AN EXTENSION OF THE METHOD OF RAPIDLY OSCILLATING SOLUTIONS

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Abstract

In this work we extend a method devised by D. Henry ([1]) to obtain explicit conditions for some pseudo-differential to be of finite rank. These operators arise as solutions operators for boundary value problems involving the Bilaplacian.

1 Introduction

In his monograph ([1]) dedicated to the study of perturbation of the domain for boundary values problems, D. Henry developed many new tools, including a generalized version of the Transversality Theorem. His version is specially well-suited to the study of 'generic properties' for solutions of boundary value problems, as it allows the consideration of semi-Fredholm operators with index $-\infty$ which often arise in these problems. However, a crucial hypothesis in Henry's version of the Transversality Theorem usually boils down to the verification that a certain (pseudo-differential) operator is not of finite rank. As it is well-known, a pseudo-differential of finite rank must have null symbols of all orders. It would be very convenient to obtain these symbols from the abstract theory of pseudo-differential operators, but such detailed computations do not seem to be available in the literature. To overcome this problem, Henry developed in [1] an alternative method for a class of operators, given by solutions of second order elliptic equations. His method is based on the computation of

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approximate solutions for a special class of boundary data - the 'rapidly oscillating functions'. It is tempting to conjecture that the conditions obtained are exactly the nullity of the symbols but his argument does not depend on this (unproved) fact. The essential point is that the conditions obtained are often in contradiction with other hypotheses present in the problem, thus establishing the needed infinite rank property. Some applications of the method to the proof of generic properties for second order elliptic boundary value problems can be found in [1], [3] and [4].

Our aim here is to extend Henry's method to some elliptic equations with the Bilaplacian as its principal part. In a forthcoming paper, we shall use this extension to prove the solutions of the semilinear problem

$$\begin{cases} \Delta^2 u + f(\cdot, u, \nabla u, \Delta u) = 0 & \text{in } \Omega \\ u = \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

are generically simple, thus extending similar results obtained in [5] and [1] for second order elliptic equations.

Since our results involve rather lenghty computations, we try to give here a general idea of the contents of this paper.

Suppose $a : \mathbb{R}^n \to \mathbb{C}$, $b : \mathbb{R}^n \to \mathbb{C}^n$ and $c : \mathbb{R}^n \to \mathbb{C}$ are smooth functions and consider the differential operator $L = \Delta^2 + a(x)\Delta + b(x) \cdot \nabla + c(x) \quad x \in \mathbb{R}^n$. Let $\mathcal{R}(L)$ and $\mathcal{N}(L)$ denote the range and the kernel of L, considered as an operator from $W^{4,p} \cap W_0^{2,p}(\Omega, \mathbb{C})$ to $L^p(\Omega, \mathbb{C})$.

Let now $\{w_1, ..., w_m\}$ be a basis for a complement of $\mathcal{R}(L)$ and $\{\phi_1, ..., \phi_m\}$ a basis for $\mathcal{N}(L)$ with associated dual basis $\{\tau_1, ..., \tau_m\}$. Define

$$\mathcal{A}_L : L^p(\Omega, \mathbb{C}) \to W^{4,p} \cap W_0^{1,p}(\Omega, \mathbb{C}) \text{ and} \tag{2}$$

$$\mathcal{C}_L : W^{3-\frac{1}{p},p}(\partial\Omega) \to W^{4,p} \cap W_0^{1,p}(\Omega, \mathbb{C}) \tag{3}$$

by

$$v = \mathcal{A}_L(f) + \mathcal{C}_L(g) \in W^{4,p} \cap W_0^{1,p}(\Omega, \mathbb{C}) \tag{4}$$

where

$$Lv - f \in [w_1, ..., w_m], \tag{5}$$

$$\frac{\partial v}{\partial N} = g \text{ on } \partial\Omega \tag{6}$$

and

$$\int_{\Omega} v\bar{\tau}_i = 0 \text{ for all } 1 \leq i \leq m. \tag{7}$$

The operators of interest in our applications are given in terms of $\mathcal{A}_L$ and $\mathcal{C}_L$. For instance, in the proof of simplicity for solutions of (1) we encounter the operator

