ON GRAPHS WITH STABILITY NUMBER EQUAL TO THE OPTIMAL VALUE OF A CONVEX QUADRATIC PROGRAM

Domingos M. Cardoso

Abstract

Since the Motzkin-Straus result on the clique number of graphs, published in 1965, where they show that the size of the largest clique in a graph can be obtained by solving a quadratic programming problem, several results on the continuous approach to the determination of the clique number of a graph or, equivalently, to the determination of the stability number of its complement, have been published. In this paper, a Motzkin-Straus-like approach to the stability number of graphs is presented and extended to the study of graphs for which the stability number is equal to the optimal value of a convex quadratic programming problem (called graphs with convex-QP stability number) as well as the determination of convex quadratic lower and upper bounds on the stability number of arbitrary graphs. In the presence of adverse conditions, it is proved that the recognition of graphs with convex-QP stability number is equivalent to the recognition of graphs with a particular combinatorial structure (called regular-stable graphs). Additionally, for particular types, as is the case of line graphs of forests or threshold graphs, the polynomial-time recognition of graphs with convex-QP stability number is introduced.

1 Introduction

A stable set of a graph is a set of mutually non-adjacent vertices. The determination of a maximum size stable set (which is called maximum stable set) and/or the determination of its size (which is called stability number) in a graph, are central combinatorial optimization problems. However, given a nonnegative integer k, to determine if a graph G has a stable set of size k is NP-complete [13]. Fortunately, there are several graph classes for which the stability number

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can be determined in polynomial-time, as is the case of perfect graphs [14], claw-free graphs [18] and [23], (P_6, C_4)-free graphs [19] and $(P_5, banner)$-free graphs [15], among many others. Several continuous optimization approaches to approximate the stability number of arbitrary graphs have been developed, since the publication of the Motzkin-Straus result [20] on the clique number of graphs, where they show that the size of the largest clique in a graph (which is equal to the stability number of its complement) may be obtained by solving an indefinite quadratic programming problem (that is, a quadratic programming problem with an objective function which has an indefinite Hessian matrix). Among several papers on continuous optimization approach to the maximum stable set problem or equivalent problems (as is the case of the maximum clique problem) we refer [11] and [3]. The focus of this paper is the study of graphs for which the stability number can be determined by solving a convex quadratic programming problem. Such graphs were introduced in [6] and are called graphs with convex-QP stability number.

In this paper we consider undirected simple graphs, $G = (V(G), E(G))$, where $V(G)$ denotes the nonempty set of vertices and $E(G)$ the set of edges. It is assumed that G is of order $n > 1$, i.e., $|V(G)| = n > 1$. An element of $E(G)$, whose endpoints are the vertices v and w, is denoted by vw and, in such case, we say that the vertex v is adjacent to the vertex w. If $v \in V(G)$, then we call neighborhood of v the vertex set denoted by $N_G(v) = \{w : vw \in E(G)\}$ and the degree of v, $d_G(v) = |N_G(v)|$. If G is such that $\forall v \in V(G) \;\; d_G(v) = k$ then we say that G is k-regular (in particular, when $k = 3$ the graph is called cubic). Given a graph G and a set of vertices $U \subseteq V(G)$, the subgraph of G induced by U, $G[U]$, is such that $V(G[U]) = U$ and $E(G[U]) = \{vw \in E(G) : v, w \in U\}$. Consider that the set of vertices $X = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ is such that $x_i x_{i+1} \in E(G) \; \forall i \in \{1, \ldots, k-1\}$. If the vertices of X are all distinct, then we say that X defines a path (and if they are all distinct but x_1 and x_k, which are equal, then we say that X defines a cycle). A graph G is connected if $\forall i, j \in V(G)$ there exists a path between i and j and is disconnected if it is not connected. A component of a graph G is H, a connected subgraph of G, such that $\forall v \in V(G) \setminus V(H)$ the induced subgraph $G[V(H) \cup \{v\}]$ is disconnected. A graph G with p vertices such that $\forall x, y \in V(G) \; xy \in E(G)$ is designated complete graph and it is denoted by K_p.

