QUADRATIC ORLIK-SOLOMON ALGEBRAS OF GRAPHIC MATROIDS

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Abstract

In this note we introduce a sufficient condition for the Orlik-Solomon algebra associated to a matroid $\mathcal{M}$ to be l-adic and we prove that this condition is necessary when $\mathcal{M}$ is binary (in particular graphic). Moreover, this result cannot be extended to the class of all matroids.

1 Introduction

Throughout this note $\mathcal{M}$ denotes a simple matroid of rank r on the ground set $[n]$. We refer to [7] as a standard source for matroids. An *(essential) arrangement of hyperplanes* in $\mathbb{K}^d$ is a finite collection $\mathcal{A}_{\mathbb{K}} = \{H_1, \dots, H_n\}$ of codimension 1 subspaces of $\mathbb{K}^d$ such that $\bigcap_{H_i \in \mathcal{A}_{\mathbb{K}}} H_i = 0$. There is a matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$, on the ground set $[n] := \{1, \dots, n\}$, canonically determined by $\mathcal{A}_{\mathbb{K}}$: i.e., a subset $X \subseteq [n]$ is independent if and only if the codimension of $\bigcap_{i \in X} H_i$ is equal to the cardinality of X. The manifolds $\mathfrak{M}(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}^d \setminus \bigcup_{H_i \in \mathcal{A}} H_i$ are important in the Aomoto-Gelfand theory of $\mathcal{A}_{\mathbb{C}}$-hypergeometric functions, see [5] for a recent introduction from the point of view of arrangement theory. We refer to the survey [3] for a recent discussion on the role of matroid theory in the study of hyperplane arrangements. Fix a set $E := \{e_1, \dots, e_n\}$ and consider the Grassmann algebra $\mathcal{E} := \bigwedge \left(\bigoplus_{i=1}^{n} \mathbb{K} e_i \right)$. We set $\mathcal{E}_\ell := \bigwedge^\ell \left(\bigoplus_{e \in E} \mathbb{K} e \right)$, $\forall \ell \in \mathbb{N}$. (By convention $\mathcal{E}_0 = \mathbb{K}$ even in the case where $E = \emptyset$.) For every linearly ordered subset $X = (i_1, \dots, i_m) \subset [n]$, $i_1 < \dots < i_m$, let e_X be the monomial

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$e_X := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$. By definition set $e_\emptyset = 1 \in \mathbb{K}$. For $\mathfrak{X}$ a subset of $2^{[n]}$, let $\Im(\mathfrak{X}) := \langle \partial(e_X) : X \in \mathfrak{X} \rangle$ be the two-side graded ideal of the Grassmann algebra $\mathcal{E}$ generated by the "differentials",

$$\partial(e_X) = \partial(e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum (-1)^j e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \cdots \wedge e_{i_m}.$$

The graded Orlik-Solomon $\mathbb{K}$-algebra $\mathrm{OS}(\mathcal{M})$ is the quotient $\mathcal{E}/\Im(\mathfrak{C})$, where $\mathfrak{C}$ denotes the set of circuits of $\mathcal{M}$. The de Rham cohomology algebra $H^\bullet\big(\mathfrak{M}(\mathcal{A}_\mathbb{C}); \mathbb{C}\big)$ is shown to be isomorphic to the Orlik-Solomon $\mathbb{C}$-algebra of the matroid $\mathcal{M}(\mathcal{A}_\mathbb{C})$, see [4]. Since then the combinatorial structure of the manifold $\mathfrak{M}(\mathcal{A}_\mathbb{C})$ have been intensively studied (i.e., the properties depending of the matroid $\mathcal{M}(\mathcal{A}_\mathbb{C})$). Quadratic OS (Orlik-Solomon) algebras (i.e., such that $\Im(\mathfrak{C}) = \Im(\mathfrak{C}_3)$ where $\mathfrak{C}_\ell$, $\ell \in [n]$, denotes the set of circuits with at most ℓ elements of $\mathcal{M}$) appear in the study of complex arrangements, in the rational homotopy theory of the manifold $\mathfrak{M}$ and the Koszul property of $\mathrm{OS}(\mathcal{M})$, see [2, 6]. It is an old open problem to find a condition on the matroid $\mathcal{M}$ equivalent to $\mathrm{OS}(\mathcal{M})$ being quadratic, or in general *ℓ-adic* $\big($i.e., such that $\Im(\mathfrak{C}) = \Im(\mathfrak{C}_{\ell+1})\big)$. Some necessary conditions are known but none of these conditions implies the quadratic property. For more details, we refer the reader to the recent survey on OS algebras [8], which also provides a large bibliography.

