SHARP BOUNDS ON THE NUMBER OF RESONANCES FOR CONFORMALLY COMPACT MANIFOLDS WITH CONSTANT NEGATIVE CURVATURE NEAR INFINITY

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A resonance is a complex number $\lambda \in \mathbb{C}$ describing a nonstable quantum state oscillating with a frequency $Re\,\lambda$, whose life-time is proportional to $1/Im\,\lambda$. Therefore, the closer a resonance is to the real axis (that is, the smaller its imaginary part is), the longer it lives, and hence the more interesting it is from the physical point of view. In the physical experiments the real parts of the resonances are observed as the points at which the first derivative, $s'(\lambda)$, of the phase $s(\lambda)$, $\lambda \in \mathbb{R}$, of the scattering matrix has peacks. The knowledge of the resonances near the real axis also enables us to deduce important information about the decay of the local energy of the solutions of the wave equation.

The systematic study of resonances associated to compactly supported perturbations of the Laplacian on $\mathbb{R}^n$ was initiated by Lax and Phillips. They have constructed a mathematical theory for the wave equation outside a compact obstacle as well as for the wave equation with compactly supported potential. In the Lax e Phillips's book [5] the resonances are defined as the poles of the meromorphic continuation of resolvent (acting on suitable spaces).

The goal of this work is to prove a sharp bound on the number of resonances for the Laplacian on conformally compact manifolds with constant negative curvature near infinite. Recall that a compact n-dimensional manifold (X, g) is called conformally compact with constant negative curvature near infinity if and only if $g = \rho^{-2}h$ where:

(i) ρ is a C^∞-function on $\overline{X}$ such that $\rho|\partial X = 0,\ d\rho|_{\partial X} \neq 0,\ \rho > 0$ in X;

(ii) h is a Riemannian metrix on X of class $C^\infty(\overline{X})$;

(iii) All sectional curvatures of g are equal to -1 in some neighborhood of the boundary of X.

Denote by Δ_X the Laplace-Beltrami operator on (X, g). The resolvent operator is defined by

$$R(s) = (\Delta_X - s(n-1-s))^{-1} : L^2(X) \longrightarrow L^2(X), \quad \text{for} \quad Re\, s \gg 1,$$

where $L^2(X) = L^2(X, dVol_g)$. The resolvent operator from $L^2_{comp}(X)$ into $L^2_{loc}(X)$ extends meromorphically to the whole complex plane. This fact was proved by Mazzeo and Melrose [6] for a larger class of conformally compact complete manifolds. The poles of this continuation are called resonances. We emphasize that the resolvent extends meromorphically, but not as an operator from L^2 to L^2, only between suitable weighted spaces.

Denote by $\mathcal{R}_X$ the set of all resonances repeated according to the multiplicity, and let $N_X(r)$ be the number of resonances in a disc of radius r with $r > 1$.

Guillopé and Zworski [2] prove that $N_X(r) = \mathcal{O}(r^{n+1})$. As they noted this bound is not optimal and the expected power is n. Indeed, in the case of $n = 2$ they obtained a better bound $N_X(r) = \mathcal{O}(r^2)$ (see [3]) as well as a lower bound $N_X(r) \geq r^2/C,\ C > 0$, under additional assumption (see [4]).

Our main result is the following:

Theorem 1.1 *For any conformally compact manifold (X, g) with constant negative curvature near infinity, the following upper bound holds:*

$$N_X(r) \leq Cr^n \tag{1.1}$$

with a constant $C > 0$.

Thus we improve the polynomial bound of Guillopé and Zworski [2].

Note that such a bound was proved by Patterson and Perry in [7] for a class of quotients by hyperbolic spaces in the even dimensional case via the properties of the dynamical zeta functions. Perry [8] has recently obtained sharp lower bound of the form $N_X(r) \geq r^n/C$ for such quotients.