Throughout this paper, A_G will denote the adjacency matrix of the graph G

with vertices $V(G) = \{v_1, \ldots, v_n\}$, that is, $A_G = (a_{ij})_{n\times n}$ is such that

$$a_{ij} = \begin{cases} 1 & \text{, if } v_iv_j \in E(G) \\ 0 & \text{, otherwise} \end{cases}$$

and $\lambda_{min}(A_G)$ the minimum eigenvalue of A_G. As it is well known, if G has at least one edge then $\lambda_{min}(A_G) \leq -1$. Indeed, $\lambda_{min}(A_G) = 0$ iff G has no edges; $\lambda_{min}(A_G) = -1$ iff G has at least one edge and each component of G is complete and, otherwise, $\lambda_{min}(A_G) \leq -\sqrt{2}$ [9]. Throughout this paper it will be consider only graphs G with at least one edge, and then with $\lambda_{min}(A_G) \leq -1$. A set of vertices is called stable (clique) set if no (every) two vertices are adjacent. A stable (clique) set is called maximum stable (clique) set if there is no other stable (clique) set with greater number of vertices. The number of vertices, in a maximum stable (clique) set of a graph G, is called the stability (clique) number of G and is denoted by $\alpha(G)$ $(\omega(G))$. The complement of G, denoted by $\bar{G}$, is such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{vw : v, w \in V(G) \wedge vw \notin E(G)\}$. It is well known that $\alpha(G) = \omega(\bar{G})$ and then the determination of the stability number is equivalent to the determination of the clique number.
As introduced in [1], a graph G is τ-regular-stable if there exists a maximum stable set of G, S, such that

$$\forall v \in V(G) \setminus S \qquad |N_G(v) \cap S| = \tau.$$

The figure 1 exemplifies a τ-regular-stable graph, with $\tau = 2$.

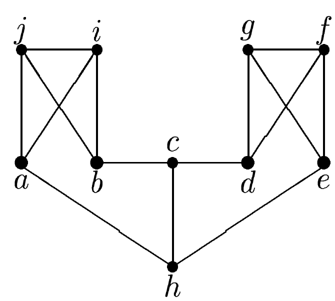

Figure 1: The cubic 2-regular-stable graph G_1, with maximum stable set $S = \{a, b, d, e\}$.

The τ-regular-stable graphs are particular cases of graphs with convex-QP stability number when $\tau = -\lambda_{min}(A_G)$. According to [1], if G is τ-regular-stable then

$$\frac{n\tau}{\Delta(G)+\tau} \leq \alpha(G) \leq \frac{n\tau}{\delta(G)+\tau},$$

where, as usually, $\Delta(G) = \max\{d_G(v) : V \in V(G)\}$ and $\delta(G) = \min\{d_G(v) : V \in V(G)\}$. Therefore, if G is k-regular and τ-regular-stable then $\alpha(G) = \frac{n\tau}{k+\tau}$.
A matching in a graph G is a subset of edges, $M \subseteq E(G)$, no two of which have a common vertex. A matching with maximum cardinality is designated maximum matching. Furthermore, if for each vertex $v \in V(G)$ there is one edge of the matching M incident with v, then M is called a perfect matching. A line graph of a graph G is the graph $L(G)$ constructed by taking the edges of G as vertices of $L(G)$, and joining two vertices in $L(G)$, by an edge, whenever the corresponding edges in G have a common vertex. Then, from [1], we may conclude that a connected graph, with more than one edge, has a perfect matching if and only if its line graph is 2-regular-stable. Note that the determination of a maximum stable set of a line graph $L(G)$ is equivalent to the determination of a maximum matching of the original graph G.
The remainder of the paper is organized in four parts. The next section contains a Motzkin-Straus-like approach to the determination of the stability number of a graph, including the determination of convex-quadratic programming upper and lower bounds to the stability number of graphs. In section 3 the graphs with stability number equal to the optimal value of a convex quadratic program (called graphs with convex-QP stability number) are characterized and their main properties are presented. In section 4 the recognition of graphs with convex-QP stability number is analyzed and, for particular types of graphs, results which allows its polynomial-time recognition are introduced. In section 5, final remarks and open problems are presented.

