HIGH FREQUENCY RESOLVENT ESTIMATES AND ENERGY DECAY OF SOLUTIONS TO THE WAVE EQUATION

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Let (M, g) be an n-dimensional non-compact, connected Riemannian manifold with a Riemannian metric g of class $C^\infty(\overline{M})$ and a compact C^∞-smooth boundary ∂M (which may be empty), of the form $M = X_0 \cup X$, where X_0 is a compact, connected Riemannian manifold with a metric $g_{|X_0}$ of class $C^\infty(\overline{X}_0)$ with a compact boundary $\partial X_0 = \partial M \cup \partial X$, $\partial M \cap \partial X = \emptyset$, $X = [r_0, +\infty) \times S$, $r_0 \gg 1$, with metric $g_{|X} := dr^2 + \sigma(r)$. Here $(S, \sigma(r))$ is an $n-1$ dimensional compact Riemannian manifold without boundary equipped with a family of Riemannian metrics $\sigma(r)$ depending smoothly on r which can be written in any local coordinates $\theta \in S$ in the form

$$\sigma(r) = \sum_{i,j} g_{ij}(r,\theta) d\theta_i d\theta_j, \quad g_{ij} \in C^\infty(X).$$

Denote $X_r = [r, +\infty) \times S$. Clearly, ∂X_r can be identified with the Riemannian manifold $(S, \sigma(r))$ with the Laplace-Beltrami operator $\Delta_{\partial X_r}$ written as follows

$$\Delta_{\partial X_r} = -p^{-1} \sum_{i,j} \partial_{\theta_i}(p g^{ij} \partial_{\theta_j}),$$

where (g^{ij}) is the inverse matrix to (g_{ij}) and $p = (\det(g_{ij}))^{1/2} = (\det(g^{ij}))^{-1/2}$. Let Δ_g denote the Laplace-Beltrami operator on (M, g). We have

$$\Delta_X := \Delta_g|_X = -p^{-1}\partial_r(p\partial_r) + \Delta_{\partial X_r} = -\partial_r^2 - \frac{p'}{p}\partial_r + \Delta_{\partial X_r},$$

where $p' = \partial p/\partial r$. We have the identity

$$\Delta_X^\sharp := p^{1/2}\Delta_X p^{-1/2} = -\partial_r^2 + \Lambda_r + q(r,\theta), \tag{1}$$

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where

$$\Lambda_r = -\sum_{i,j} \partial_{\theta_i}(g^{ij}\partial_{\theta_j}),$$

and q is an effective potential given by

$$q(r,\theta) = (2p)^{-2}\left(\frac{\partial p}{\partial r}\right)^2 + (2p)^{-2}\sum_{i,j}\frac{\partial p}{\partial \theta_i}\frac{\partial p}{\partial \theta_j}g^{ij} + 2^{-1}p\Delta_X(p^{-1}).$$

We suppose that $q = q_1 + q_2$, where q_1 and q_2 are real-valued functions satisfying

$$|q_1(r,\theta)| \le C, \quad \frac{\partial q_1}{\partial r}(r,\theta) \le Cr^{-1-\delta_0}, \quad |q_2(r,\theta)| \le Cr^{-1-\delta_0}, \tag{2}$$

with constants $C, \delta_0 > 0$. Denote

$$h(r,\theta,\xi) = \sum_{i,j} g^{ij}(r,\theta)\xi_i\xi_j, \quad (\theta,\xi) \in T^*S.$$

We also suppose that

$$-\frac{\partial h}{\partial r}(r,\theta,\xi) \ge \frac{C}{r}h(r,\theta,\xi), \quad \forall(\theta,\xi) \in T^*S, \tag{3}$$

with a constant $C > 0$. Note that these assumptions are fulfilled for both asymptotically Euclidean and asymptotically hyperbolic manifolds. Denote by G the selfadjoint realization of Δ_g on the Hilbert space $H = L^2(M, d\mathrm{Vol}_g)$ with Dirichlet or Neumann boundary conditions, $Bu = 0$, on ∂M. Given a real $s > 1/2$, choose a real-valued function $\chi_s \in C^\infty(\overline{M})$, $\chi_s = 1$ on $M \setminus X_{r_0+1}$, $\chi_s = r^{-s}$ on X_{r_0+2}. Also, given $a > r_0$ choose a real-valued function $\eta_a \in C^\infty(\overline{M})$, $\eta_a = 0$ on $M \setminus X_a$, $\eta_a = 1$ on X_{a+1}. The following resolvent estimates are proved in [3].