We say that a circuit C of a matroid has a *chord* $i_\alpha \in C$, if there are two circuits C_1 and C_2 such that $C_1 \cap C_2 = i_\alpha$ and $C = C_1 \Delta C_2$. We say that a matroid is *ℓ-chordal*, $\ell \geq 4$, if every circuit with at least ℓ elements has a chord. $\mathcal{M}$ is said *chordal* if it is 4-chordal. We prove that if all the circuits of $\mathcal{M}$ with at least $\ell+2$ elements have a chord, then the algebra $\mathrm{OS}(\mathcal{M})$ is ℓ-adic. The 2-adic case with $\mathrm{rk}(\mathcal{M}) = 3$ is known and correspond to the "parallel 3-arrangements", see [1] for details. A matroid $\mathcal{M}$ is said to be *binary* if it is realizable over $\mathbb{Z}_2$, or equivalently if the symmetric difference of two different circuits of $\mathcal{M}$ contains a circuit. Graphic (and cographic) matroids are important examples of binary matroids. We prove also that in the case of binary matroids the reverse is also true. The proofs use only simple algebraic arguments and basic matroid theory.

2 ℓ-adic OS algebras

Let $G = (V; S)$ be a connected simple graph and suppose the vertices [resp. edges] of G labelled with the integers $1, \ldots, d$, [resp. $1, \ldots, n$,] i.e., $V = [d]$ and

$S = [n]$. Consider the hyperplane arrangement in $\mathbb{K}^d$ given by

$$\mathcal{A}_{\mathbb{K}} := \{\mathrm{Ker}(x_i - x_j) : \ \{i,j\} \in S\}.$$

We say that $\mathcal{A}_{\mathbb{K}}$ is the *graphic arrangement* determined by G. It is clear that the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ on the ground set $[n]$ is the cycle matroid of the graph G. We recall some of our notations. Let $\mathfrak{X}$ be a subset of $2^{[n]}$. Let $\Im(\mathfrak{X}) = \big\langle \partial(e_X) : X \in \mathfrak{X} \big\rangle$ be the two-side ideal of the Grassmann $\mathbb{K}$-algebra $\mathcal{E}$ generated by the elements $\{\partial(e_X) : X \in \mathfrak{X}\}$. We recall that the differential of degree -1, $\partial : \mathcal{E} \to \mathcal{E}$, can be defined by the equalities

$$\partial(e_i) = 1 \quad \text{for every} \quad e_i,\ i = 1, \ldots, n \tag{2.1}$$

and the *Leibniz rule*

$$\partial(a \wedge b) = \partial(a) \wedge b + (-1)^{\deg(a)} a \wedge \partial(b) \tag{2.2}$$

for every pair of homogeneous elements $a, b \in \mathcal{E}$. From Equations (2.1) and (2.2) we conclude that

$$\partial^2(e) = 0 \quad \text{for every element} \quad e \in \mathcal{E}. \tag{2.3}$$

We make use of the following technical (but fundamental) lemma.

Lemma 2.1 *Consider two subsets X and X' of $[n]$ of at least two elements and an element $i_\alpha \in X$. Let $\mathfrak{X}$ be a subset of $2^{[n]}$. Then the following properties hold:*

(2.1.1) $\partial(e_X) \in \Im(\mathfrak{X}) \iff e_X \in \Im(\mathfrak{X})$.

(2.1.2) $\big(e_{X \setminus i_\ell} \in \Im(\mathfrak{X}),\ \forall i_\ell \in X \setminus i_\alpha\big) \implies e_{X \setminus i_\alpha} \in \Im(\mathfrak{X})$.

(2.1.3) *If* $X \cap X' = i_\alpha$, *then* $e_{X \Delta X'} \in \Im(\{X, X'\})$.

Proof: (2.1.1) Note that

$$\partial(e_X) \in \Im(\mathfrak{X}) \implies e_{i_\alpha} \wedge \partial(e_X) = \pm e_X \in \Im(\mathfrak{X}).$$

Conversely suppose that

$$e_X = \sum_i y_i \wedge \partial(e_{Y_i}), \text{ where } y_i \in \mathcal{E}, \text{ and } Y_i \in \mathfrak{X}.$$

From Equations (2.1), (2.2) and (2.3) we conclude that

$$\partial(e_X) = \sum_i \partial(y_i) \wedge \partial(e_{Y_i}) \in \Im(\mathfrak{X}).$$