2 A Motzkin-Straus-like approach to the stability number of a graph

Consider the quadratic programming problem

$$(P_G(\tau)) \qquad \upsilon_G(\tau) = \max\{2\hat{e}^T x - x^T(\frac{1}{\tau}A_G + I_n)x : x \geq 0\},$$

with $\tau > 0$, where $\hat{e}$ is the all ones vector and I_n is the identity matrix of order n (these notations will be used throughout the paper).
Given a subset of vertices of a graph G, $S \subseteq V(G)$, the vector $x \in \mathbb{R}^V$ with

$x_v = 1$ if $v \in S$ and $x_v = 0$ if $v \notin S$ is called the characteristic vector of S. An optimal solution, x^*, of $(P_G(\tau))$ is called spurious when $\upsilon_G(\tau) = \alpha(G)$ but x^* not defines the characteristic vector of a maximum stable set. The presence of spurious solutions among the optimal ones of the Motzkin-Straus quadratic programming problem [20] first observed in [21], has been deeply studied (see, [22] and [4]).
According to [7], we may conclude that

$$\forall \tau > 0 \qquad 1 \leq \upsilon_G(\tau) \leq n,$$

with $\upsilon_G(\tau) = 1$ if G is a clique and $\upsilon_G(\tau) = n$ if G has no edges. Furthermore, the following holds,

- $\forall \tau \geq -\lambda_{min}(A_G)$ $(P_G(\tau))$ is a convex program.
- If $x^*(\tau)$ is an optimal solution for $(P_G(\tau))$ then

  $$\forall i \in V(G) \;\; 0 \leq [x^*(\tau)]_i \leq 1,$$

  where $[x^*(\tau)]_i$ denotes the i-the component of $x^*(\tau)$.

Theorem 2.1 *[7] Given a graph, G, the function*

$$\begin{array}{rcl} \upsilon_G :]0, +\infty[& \mapsto & [1, n] \\ \tau & \rightsquigarrow & \upsilon_G(\tau) \end{array}$$

verifies the properties:

1. $\forall \tau > 0 \;\; \alpha(G) \leq \upsilon_G(\tau)$.
2. $0 < \tau_1 < \tau_2 \Rightarrow \upsilon_G(\tau_1) \leq \upsilon_G(\tau_2)$.
3. *Assuming $\tau^* > 0$, then the following statements are equivalent.*
   - $\exists \bar{\tau} \in]0, \tau^*[$ *such that* $\upsilon_G(\bar{\tau}) = \upsilon_G(\tau^*)$;
   - $\upsilon_G(\tau^*) = \alpha(G)$ *and* $\forall \tau \in]0, \tau^*[$ $(P_G(\tau))$ *has no spurious optimal solutions;*
   - $\forall \tau \in]0, \tau^*]$ $\upsilon_G(\tau) = \alpha(G)$.
4. $\forall U \subset V(G)$ $\forall \tau > 0$ $\upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$, *where $G - U$ denotes the subgraph of G induced by $V(G) \setminus U$.*

Given an arbitrary graph G, from the above properties, we conclude that $\upsilon_G(\tau)$ is a monotone upper bound on the stability number of G. Furthermore, being $\tau^* = \max\{\tau : \upsilon_G(\tau) = \alpha(G)\}$ (and then, as will be seen later on, $\tau^* \geq 1$), we may conclude that $\forall \tau \in]0, \tau^*[$ $(P_G(\tau))$ has no spurious optimal solutions. The figure 2 illustrates the graph of the function $\upsilon_G(\tau)$, obtained for the graph G depicted in figure 1, for which $\alpha(G) = 4$.

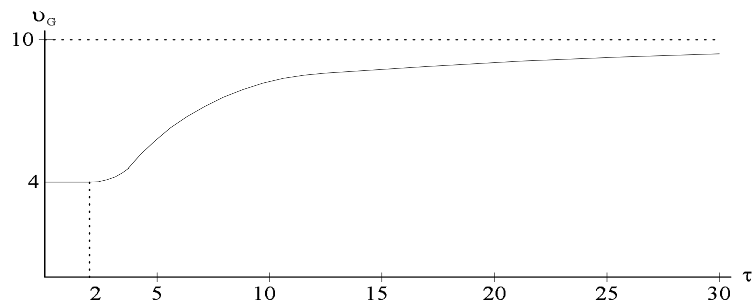

Figure 2: Graph of $\upsilon_G(\tau)$, where G is the graph of the figure 1, for which $\alpha(G) = 4$.