Theorem 1. *Assume (2) and (3) fulfilled. Then, for every $s > 1/2$ there exist constants $C_0, C, C' > 0$ and $a > r_0$ such that for $z \ge C_0$, $0 < \varepsilon \le 1$, we have the estimates*

$$\|\chi_s(G - z \pm i\varepsilon)^{-1}\chi_s\|_{\mathcal{L}(H)} \le e^{Cz^{1/2}}, \tag{4}$$

$$\|\eta_a\chi_s(G - z \pm i\varepsilon)^{-1}\chi_s\eta_a\|_{\mathcal{L}(H)} \le C'z^{-1/2}. \tag{5}$$

Note that such estimates have first been proved by Burq [1], [2] for a class of perturbations of the Euclidean Laplacian. This kind of high-frequency resolvent estimates are very important for the study of the local energy decay of the solutions of the mixed problem for the wave equation

$$\begin{cases} (\partial_t^2 + \Delta_g)u(t,x) = 0 \quad \text{in} \quad \mathbf{R} \times M, \\ Bu(t,x) = 0 \quad \text{on} \quad \mathbf{R} \times \partial M, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x), \quad x \in M. \end{cases} \tag{6}$$

Recall that the solutions to (6) can be expressed by the formula

$$u = \cos\left(t\sqrt{G}\right) f_1 + \frac{\sin\left(t\sqrt{G}\right)}{\sqrt{G}} f_2. \tag{7}$$

It is not hard to deduce from (5) the following

Corollary 2. *Assume (2) and (3) fulfilled, and let the functions χ_s and η_a be as in Theorem 1. Then, $\forall f \in H$, the following inequality holds:*

$$\int_0^\infty \left\| \eta_a \chi_s \cos\left(t\sqrt{G}\right) \phi(G) \chi_s \eta_a f \right\|_H^2 dt \le C \|f\|_H^2, \tag{8}$$

with a constant $C > 0$ independent of f, where $\phi \in C^\infty(\mathbf{R})$, $\phi(\sigma) = 0$ for $\sigma \le C_0 + \varepsilon_0$, $\phi(\sigma) = 1$ for $\sigma \ge C_0 + 2\varepsilon_0$, $0 < \varepsilon_0 \ll 1$, C_0 being as in Theorem 1.

Remark 1. If the metric g is non-trapping (i.e. every generalized geodesics leaves any compact in a finite time), then the estimate (5) holds with $\eta_a \equiv 1$ (see Theorem 1.1 in [7]). Therefore, in this case (8) holds with $\eta_a \equiv 1$.

Denote by G_0 the Dirichlet selfadjoint realization of Δ_X on $H_0 = L^2(X, d\mathrm{Vol}_g)$. Suppose that there exist constants $s > 1/2$, $C > 0$, $\beta \ge -1$ and $0 < \mu \le 1$ so that the following estimate holds, $\forall\, 0 < \varepsilon \le 1$,

$$\left\| r^{-s}(G_0 - z \pm i\varepsilon z^{1/2})^{-2} r^{-s} \right\|_{\mathcal{L}(H_0)} \le C z^\beta \varepsilon^{-1+\mu}. \tag{9}$$

This assumption allows to improve Theorem 1 above showing that the weitghed resolvent is a Hölder function. We first have the following