The bound (1.1) follows from the following upper bounds:

Proposition 1.2 *For each $0 < \varepsilon \ll 1$ there exist a positive constant C_ε such that*

$$\#\left\{s \in \mathcal{R}_X : \frac{r}{2} \leq |s| \leq r,\ s \in \mathbb{C}_\varepsilon\right\} \leq C_\varepsilon r^n,\ r > 1 \tag{1.2}$$

where

$$\mathbb{C}_\varepsilon = \mathbb{C} \backslash \{s \in \mathbb{C} : \pi - \varepsilon \leq Arg\left(s - \frac{n-1}{2}\right) \leq \pi + \varepsilon\}$$

Proposition 1.3 *For each $0 < \varepsilon \ll 1$ there exist a positive constant $\tilde{C}_\varepsilon$ such that*

$$\#\{s \in \mathcal{R}_X : |s| \leq r, \ s \in \tilde{\mathbb{C}}_\varepsilon\} \leq \tilde{C}_\varepsilon r^n, \ r > 1 \tag{1.3}$$

where

$$\tilde{\mathbb{C}}_\varepsilon := \left\{s \in \mathbb{C} : \frac{\pi}{2} + \varepsilon \leq Arg\left(s - \frac{n-1}{2}\right) \leq \frac{3\pi}{2} - \varepsilon\right\}.$$

The number of resonances in the shadow region is of order $\mathcal{O}_\varepsilon(r^n)$.