From the next theorem follows that the indefinite quadratic program of Motzkin-Straus [20] is a particular case of the family of quadratic programming problems $(P_G(\tau))$.

Theorem 2.2 *[7] Let us define the quadratic programming problem*

$$(Q_G(\tau)) \qquad \nu_G(\tau) = \min\{z^T(\frac{A_G}{\tau} + I_n)z : \hat{e}^T z = 1, z \geq 0\},$$

with $\tau > 0$. If x^* and z^* are optimal solutions for $P_G(\tau)$ and $Q_G(\tau)$, respectively, then $\frac{z^*}{\nu_G(\tau)}$ and $\frac{x^*}{\upsilon_G(\tau)}$ are optimal solutions for $P_G(\tau)$ and $Q_G(\tau)$, respectively, and $\upsilon_G(\tau) = \frac{1}{\nu_G(\tau)}$.

Considering the Motzkin-Straus result on the stability number of graphs [20], which is equivalent to the equality

$$\frac{1}{\min\{z^T(A_G + I_n)z : \hat{e}^T z = 1, z \geq 0\}} = \alpha(G), \tag{1}$$

(see proposition 2 in [11]) and, combining this result with theorem 2.2, we may conclude that $\upsilon_G(1) = \frac{1}{\nu_G(1)} = \alpha(G)$.

If x^* is an optimal solution of $(P_G(\tau))$, with $\tau \geq 1$, then

$$\frac{\upsilon_G(\tau)^2}{x^{*T}(A_G+I_n)x^*} \leq \alpha(G) \leq \upsilon_G(\tau), \tag{2}$$

$$||x^*||^2 - (\tau-2)(\upsilon_G(\tau) - ||x^*||^2) \leq \alpha(G) \leq \upsilon_G(\tau). \tag{3}$$

The lower bounds in (2) and (3) were obtained in [6] and [7], respectively. The graphs G such that $\upsilon_G(-\lambda_{min}(A_G)) = \alpha(G)$, will be called, as in [6], graphs with convex-QP stability number (where QP means quadratic programming).

3 Characterization of graphs with convex-QP stability number

The upper bound $\upsilon_G(\tau)$ on the stability number of G, with $\tau = -\lambda_{min}(A_G)$, was introduced in [17], where the following necessary and sufficient condition to obtain the equality is given.

- $\alpha(G) = \upsilon_G(-\lambda_{min}(A_G))$ if and only if for a maximum stable set S of G (and then for all),

$$-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\}. \tag{4}$$

There exists an infinite number of graphs with convex-QP stability number. In fact, according to [6], a connected graph, with at least one edge, which is nor a star neither a triangle, has a perfect matching if and only if its line graph has convex-QP stability number. Note that, as referred in section 1, the line graphs of graphs with a perfect matching are 2-regular-stable and then, since one of the basic properties of line graphs $L(G)$ is that $\lambda_{min}(A_{L(G)}) \geq -2$, applying condition (4) the conclusion follows.

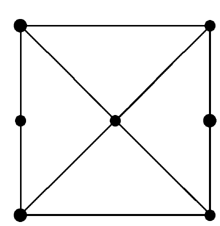

Figure 3: The 2-regular-stable graph G_2, for which $\upsilon_{G_2}(2) = \alpha(G_2) = 3$.

The graphs of figures 1, 3 and 4 are examples of graphs with convex-QP stability number. The graphs G_1 and G_2 of figures 1 and 3, respectivelly, are both 2-regular-stable and $\lambda_{min}(A_{G_1}) = \lambda_{min}(A_{G_2}) = -2$. The graph G_3 of figure 4 is 6-regular-stable and $\lambda_{min}(A_{G_3}) = -3$.

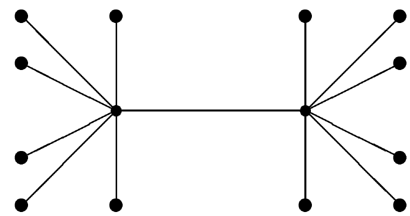

Figure 4: The 6-regular-stable graph G_3, for which $\upsilon_{G_3}(3) = \alpha(G_3) = 12$.