Proposition 3. *Assume (2) and (3) fulfilled. Then, for every $s > 1/2$, there exist constants $C, C_0 > 0$ so that for $z \geq C_0$, $0 < \varepsilon \leq 1$, we have*

$$\left\| r^{-s}(G_0 - z \pm i\varepsilon)^{-1} r^{-s} \right\|_{\mathcal{L}(H_0)} \leq C z^{-1/2}. \tag{10}$$

Assume additionally (9) fulfilled. Then, for every $s > 1/2$, $z \geq C_0$, the limit

$$R^{\pm}_{s,0}(z) := \lim_{\varepsilon \to 0^+} r^{-s}(G_0 - z \pm i\varepsilon)^{-1} r^{-s} : H_0 \to H_0$$

exists and satisfies the estimate, for $C_0 \leq z_1 \leq z$, $C_0 \leq z_2 \leq z$,

$$\| R^{\pm}_{s,0}(z_2) - R^{\pm}_{s,0}(z_1) \|_{\mathcal{L}(H_0)} \leq C' |z_2 - z_1|^{\mu'} z^{\beta'}, \tag{11}$$

with a constant $C' > 0$, where $0 < \mu' < 1$ depends on s and μ, while $\beta' \geq 0$ depends on s and β.

Using this proposition together with Theorem 1, one can prove the following theorem (see[4]).

Theorem 4. *Assume (2), (3) and (9) fulfilled. Then, for every $s > 1/2$, there exist constants $a > r_0$ and $C_0, C, C' > 0$ so that for $z \geq C_0$, the limit*

$$R^{\pm}_{s}(z) := \lim_{\varepsilon \to 0^+} \chi_s (G - z \pm i\varepsilon)^{-1} \chi_s : H \to H$$

exists and satisfies the estimates, for $C_0 \leq z_1 \leq z$, $C_0 \leq z_2 \leq z$,

$$\| R^{\pm}_{s}(z_2) - R^{\pm}_{s}(z_1) \|_{\mathcal{L}(H)} \leq C' |z_2 - z_1|^{\mu'} e^{C z^{1/2}}, \tag{12}$$

$$\| \eta_a R^{\pm}_{s}(z_2) \eta_a - \eta_a R^{\pm}_{s}(z_1) \eta_a \|_{\mathcal{L}(H)} \leq C' |z_2 - z_1| e^{C z^{1/2}} + C' |z_2 - z_1|^{\mu'} z^{\beta'}, \tag{13}$$

where μ' and β' are as in Proposition 3.

The estimate (9) can be easily proved in the case of asymptotically Euclidean manifolds. More precisely, instead of the assumptions (2) and (3), we make the following more restrictive assumptions:

$$\left|\frac{\partial^k q}{\partial r^k}(r,\theta)\right| \le C r^{-k-\delta}, \quad k = 0, 1, \tag{14}$$

with constants $C > 0$, $0 < \delta \le 1$. Set $g_\flat^{ij} := r^2 g^{ij}$ and denote

$$h^\flat(r,\theta,\xi) = \sum_{i,j} g_\flat^{ij}(r,\theta)\xi_i\xi_j, \quad (\theta,\xi) \in T^*S.$$

We suppose that

$$\left|\frac{\partial h^\flat}{\partial r}(r,\theta,\xi)\right| \le C r^{-1-\delta} h^\flat(r,\theta,\xi), \quad \forall(\theta,\xi) \in T^*S, \tag{15}$$

with constants $C > 0$, $0 < \delta \le 1$. The following proposition is proved in [4].

Proposition 5. *Assume (14) and (15) fulfilled. Then (9) holds with $\beta = 0$, $s = 1 + \delta/2, \forall\ 0 < \mu < \delta$, and C_0 being as in Proposition 3.*

Remark 2. One can improve the above result if the assumption (15) is replaced by a little bit stronger one (see Proposition 1.4 of [7]). To describe it, denote

$$\Lambda_r^\flat = -r^{-1} \sum_{i,j} \partial_{\theta_i} \left(\frac{\partial g_\flat^{ij}}{\partial r} \partial_{\theta_j} \right).$$

Let $G_0^\sharp$ be the self-adjoint Dirichlet realization of the operator $\Delta_X^\sharp$ (defined in (1)) on the Hilbert space $H_0^\sharp = L^2(X, drd\theta)$. Instead of (15), suppose that

$$r^\delta \Lambda_r^\flat (G_0^\sharp - i)^{-1} \in \mathcal{L}(H_0^\sharp). \tag{16}$$

Then, Proposition 5 holds with $\beta = -1$.