$$\begin{aligned} \Upsilon(\dot{h}) \;=\; & \Big\{\dot{h}\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v) - \Delta v\Delta\Big(\mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)\Big) \\ & +\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)\Big)\Big) - \mathcal{C}_{L^*(u)}(\dot{h}\cdot N\Delta v)\Big]\Big\}\Big|_{\partial\Omega} \end{aligned} \tag{8}$$

where $L(u)$ is the linearisation around a solution u of (1), $L^*(u)$ its adjoint, v is a solution of

$$\left\{\begin{array}{ll} L^*(u)v = 0 & \text{in } \Omega \\ v = \frac{\partial v}{\partial N} = 0 & \text{on } \partial\Omega \end{array}\right. \tag{9}$$

and $\Delta u\Delta v|_{\partial\Omega} = 0$. We then compute the approximate value of Υ at special points, the 'rapidly oscillating functions'. More precisely, we show that

$$\Upsilon\Big(\cos(\omega\theta)\Big) = \cos(\omega\theta)\frac{\partial}{\partial N}(\Delta u\Delta v)\Big|_{\partial\Omega} + O(\omega^{-1}) \text{ as } \omega \to +\infty$$

where θ is a smooth real function on $\partial\Omega$ with $|\nabla_{\partial\Omega}\theta| \equiv 1$. If Υ is assumed to have finite rank, then using lemma (1) below (see [1] for a proof), it follows that $\frac{\partial}{\partial N}(\Delta u\Delta v)\Big|_{\partial\Omega} = 0$ implying that u or v must be identically null by uniqueness in the Cauchy problem.

Lemma 1 *Suppose S is a C^1 manifold; A, $B \in L^2(S)$ with compact support; θ is C^1 on S and real valued with $\nabla_S\theta \neq 0$ in $suppA \cup suppB$; E is a finite dimensional subspace of $L^2(S)$ and $u(\omega) \in E$ for all large $\omega \in \mathbb{R}$ satisfying*

$$u(\omega) = A\cos(\omega\theta) + B\sin(\omega\theta) + o(1) \textit{ in } L^2(S)$$

as $\omega \to \infty$. Then $A = 0$, $B = 0$.

To compute approximate values of Υ we need to compute $\mathcal{A}_L$ and $\mathcal{C}_L$ and, therefore, we look for approximate solutions of the boundary value problem

$$\left\{\begin{array}{lcl} Lu & = & f \text{ in } \Omega \\ \frac{\partial u}{\partial N} & = & g \text{ on } \partial\Omega \\ u & = & 0 \text{ on } \partial\Omega \end{array}\right. \tag{10}$$

for ‘rapidly oscillating’ functions f and g.

We now proceed as follows: in the next section we compute formal asymptotic solutions of (10) for ‘rapidly oscillating functions’ f and g. In section (3) we show these solutions are close to real solutions and finally, in section (4), we apply these results to the operator Υ.

2 Formal asymptotic solutions

We seek a formal asymptotic solution $u(x) = e^{\omega S(x)} \sum_{k\geq 0} \frac{U_k(x)}{(2\omega)^k}$ of

$$\left\{\begin{array}{lcl} Lu & = & (2\omega)^2 e^{\omega S} F \text{ in } \Omega \\ \frac{\partial u}{\partial N} & = & e^{\omega i\theta} G \text{ on } \partial\Omega \\ u & = & 0 \text{ on } \partial\Omega \end{array}\right. \tag{11}$$

when $\omega \to +\infty$, where U_k is a complex-valued smooth function, $\Omega \subset \mathbb{R}^n$ is an open, bounded, connected regular region and N is its exterior normal;

$$F(x) = \sum_{k\geq 0} \frac{F_k(x)}{(2\omega)^k}, \quad G(x) = \sum_{k\geq 0} \frac{G_k(x)}{(2\omega)^k}$$