Let us denote by $\mathcal{Q}$ the class of graphs with convex-QP stability number. This class of graphs is not hereditary (that is, is not closed for induced subgraphs) [16]. However, according to [6], $\mathcal{Q}$ is closed under deletion of α-redundant subsets of vertices, defining an α-redundant subset of vertices as being a subset $U \subseteq V(G)$, such that $\alpha(G) = \alpha(G - U)$, where $G - U$ denotes the subgraph of G induced by $T = V(G) \setminus U$.

Theorem 3.1 *[6] Let G be a graph and $\tau = -\lambda_{min}(A_G)$.*

1. *If $G \in \mathcal{Q}$ and $\exists U \subseteq V(G)$ such that $\alpha(G) = \alpha(G - U)$ then $G - U \in \mathcal{Q}$.*
2. *If $\exists v \in V(G)$ such that $\upsilon_G(\tau) \neq \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\}$ then $G \notin \mathcal{Q}$.*
3. *Consider that $\upsilon_{G-\{v\}}(\tau) \neq \upsilon_{G-N_G(v)}(\tau)$.*
   (a) *If $\upsilon_G(\tau) = \upsilon_{G-\{v\}}(\tau)$ then $G \in \mathcal{Q}$ iff $G - \{v\} \in \mathcal{Q}$.*
   (b) *If $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$ then $G \in \mathcal{Q}$ iff $G - N_G(v) \in \mathcal{Q}$.*

Assuming $\tau = -\lambda_{min}(A_G)$, we may conclude the following:

- Every graph G has a subgraph H such that $\upsilon_H(\tau) = \alpha(H) = \alpha(G)$ (note that deleting as many α-redundant vertices of G as necessary such subgraph H is obtained [1]).

[1] In the worst case we may delete all vertices not belonging to a maximum stable set and then the remaining subgraph H has no edges. Therefore, the corresponding adjacency matrix is the null one and $\upsilon_H(\tau) = |V(H)|$.

- If G has convex-QP stability number then $\forall v \in V(G)$

$$\upsilon_G(\tau) = \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\}.$$

- Assume that $\forall v \in V(G)$, $\upsilon_G(\tau) = \max\{\upsilon_{G-\{v\}}(\tau), \upsilon_{G-N_G(v)}(\tau)\}$ and also that $\exists v \in V(G)$, such that $\upsilon_{G-\{v\}}(\tau) \neq \upsilon_{G-N_G(v)}(\tau)$. Then to recognize if $G \in \mathcal{Q}$ is equivalent to recognize if an induced subgraph ($G - \{v\}$ if $\upsilon_G(\tau) = \upsilon_{G-\{v\}}(\tau)$ or $G - N_G(v)$ if $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$) has or not convex-$QP$ stability number.

Therefore, concerning the recognition of graphs with convex-QP stability number, the problem is what to do when

$$\begin{aligned} \forall v \in V(G) \ \ \upsilon_G(\tau) &= \upsilon_{G-\{v\}}(\tau) && (5) \\ &= \upsilon_{G-N_G(v)}(\tau), && (6) \end{aligned}$$

with $\tau = -\lambda_{min}(A_G)$. However, supposing that G has no isolated vertices (that is, there is no vertex $j \in V(G)$, such that $N_G(j) = \emptyset$), the equalities (6) imply the equalities (5). In fact, assuming that $\forall v \in V(G)$, $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$, since G has no isolated vertices, $\forall u \in V(G)$ $\exists w \in N_G(u)$ and then, from the inequalities $\upsilon_{G-N_G(w)}(\tau) \leq \upsilon_{G-\{u\}}(\tau) \leq \upsilon_G(\tau)$, the equality $\upsilon_{G-\{u\}}(\tau) = \upsilon_G(\tau)$ is obtained. So, aiming to recognize if an arbitrary graph has or not convex-QP stability number, it remains what to do only when the equalities (6) hold. Designating the equalities (6) by *adverse conditions*, the next theorem states that, in the presence of such adverse conditions, to recognize if a graph has convex-QP stability number is equivalent to recognize if it is τ-regular-stable.