Note that the proof of Proposition 5 is based on the estimate (10) and the fact that, because of (14) and (15), the operator $2\Delta_X^\sharp + [r\partial_r, \Delta_X^\sharp]$ is $O(r^{-\delta})$ for $r \gg 1$.

One can use Theorem 4 to extend a result by Burq [1] on the behaviour of the local energy of the solutions of (6). The following theorem is proved in [4].

Theorem 6. *Under the assumptions of Theorem 4, for every $s > 1/2$, and $\forall m > 0$, the following estimate holds for $t \gg 1$:*

$$\|\chi_s \cos\left(t\sqrt{G}\right)\psi(G)(G+1)^{-m/2}\chi_s\|_{\mathcal{L}(H)} \le C_{m,s}(\log t)^{-m}, \tag{17}$$

with a constant $C_{m,s} > 0$, where ψ denotes the characteristic function of the interval $[C_0 + \varepsilon_0, +\infty)$, $0 < \varepsilon_0 \ll 1$.

Remark 3. It follows easily by an interpolation argument that we have analogues of (17) for $0 < s \le 1/2$ as well, but with $O_\epsilon\left((\log t)^{-m(2s)^2+\epsilon}\right)$, $\forall 0 < \epsilon \ll 1$, in place of $(\log t)^{-m}$.

Remark 4. Clearly, the above results still hold true for the selfadjoint realization of $\Delta_g + V(x)$, where V is a real-valued potential, $V(x) \ge 0$, provided the assumptions (2) and (14) are satisfied with q replaced by $q + V|_X$.

Remark 5. When the metric g is nontrapping it is shown in [7] that (17) holds with $m = 0$ and $O(t^{-\mu})$ in the RHS, provided the function ψ is taken smooth.

Remark 6. We can take $\psi \equiv 1$ in Theorem 6 if the resolvent satisfies the following estimates:

$$\|\lambda R_s^{\pm}(\lambda^2)\|_{\mathcal{L}(H)} \le C,$$

$$\|\lambda_2 R_s^{\pm}(\lambda_2^2) - \lambda_1 R_s^{\pm}(\lambda_1^2)\|_{\mathcal{L}(H)} \le C|\lambda_2 - \lambda_1|^{\mu},$$

for all $0 < \lambda, \lambda_1, \lambda_2 \le \sqrt{C_0}$, with some constants $C, \mu > 0$, where C_0 is as in Theorem 4.

It is easy to see that a long-range perturbation of the Euclidean metric on $\mathbf{R}^n$, $n \ge 2$, provides an example of an asymptotically Euclidean manifold satisfying the assumptions (14) and (15), and hence the above results apply to.

More precisely, let $\mathcal{O} \subset \mathbf{R}^n$ be a bounded domain with a C^∞-smooth boundary and a connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Let g be a Riemannian metric in Ω of the form

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad g_{ij}(x) \in C^\infty(\overline{\Omega}),$$

satisfying the estimates

$$|\partial_x^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha \langle x \rangle^{-\gamma_0 - |\alpha|}, \tag{18}$$

for every multi-index α, with constants $C_\alpha, \gamma_0 > 0$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and δ_{ij} denotes the Kronecker symbol. It is easy to see that (Ω, g) is isometric to a Riemannian manifold of the form described above satisfying assumptions (14) (with $\delta = 2$) and (15) (with $\delta = \gamma_0$) because of (18) and the fact that they are satisfied for the Euclidean metric on $\mathbf{R}^n$.

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