F_k and G_k smooth complex valued; $S|_{\partial\Omega} = i\theta$, $Re(\frac{\partial S}{\partial N}) > 0$ with $\theta : \partial\Omega \to \mathbb{R}$ smooth and $|\nabla_{\partial\Omega}\theta| = 1$ in the region of interest. Note that there exists a neighborhood V of $\partial\Omega$ such that $Re(S) < 0$ in $V \cap \Omega$ and, therefore, u and $(2\omega)^2 e^{\omega S} F$ tend very fast to 0 in the interior of Ω as $\omega \to +\infty$ (except possibly at points in or close to $\partial\Omega$). Since $u|_{\partial\Omega} = 0$, we have $U_k|_{\partial\Omega} = 0$ for all $k \geq 0$

and, therefore

$$\begin{aligned}\frac{\partial u}{\partial N}\Big|_{\partial\Omega} &= \frac{\partial}{\partial N}\Big(e^{\omega S}\sum_{k\geq 0}\frac{U_k}{(2\omega)^k}\Big)\Big|_{\partial\Omega}\\ &= e^{\omega S}\Big(\omega\sum_{k\geq 0}\frac{U_k}{(2\omega)^k}+\sum_{k\geq 0}\frac{\frac{\partial U_k}{\partial N}}{(2\omega)^k}\Big)\Big|_{\partial\Omega}\\ &= e^{\omega i\theta}\sum_{k\geq 0}\frac{\frac{\partial U_k}{\partial N}}{(2\omega)^k}\text{ in }\partial\Omega.\end{aligned}$$

We also have

$$\begin{aligned}&\Delta^2 u = e^{\omega S}\Big\{\Big[\omega^4(\nabla S\cdot\nabla S)^2+2\omega^3\Big((\nabla S\cdot\nabla S)\Delta S+\nabla S\cdot\nabla(\nabla S\cdot\nabla S)\Big)\\ &+4\omega^3(\nabla S\cdot\nabla S)\nabla S\cdot\nabla+2\omega^2\Big(\nabla(\nabla S\cdot\nabla S)+\nabla S\cdot\nabla S\Delta+\frac{1}{2}\Delta(\nabla S\cdot\nabla S)\Big)\Big]u\\ &+\sum_{k\geq 0}\Big[(2\omega)^{2-k}\Big[\Big(\frac{1}{4}(\Delta S)^2+\frac{1}{2}\nabla S\cdot\nabla(\Delta S)+\Delta S\nabla S\cdot\nabla\Big)U_k+\nabla S\cdot\nabla(\nabla S\cdot\nabla U_k)\Big]\\ &+(2\omega)^{1-k}\Big(\frac{1}{2}\Delta^2 S+\Delta S\Delta+\nabla(\Delta S)\cdot\nabla+\nabla S\cdot\nabla\Delta\Big)U_k\\ &+(2\omega)^{1-k}\Delta(\nabla S\cdot\nabla U_k)+(2\omega)^{-k}\Delta^2 U_k\Big]\Big\}\end{aligned}$$

$$\begin{aligned}a\Delta u &= ae^{\omega S}\Big[\sum_{k\geq 0}(2\omega)^{-k}\Delta U_k+\sum_{k\geq 0}(2\omega)^{1-k}\Big(\nabla S\cdot\nabla U_k+\Delta S U_k\Big)\\ &\quad+\frac{1}{4}\sum_{k\geq 0}(2\omega)^{2-k}(\nabla S\cdot\nabla S)U_k\Big]\end{aligned}$$

$$b\cdot\nabla u = e^{\omega S}\Big(\frac{b\cdot\nabla S}{2}\sum_{k\geq 0}\frac{U_k}{(2\omega)^{k-1}}+\sum_{k\geq 0}\frac{b\cdot\nabla U_k}{(2\omega)^k}\Big).$$