(2.1.2) Choose an element $i_\ell \in X \backslash i_\alpha$. By hypothesis we know that $e_{X\backslash i_\ell} \in \Im(\mathfrak{X})$ and so $e_X = \pm e_{i_\ell} \wedge e_{X\backslash i_\ell} \in \Im(\mathfrak{X})$. From (2.1.1) we know that $\partial(e_X) \in \Im(\mathfrak{X})$ and so

$$e_{X\backslash i_\alpha} = \pm\partial(e_X) + \sum_{i_\ell \neq i_\alpha} \pm e_{X\backslash i_\ell} \in \Im(\mathfrak{X}).$$

(2.1.3) We know that $|X \cup X'| \geq 3$. It is clear that $e_{X\cup X'\backslash i_j} \in \Im(\{X, X'\})$, for all $i_j \neq i_\alpha$. From (2.1.2) we conclude that $e_{X\cup X'\backslash i_\alpha} \in \Im(\{X, X'\})$. By hypothesis we know that $X \cap X' = i_\alpha$ so $X\Delta X' = X \cup X' \setminus i_\alpha$.

□

Definition 2.2 A set $\mathfrak{X} \subseteq 2^{[n]}$ is said to be Δ-*closed* if, for all subsets $X, X' \subseteq [n]$, we have:

(2.2.1) $\big(X \subseteq X'$ and $X \in \mathfrak{X}\big) \implies X' \in \mathfrak{X}$,

(2.2.2) $(X \cap X' = i_\alpha,\ |X| \geq 2,\ |X'| \geq 2$ and $X, X' \in \mathfrak{X}) \implies X\Delta X' \in \mathfrak{X}$.

Given a subset $\mathfrak{Z} \subseteq 2^{[n]}$ we will note $c\ell_\Delta(\mathfrak{Z})$ the smallest Δ-closed subset of $2^{[n]}$ containing $\mathfrak{Z}$.

Proposition 2.3 *$\mathcal{M}$ is ℓ-chordal then $\mathfrak{C} \subseteq c\ell_\Delta(\mathfrak{C}_{\ell-1})$.*

Proof: As every circuit with ℓ or more elements has a chord the result is clear by induction on the number of elements of the circuits.

□

As a consequence of Lemma 2.1 and Definition 2.2 we get:

Proposition 2.4 *For every subset $\mathfrak{C}' \subseteq \mathfrak{C}(\mathcal{M})$, we have $\Im(\mathfrak{C}') = \Im(c\ell_\Delta(\mathfrak{C}'))$.*

The following Corollary extends Proposition 3.2 in [1] (case $\ell = 4$ and $\mathrm{rk}(\mathcal{M}) = 3$). It is a consequence of Propositions 2.2 and 2.3.

Corollary 2.5 *If the matroid $\mathcal{M}$ is ℓ-chordal then the associated* OS *algebra* $\mathrm{OS}(\mathcal{M})$ *is* $(\ell-2)$*-adic.*

Proposition 2.6 *Let $\mathcal{M}$ be a binary matroid. For every circuit C of $\mathcal{M}$ with k elements, $k \geq 4$, the following statements are equivalent:*

(2.6.1) *C has a chord,*

(2.6.2) $\partial(e_C) \in \Im(\mathfrak{C}_{k-1})$,

(2.6.3) $e_C \in \Im(\mathfrak{C}_{k-1})$.

Proof: (2.6.2) $\iff$ (2.6.3) is a consequence of (2.1.1).
(2.6.1) $\Longrightarrow$ (2.6.3). Suppose that C has a chord i_α and let C_1, C_2 be two circuits such that $C_1 \cap C_2 = i_\alpha$ and $C_1 \Delta C_2 = C$. From the equivalence (2.1.1) $\iff$ (2.1.2) we know that $e_{C_1}, e_{C_2} \in \Im(\mathfrak{C}_{k-1})$. It follows from (2.1.3) that $e_{C\setminus i_\alpha} \in \Im(\mathfrak{C}_{k-1})$.
(2.6.3) $\Longrightarrow$ (2.6.1). We will prove it by contradiction. Let C be a circuit without a chord and suppose that $e_C \in \Im(\mathfrak{C}_{k-1})$. From the definitions we know that

$$e_C = \sum_j a_j \wedge \partial(e_{C_j}),\ a_j \in \mathcal{E},\ C_j \in \mathfrak{C}_{k-1}.$$

For this to be possible, there must exist a circuit C_{j_ℓ} such that $|C \cap C_{j_\ell}| = |C_{j_\ell}| - 1$. Set $x = C_{j_\ell} \setminus C$. As $\mathcal{M}$ is binary we know that $C' = C \Delta C_{j_\ell} = C \setminus C_{j_\ell} \cup x$ is also a circuit. We conclude that $C' \cap C_{j_\ell} = x$ and $C = C' \Delta C_{j_\ell}$ and so x is a chord of C, a contradiction.