Theorem 3.2 *Let G be a graph such that $\forall v \in V(G)$ $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$, with $\tau = -\lambda_{min}(A_G)$. Then $G \in \mathcal{Q}$ if and only if G is τ-regular-stable.*

Proof: Considering the equalities $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$ $\forall v \in V(G)$, with $\tau = -\lambda_{min}(A_G)$, and assuming that $\tilde{x}^{v*}$ is an optimal solution for $(P_{G-N_G(v)}(\tau))$, then, for each $v \in V(G)$, we may conclude that x^{v*} defined by

$$x_i^{v*} = \begin{cases} \tilde{x}_1^{v*} & \text{, if } i \notin N_G(v) \\ 0 & \text{, otherwise} \end{cases}$$

is an optimal solution for $(P_G(\tau))$. On the other hand, by Karush-Kuhn-Tucker optimality conditions (see, for instance, [2]), $\exists y^* \geq 0$ such that for any optimal solution x^* of $(P_G(\tau))$,

$$A_G x^* = \tau(\hat{e} - x^*) + y^*, \tag{7}$$

$$y^{*T} x^* = 0. \tag{8}$$

Therefore, from (7), $\forall v \in V(G)$

$$\sum_{j \in N_G(v)} x_j^{v*} = \tau(1 - x_v^{*v}) + y_v^* \ \Leftrightarrow\ x_v^{*v} = 1 + \frac{y_v^*}{\tau} \ \Leftrightarrow\ x_v^{*v} = 1$$

and thus we may conclude that $y_v^* = 0\ \forall v \in V(G)\ \Leftrightarrow y^* = 0$. Note that from (8) $\forall v \in V(G)\quad x_v^* > 0 \quad \Rightarrow \quad y_v^* = 0$.

- Let us suppose that $G \in \mathcal{Q}$ and that $\bar{x}$ is the characteristic vector of a maximum stable set S of G. Then $\bar{x}$ is an optimal solution of $(P_G(\tau))$ and, since by (7) $A_G\bar{x} = \tau(\hat{e} - \bar{x})$, it follows that

$$\forall v \in V(G) \quad \sum_{j \in N_G(v)} \bar{x}_j = \tau(1 - \bar{x}_v).$$

  Hence, $\forall v \notin S$

$$\sum_{j \in N_G(v)} \bar{x}_j = \tau \ \Leftrightarrow\ |N_G(v) \cap S| = \tau.$$

- Conversely, supposing that G is τ-regular-stable, it is immediate that the condition (4) holds and then $G \in \mathcal{Q}$.

□

As a consequence of this theorem, in order to recognize if a graph G has or not convex-QP stability number, in the worst case (that is, when $\forall v \in V(G) \quad \upsilon_G(-\lambda_{min}(A_G)) = \upsilon_{G-N_G(v)}(-\lambda_{min}(A_G)))$, we must recognize if G is τ-regular-stable, with $\tau = -\lambda_{min}(A_G)$.

4 Recognition of graphs with convex-QP stability number

When $\lambda_{min}(A_G) \notin \mathbb{Z}$ it is easy to recognize if G has or not convex-QP stability number. In fact, assuming that $G \in \mathcal{Q}$, if $\tau = -\lambda_{min}(A_G) \notin \mathbb{Z}$ then every optimal solution x^* for $(P_G(\tau))$ verifies the Karush-Kuhn-Tucker optimality condition (7), with $y^* \neq 0$, and hence $\exists v \in V(G)$ such that $\upsilon_G(\tau) \neq \upsilon_{G-N_G(v)}(\tau)$. Therefore, applying theorem 3.1, we may proceed as follows.

- If $\upsilon_G(\tau) = \upsilon_{G-\{v\}}(\tau)$ then $G \in \mathcal{Q}$ iff $G - \{v\} \in \mathcal{Q}$ and thus the recognition may be obtained analyzing $G - \{v\}$ instead of G.

- If $\upsilon_G(\tau) = \upsilon_{G-N_G(v)}(\tau)$ then $G \in \mathcal{Q}$ iff $G - N_G(v) \in \mathcal{Q}$ and thus the recognition may be obtained analyzing $G - N_G(v)$ instead of G.

