v_0 \cdots v_q \cdots v_{q+d}$ is a path, which we call R. So $Q \subseteq R$, and $Q = R$ if and only if $d = 0$. If $d \ge 1$ path R is not necessarily chordless, as there may be chords between Q and $v_{q+1} \cdots v_{q+d}$. We call C the hole $P_x[x_1, x_i] \cup P_y[y_1, y_j]$, and recall that v_0 sees x $(= y_1)$ and y $(= x_1)$, which lie on C. Considering the adjacency of v_0 to C, we distinguish between two cases.

Case 1. Vertex v_0 has a neighbour on $C \setminus \{x, y\}$.

By Lemma 3, C must contain an even number of v_0-edges.

If u has no neighbour on $C \setminus \{x, y\}$ (so $d \ge 1$), we find a stretcher induced by the two triangles uxy and $x_i y_j v_{q+d}$ with the three paths $P_x[x_1, x_i]$, $P_y[y_1, y_j]$ and $u \cup R[v_h, v_{q+d}]$, where h is the largest integer such that uv_h is an edge ($q + 1 \le h \le q + d$). So u has at least one other neighbour than x, y on hole C, and so, by Lemma 3, C must contain an even number of u-edges (note that $x_1 y_1$ is one of them).

If $x_i y_j$ is not a u-edge (so $d \ge 1$), we may assume that $P_x[x_1, x_i]$ has an odd number of u-edges and $P_y[y_1, y_j]$ has an even number of u-edges (or vice-versa). However, the two paths $P_x[x_1, x_r]$ and $P_y[y_1, y_s]$ must both have an even number of u-edges by Lemma 2. This is possible if and only if exactly one of $x_i v_{q+d}, y_j v_{q+d}$ is a u-edge; so u sees v_{q+d}. Thus v_0 misses v_{q+d} by (2), and so none of $x_i v_{q+d}, y_j v_{q+d}$ is a v_0-edge. Since u sees exactly one of x_i, y_j, vertex v_0 is not adjacent to both, that is, $x_i y_j$ is not a v_0-edge. Now (5) and (6) imply that $P_x[x_1, x_i]$ and $P_y[y_1, y_j]$ both have an even number of v_0-edges, and consequently C has an odd number of v_0-edges, a contradiction.

If $x_i y_j$ is a u-edge, a similar conclusion arises even more immediately: both

x_i, y_j are non-neigbhours of v_0, thus again by (5) and (6) the paths $P_x[x_1, x_i]$ and $P_y[y_1, y_j]$ both have an even number of v_0-edges, and consequently C has an odd number of v_0-edges, a contradiction.

Case 2. Vertex v_0 has no neighbour on $C \setminus \{x, y\}$.

If no vertex of Q has any neighbour in $C \setminus \{x, y\}$, we find a stretcher consisting of the two triangles v_0xy, $x_iy_jv_{q+d}$ with the three paths $P_x[x_1, x_i]$, $P_y[y_1, y_j]$ and R', where R' is any shortest path from v_0 to v_{q+d} contained in R. So we may assume that some vertex of Q has a neighbour in $C \setminus \{x, y\}$. Let p be the smallest integer $(0 \leq p \leq q)$ such that v_p has a neighbour in $C \setminus \{x, y\}$; by symmetry we may assume that v_p has a neigbhour on $P_x[x_2, x_i]$; let m be the smallest integer (with $2 \leq m \leq i$) such that v_px_m is an edge. Note that $p \geq 1$ since v_0 has no neighbour on $C \setminus \{x, y\}$.

Suppose $m < i$. Let P'' be any shortest path from y_1 to v_p contained in $P_y[y_1, y_s] \cup Q[v_q, v_p]$. Note that x_m has no neighbour along P'' except possibly v_{p+1} and v_{p+2}. If x_m misses v_{p+2}, or if v_{p+2} does not lie on P'', then the triangle v_0xy and the three paths $Q[v_0, v_p]$, $P_x[x_1, x_m] \cup v_p$, P'' form either a $\Delta P(v_0xy, v_p)$ (if x_m misses v_{p+1}, or v_{p+1} does not lie on P'') or a stretcher (if x_m sees v_{p+1} and v_{p+1} lies on P''). If x_m sees v_{p+2} and v_{p+2} lies on P'' then we find a $\Delta P(v_0xy, x_m)$ formed by the triangle v_0xy and the three paths $Q[v_0, v_p] \cup x_m$, $P_x[x_1, x_m]$, $x_m \cup (P'' \setminus \{v_p, v_{p+1}\})$, a contradiction.

Suppose $m = i$. By symmetry we may assume that if v_p has a neighbour along $P_y[y_1, y_j]$ it is only y_j. If v_p misses y_j we find a $\Delta P(v_0xy, x_i)$ with paths $P_x[x_1, x_i]$, $P_y[y_1, y_j] \cup x_i$ and $Q[v_0, v_p] \cup x_i$. If v_p sees y_j then the same vertices induce a stretcher with the two triangles v_0xy and $v_px_iy_j$. This completes the proof of the lemma.