□

Corollary 2.7 *Let $\mathcal{M}$ be a binary matroid and ℓ, $\ell \geq 4$, a natural number. Then the following statements are equivalent:*

(2.7.1) *The matroid $\mathcal{M}$ is ℓ-chordal.*

(2.7.2) *The algebra* $\mathrm{OS}(\mathcal{M})$ *is* $(\ell-2)$*-adic.*

In particular $\mathcal{M}$ is chordal iff the algebra $\mathrm{OS}(\mathcal{M})$ *is quadratic.*

Proof: (2.7.1) $\Longrightarrow$ (2.7.2) is a special case of Corollary 2.5.
(2.7.2) $\Longrightarrow$ (2.7.1) From the definitions we know that for all k, $k \geq \ell$, $\Im(\mathfrak{C}_{k-1}) = \Im(\mathfrak{C}_{\ell-1}) = \Im(\mathfrak{C})$. We conclude from the Proposition 2.6 that every circuit with k, $k \geq \ell$, elements has a chord, i.e., $\mathcal{M}$ is ℓ-chordal.

□

Corollary 2.5 gives a condition which makes a matroid quadratic and we have shown in Corollary 2.7 that this condition is also necessary in the case of binary matroids. It is known that this condition is not necessary in general, see [3]. Close related with our results we propose a weaker sufficient condition for quadraticity.

Definition 2.8 A set $\mathfrak{X} \subseteq 2^{[n]}$ is said to be Δ'-*closed* if, for all subsets $X, X' \subseteq [n]$ and $i_\alpha \in [n]$ we have:

(S_1) $\big(X \subseteq X'$ and $X \in \mathfrak{X}\big) \Longrightarrow X' \in \mathfrak{X}$,

(S_3) $(X \setminus i_\ell \in \mathfrak{X},\ \forall i_\ell \in X \setminus i_\alpha) \Longrightarrow X \setminus i_\alpha \in \mathfrak{X}$.

Given a subset $\mathfrak{Z} \subseteq 2^{[n]}$ we will note $c\ell_{\Delta'}(\mathfrak{Z})$ the smallest Δ'-closed subset of $2^{[n]}$ containing $\mathfrak{Z}$.

Proposition 2.9 *For every subset $\mathfrak{C}' \subseteq \mathfrak{C}(\mathcal{M})$, we have $c\ell_{\Delta}(\mathfrak{C}') \subseteq c\ell_{\Delta'}(\mathfrak{C}')$ and $\Im(\mathfrak{C}') = \Im(c\ell_{\Delta'}(\mathfrak{C}'))$. In particular if $\mathfrak{C} \subseteq c\ell_{\Delta'}(\mathfrak{C}_3)$ then the algebra* $\mathrm{OS}(\mathcal{M})$ *is quadratic.*

Proof: Remark that Conditions (2.2.1) and (2.2.3) imply Condition (2.2.2). So the result is a consequence of Lemma 2.1 and Definition 2.8.

□

The following example is a case where $\mathfrak{C} \not\subseteq c\ell_{\Delta}(\mathfrak{C}_3)$ but $\mathfrak{C} \subseteq c\ell_{\Delta'}(\mathfrak{C}_3)$. So $\mathrm{OS}(\mathcal{M})$ is quadratic from Proposition 2.9. We know no example where $\mathfrak{C} \not\subseteq c\ell_{\Delta'}(\mathfrak{C}_3)$ and $\mathrm{OS}(\mathcal{M})$ is quadratic.

Example 2.10 Consider the rank 3 (simple) matroid $\mathcal{M}$ of Figure 1 on 7 elements whose circuits of 3 elements are 123, 145, 167, 246 and 357. It is easy to see that $\mathfrak{C} \subseteq c\ell_{\Delta'}(\mathfrak{C}_3)$ (i.e., OS($\mathcal{M}$) is quadratic) but $\mathfrak{C} \not\subseteq c\ell_{\Delta}(\mathfrak{C}_3)$ (i.e., the matroid is not chordal). For example we can check that the circuit 2356 is not in $c\ell_{\Delta}(\mathfrak{C}_3)$ but is in $c\ell_{\Delta'}(\mathfrak{C}_3)$.

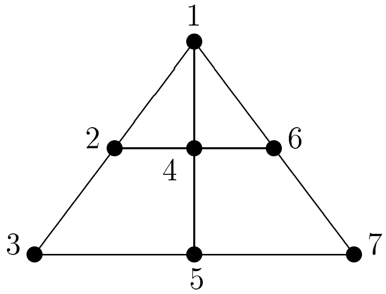

Figure 1

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