- If $\upsilon_{G-\{v\}}(\tau) \neq \upsilon_G(\tau) \neq \upsilon_{G-N_G(v)}(\tau)$ then $G \notin \mathcal{Q}$.

A tree is a connected graph without cycles and a forest is a graph whose components are trees. For instance, assuming that G is a forest with at least one edge, it follows that $-2 < \lambda_{min}(L(G)) < -1$ [2] and then we may recognize easily if $L(G) \in \mathcal{Q}$. When the forest G has no isolated vertices and has at least one component with more than one edge, to recognize if $L(G) \in \mathcal{Q}$ is equivalent to recognize if the forest G has or not a perfect matching and also to determine such perfect matching, by convex programming, if there exists.

According to the definition of a regular-stable graph, it is immediate that if G is a τ-regular-stable graph then there exists a maximum stable set S for which the characteristic vector is a solution of the linear system

$$(\frac{A_G}{\tau} + I_n)x = \hat{e}. \tag{9}$$

On the other hand, if $\tau = -\lambda_{min}(A_G)$ and the system (9) has a $0-1$ solution $\bar{x}$, then $\bar{x}$ is the characteristic vector of a maximum stable set of G. The first implication is obvious and the second follows from the fact that the system (9) is equivalent to the system $A_G x = \tau(\hat{e} - x)$ and, therefore, if $\bar{x}$ is a $0-1$ solution of (9) then $\bar{x}^T A_G \bar{x} = 0$, which is equivalent to say that $\bar{x}$ is the characteristic vector of a stable set S of G. Hence, since $\tau = -\lambda_{min}(A_G)$, then $\bar{x}$ is the optimal solution of the convex quadratic programming problem $(P_G(\tau))$ and $|S| = \upsilon_G(\tau) = \alpha(G)$.
For particular type of graphs, even when $\lambda_{min}(A_G) \neq -\tau$, the resolution of the system (9) allows to conclude if the graph is or not τ-regular-stable. Indeed, considering again a forest without isolated vertices and with at least one component which has more than one edge, we have the following result:

[2] Note that the minimum eigenvalue of any graph with at least one edge is not greater than -1, with equality iff all of its components are complete. On the other hand, in case of line graphs, $L(H)$, the minimum eigenvalue is not less than -2, with equality iff the original graph H has an even cycle or two odd cycles [8].

Theorem 4.1 *Let G be a forest for which each component has more than one edge. Then G has a perfect matching iff the unique solution of the system $(\frac{A_{L(G)}}{2} + I_m)x = \hat{e}$, where $m = |V(L(G))| = |E(G)|$, has $0-1$ components. Furthermore, if such $0-1$ solution exists then it is the characteristic vector of the maximum stable set of $L(G)$ corresponding to the edges of the perfect matching of G.*

Proof: Since G is a forest, $-2 \notin \sigma(A_{L(G)})$, where $\sigma(A_{L(G)})$ denotes the spectrum of the matrix $A_{L(G)}$, and then the system

$$(\frac{A_{L(G)}}{2} + I_m)x = \hat{e} \tag{10}$$

is determined. On the other hand, according to [1], a component of G has a perfect matching iff the corresponding component of $L(G)$ is 2-regular-stable. Therefore, since G has a perfect matching iff each component has a perfect matching it follows that G has a perfect matching iff each component and then $L(G)$ is 2-regular-stable or, equivalently, iff the solution of (10) is the characteristic vector of a maximum stable set of $L(G)$.

- If G has a perfect matching then $L(G)$ is 2-regular-stable and hence the characteristic vector of the maximum stable set of $L(G)$ is the solution of the system (10).
- If the solution of (10) has $0-1$ components then it is also the characteristic vector of a stable set $L(M) \subset V(L(G))$ such that $\forall w \notin L(M)$ $|N_{L(G)}(w) \cap L(M)| = 2$. Therefore, the corresponding set of edges $M \subset E(G)$ is a matching of G such that $\forall xy \notin M$ $\exists xv, yw \in M$. Thus M is a perfect matching (which implies that $L(M)$ is a maximum stable set of $L(G)$).

The last part follows, immediately, from the above considerations.

□

Additionally to the line graphs of forests, in what follows, we analyze a few more types of graphs for which we may recognize, in polynomial-time, if a graph has or not convex-QP stability number.