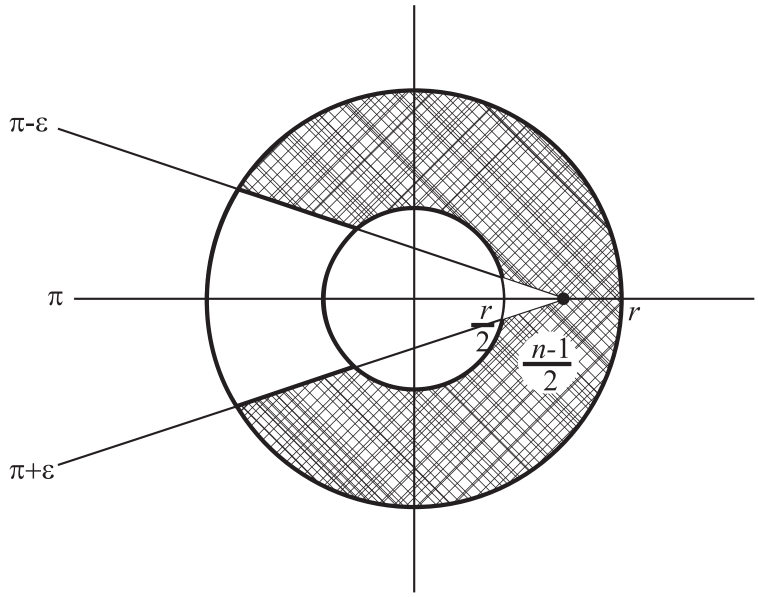

Figure 1. Proposition 1.2

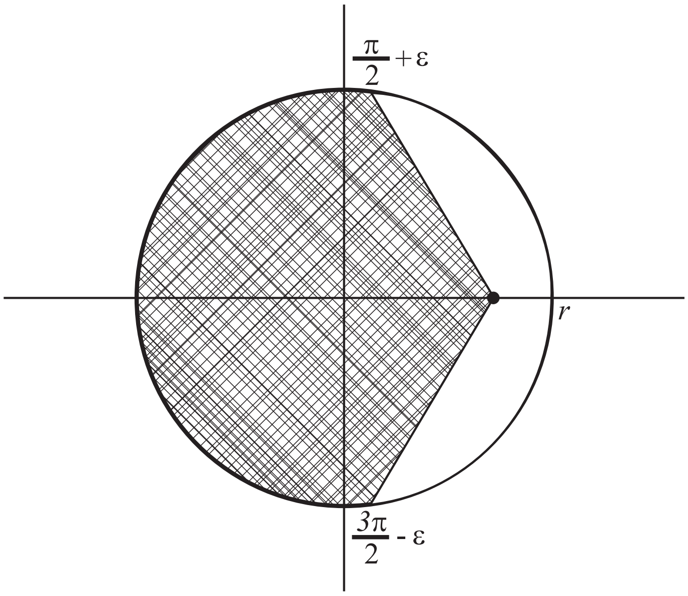

Figure 2. Proposition 1.3

Number of resonances in the Crown

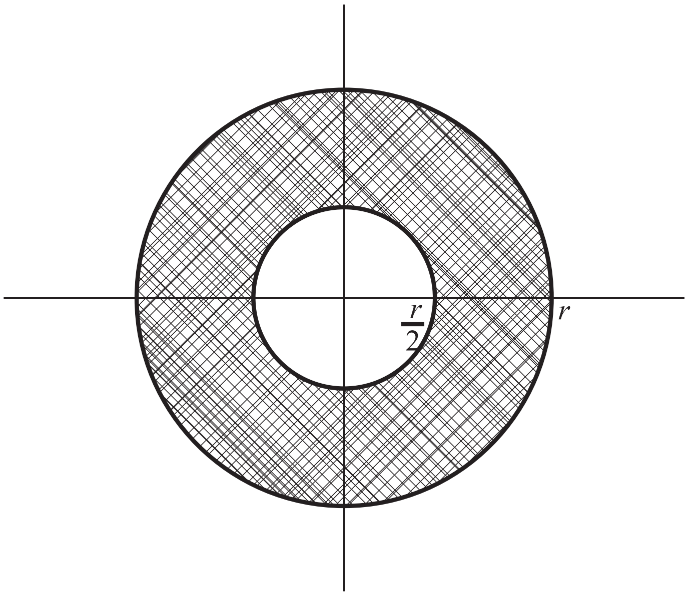

Figure 3.

From Proposition 1.2 and 1.3 we can deduce that

$$\boxed{N_X(r) - N_X\left(\frac{r}{2}\right) \leq Cr^n + C \quad \forall r > 0}$$

with some constant $C > 0$. Using the above estimate we can infer that $N_X(r) \leq Cr^n$.

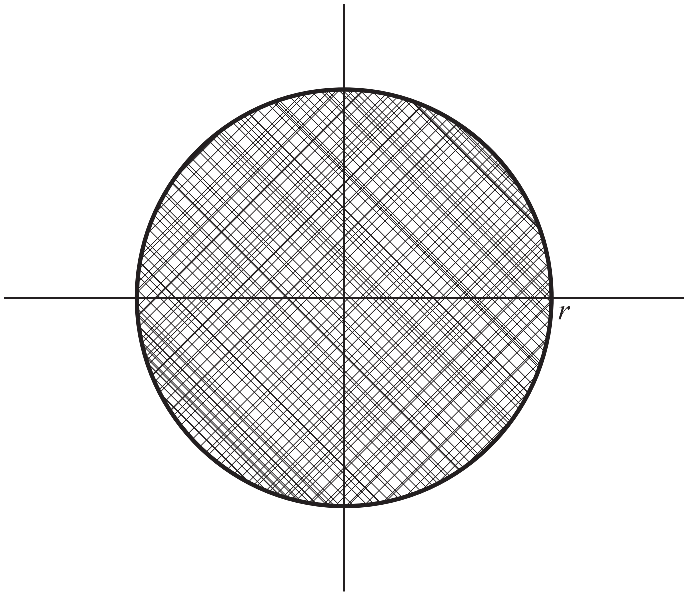

Figure 4.

The proofs of Proposition 1.2 and 1.3 are published in [1]. They are based on Carleman's theorem but we need to proceed differently in each case. To prove Proposition 1.2 we modify the parametrix for the resolvent constructed by Guillopé and Zworski (see [2]), who followed the more general construction of Mazzeo and Melrose ([6]). This modification provides us, given any integer $N \gg 1$, a function $h_N(s) = \mathcal{O}(e^{CN^n})$ analytic in $\{s \in \mathbb{C}_\varepsilon : |s| \leq N\}$, such that the resonances in the region (with multiplicities) are among the zeros of h_N. Our approach is based on the observation that the operator responsible for the non-optimal estimate in Guillopé and Zworski's paper have a factor which is the commutator of the Laplacian with cutoff functions. Then the idea is to take these functions characteristic thus obtaining for the commutator delta densities. This enables us to achieve the dimension reduction needed for obtaining the optimal upper bound for the function h_N. A similar idea has already been used in Vodev [9] to get sharp upper bound on the number of resonances for arbitrary elliptic first order system in $\mathbb{R}^n$. This approach does not allow to count properly the poles in a conical neighborhood of the half line $\{s \in \mathbb{R} : s < (n-1)/2\}$ because the resolvent may have poles on this line coming from the parametrix construction (and from the poles of the free resolvent in the even dimensional case), whose multiplicities are hard to control. That is why we count the resonances in the conical region defined in Proposition 1.3 in a different way, using that they are also poles of the scattering matrix.

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