□

Proof of Lemma 7 (transitivity of $<_A$). Let a, a', a'' be three vertices of A such that $a <_A a'$ and $a' <_A a''$. Thus there exists a chordless odd path $y_0y_1 \cdots y_s$ such that $y_0 = a'$, $y_1 = a''$, and $y_s \in B$ (with s odd, $s \geq 3$).

If a has no neighbour along $y_2 \cdots y_s$ then $ay_1y_2 \cdots y_s$ is a chordless odd path to B, implying $a <_A a''$ as desired. Let us now assume that a has a neighbour along $y_2 \cdots y_s$, and let i be the largest integer such that ay_i is an edge $(2 \leq i \leq s)$. We have $i < s$ as there is no edge between A and B.

If i is odd $(3 \leq i \leq s - 2)$, then $ay_i \cdots y_s$ is a chordless odd path with $a \in A$ and $y_s \in B$; applying Lemma 5 to this path, we have either $y_i \in A$ or $y_{s-1} \in B$. The former is impossible because A is a clique. So $y_{s-1} \in B$.

But then $y_0ay_i\cdots y_{s-1}$ is a chordless odd path to a vertex in B, which implies $a' <_A a$, contradicting Lemma 6.

If i is even ($2 \le i \le s-1$), then $y_0ay_i\cdots y_s$ is a chordless odd path to a vertex in B, again implying $a' <_A a$ and contradicting Lemma 6. This completes the proof of the lemma.

□

Since $<_A$ defines a strict partial order on A, we can find a linear extension of this order, thus defining a total order which we still denote by $<_A$.

Likewise, we can define a strict partial order $<_B$ on B as follows: for $x, y \in B$, write $x <_B y$ if there exists a chordless odd path from x to a vertex of A such that the second vertex of this path is y. This order is extended arbitrarily to a total order on B, still denoted by $<_B$.

Lemma 9 *Let a be the maximal vertex of the totally ordered set $(A, <_A)$ and b be the maximal vertex of $(B, <_B)$. Then $\{a, b\}$ is an even pair of G.*

Proof: Suppose on the contrary that there exists a chordless odd path $x_0\cdots x_k$ with $x_0 = a$ and $x_k = b$. We have $k \ge 3$ as there is no edge between A and B. Lemma 5 implies $x_1 \in A$ or $x_{k-1} \in B$. However, If $x_1 \in A$ then $ax_1\cdots x_k$ is a chordless odd path implying $a <_A x_1$, which contradicts the choice of a as a maximal vertex of $(A, <_A)$, while if $x_{k-1} \in B$ the choice of b is similarly contradicted.

□

This lemma completes the proof of Theorem 1.

We finish this section with remarks on the algorithmic aspects. It is easy to detect a shortest hole (if any) in a graph $G = (V, E)$: for any three vertices x, y, z such that $xy \in E$, $yz \in E$, $xz \notin E$, look for a shortest path from x to z in $G \setminus (N(y) \setminus \{x, z\})$. Once a shortest hole H is found, determining the sets A and B as defined above is easy, by neighbourhood examination. Next, we can determine the relation $<_A$ on A (and similarly $<_B$) as follows: for any two vertices $a, a' \in A$, and for each $b \in B$, look for a shortest path from a' to b in $G \setminus [(N(a) \setminus \{a'\}) \cup (B \setminus \{b\})]$; if there is such a path P and it has odd length, the proof of Lemma 5 finds in polynomial time an induced subgraph of $H^{a',b} \cup P$ that is a C_4, an odd hole, or a stretcher; if there is such a path and it has even length, that means $a <_A a'$; if there is no such path for any $b \in B$ then we have $a \not<_A a'$; we can then repeat this for every ordered pair of vertices of A. Furthermore, if two vertices of A violate antisymmetry (respectively if three vertices of A violate transitivity), the proof of Lemma 6 (resp. of Lemma 7) finds in polynomial time an induced subgraph of $H \cup P$ that is a C_4, an odd hole, or a stretcher. In summary, there exists a polynomial-time algorithm which, given any graph G different from a clique, returns either an even pair of G or an induced subgraph of G that is a C_4, an odd hole or a stretcher.

4 Remarks

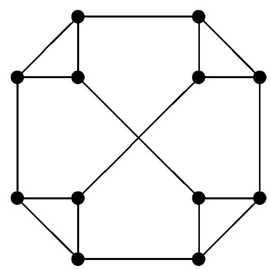


Figure 2: A C_4-free Berge graph with no even pair.

A family of graphs of interest is given by the line-graphs of bipartite subdivisions of K_4 (in short LGBSK4). It is easy to see that every LGBSK4 contains a stretcher, so the class of graphs not containing any LGBSK4 is larger than the class of graphs not containing a stretcher. We conjecture that the existence of an even pair remains true if our C_4-free Berge graph G is allowed to contain a stretcher, but under the condition that G does not contain any LGBSK4. This conjectured fact would be stronger than our theorem, but the proof of such a fact escapes us. On the other hand, forbidding induced LGBSK4's is essential. To see this, observe in Figure 2 a C_4-free Berge graph G that is not a clique and has no even pair (actually G is the smallest such graph; also $\overline{G}$ has no even pair). We remark that G is the line-graph of a bipartite subdivision of K_4.

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