According to [1] a graph G is 1-regular-stable if and only if each vertex belongs to exactly one simplex (which is a clique of a graph induced by the neighbors of some vertex). This is the case, for instance, of the graphs in which each component is complete. Then, since for such graphs $\lambda_{min}(A_G) = -1$, applying condition (4), it follows that they have convex-QP stability number. Actually, such graphs are the only 1-regular-stable graphs with convex-QP stability number.

From now on, let us denote the closed neighborhood of a vertex v by $N_G[v]$ (that is, $N_G[v] = N_G(v) \cup \{v\}$).

Theorem 4.2 *Let G be a τ-regular-stable graph with $\tau > 1$. If $\exists u, v \in V(G)$ such that $N_G[u] \subseteq N_G[v]$ then $\alpha(G) = \alpha(G - \{v\})$ and $G - \{v\}$ remains τ-regular-stable.*

Proof: Since G is τ-regular-stable then there exists a maximum stable set S such that $\forall w \notin S \quad |N_G(w) \cap S| = \tau$. Let us assume that $v \in S$. Then $N_G(v) \subset V(G) \setminus S$ and, since $N_G[u] \subseteq N_G[v]$, it follows that $|N_G(u) \cap S| = 1$, which is a contradiction. Therefore $v \notin S$ and, consequently, $\alpha(G) = \alpha(G - \{v\})$ and $G - \{v\}$ is τ-regular-stable.

□

Given two vertices $u, v \in V(G)$, the vertex v dominates vertex u if $N_G(u) \subseteq N_G[v]$ and then we say that the vertices v and u are comparable. This binary relation is a *preorder* (that is, is reflexive and transitive) and is called, in [10], *vicinal preorder*. The graph $D(G)$ such that $V(D(G)) = V(G)$ and

$$E(D(G)) = \{vw : v, w \in V(G) \wedge N_G(v) \subseteq N_G[w] \text{ or } N_G(w) \subseteq N_G[v]\},$$

is the comparability graph of the vicinal preorder of G. Considering the Dilworth number of a graph G, dilw(G), introduced in [10] as the largest number of pairwise incomparable vertices of G, then dilw$(G) = \alpha(D(G))$. Thus, taking into account that if dilw$G) \leq \omega(G)$ then there exists $u, v \in V(G)$ such that $N_G[u] \subseteq N_G[v]$, we may conclude the following corollary of theorem 4.2.

Corollary 4.1 *Let G be a graph for which dilw$(G) \leq \omega(G)$. If G is τ-regular-stable, with $\tau > 1$, then $\exists u, v \in V(G)$ such that $N_G[u] \subseteq N_G[v]$ and hence $\alpha(G) = \alpha(G - \{v\})$ and $G - \{v\}$ remains τ-regular-stable.*

As immediate consequence of the above corollary, we may recognize in polynomial-time if a connected graph without induced connected subgraphs H such that $dilw(H) > \omega(H)$, has or not convex-QP stability number. This is the case of threshold graphs which are graphs with Dilworth number equal to one (for details about graphs with Dilworth number not greater than two see [5]).

5 Final remarks and open problems

According to [24], a (ρ, σ)-set of a graph G is a subset of vertices $T \subseteq V(G)$ such that $|N_G(v) \cap T| \in \rho$ if $v \in T$ and $|N_G(v) \cap T| \in \sigma$ if $v \notin T$, where ρ and σ are subsets of $\{0, 1, \cdots, n\}$. Taking into account this definition, a graph G is τ-regular-stable if and only if there exists a maximum stable set S which is a $(\{0\}, \{\tau\})$-set. Then, as a consequence of the results obtained in [12] we may conclude that, in general, the recognition of τ-regular-stable graphs is $\mathcal{NP}$-complete. However, it is expected that, additionally to the above referred polynomial cases, there are many other graph classes in which we may recognize in polynomial-time if a graph has or not convex-QP stability number. The determination of such graph classes, namely, the hereditary ones (that is, the graph classes which are closed under vertex deletion) remains open.

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Universidade de Aveiro
Departamento de Matemática
3810-193 Aveiro, Portugal
E-mail: dcardoso@mat.ua.pt