Substitution in (11) then gives

$$U_k = 0,\qquad \frac{\partial U_k}{\partial N} = G_k$$

on $\partial\Omega$ for all $k \geq 0$ and

$$\begin{aligned} 0 &= Lu - (2\omega)^2 e^{\omega S} F \\ &= e^{\omega S}\Big\{\Big[\omega^4(\nabla S \cdot \nabla S)^2 + \\ &\quad +4\omega^3\Big(\frac{1}{2}(\nabla S \cdot \nabla S)\Delta S + \frac{1}{2}\nabla S \cdot \nabla(\nabla S \cdot \nabla S) + (\nabla S \cdot \nabla S)\nabla S \cdot \nabla\Big) \\ &\quad +2\omega^2\Big(\nabla(\nabla S \cdot \nabla S) \cdot \nabla + \nabla S \cdot \nabla S \Delta + \frac{1}{2}\Delta(\nabla S \cdot \nabla S) + \frac{1}{2}\nabla S \cdot \nabla S\Big)\Big] \\ &\qquad \sum_{k\geq 0} \frac{U_k}{(2\omega)^k} + \sum_{k \geq 0}(2\omega)^{2-k}\Big[\Lambda U_k + \Gamma U_{k-1} + LU_{k-2} - F_k\Big]\Big\} \end{aligned}$$

in Ω, where $U_{-1} = U_{-2} \equiv 0$,

$$\Lambda\phi = \frac{1}{4}(\Delta S)^2\phi + \frac{1}{2}\nabla S \cdot \nabla(\Delta S)\phi + \Delta S \nabla S \cdot \nabla\phi + \nabla S \cdot \nabla(\nabla S \cdot \nabla \phi)$$

and

$$\begin{aligned} \Gamma\phi &= \frac{1}{2}\Delta^2 S\phi + \Delta S \Delta\phi + \nabla(\Delta S) \cdot \nabla\phi + \nabla S \cdot \nabla(\Delta\phi) \\ &\quad +\Delta(\nabla S \cdot \nabla\phi) + a\Big(\nabla S \cdot \nabla \phi + \frac{1}{2}\Delta S\phi\Big) + \frac{1}{2}(b \cdot \nabla S)\phi. \end{aligned}$$

Choosing a (complex-valued) S satisfying

$$(\nabla S)^2 = \nabla S \cdot \nabla S = 0 \tag{12}$$

in a neighborhood of $\partial\Omega$ in $\mathbb{R}^n$ we obtain, for all $k \geq 0$

$$\begin{cases} \Lambda U_k + \Gamma U_{k-1} + LU_{k-2} &= F_k \\ \frac{\partial U_k}{\partial N}\big|_{\partial\Omega} &= G_k \\ U_k\big|_{\partial\Omega} &= 0 \end{cases} \tag{13}$$

with $U_{-1} = U_{-2} \equiv 0$.

The computations above are merely formal, but we may find approximate solutions of (11) in a neighborhood of $\partial\Omega$, where Ω is a C^m, $m \geq 2$ region, using the 'normal coordinates' given by $x = y + tN(y)$, where $y \in \partial\Omega$ and $t \in (-r, r)$, with $r > 0$ small.

Writing $\tilde{u}(y,t) = u(y + tN(y))$, we have for u sufficiently smooth in a neighborhood of $\partial\Omega$ that

$$\nabla u(y + tN(y)) = (1 + tK(y))^{-1}\tilde{u}_y(y,t) + \tilde{u}_t(y,t)N(y)$$

and

$$\begin{aligned}\Delta u(y+tN(y)) &= \tilde{u}_{tt}(y,t)) + \lambda_t(t,y)\tilde{u}_t(y,t) \\ &\quad +(1+tK(y))^{-2}\lambda_y(t,y)\cdot\tilde{u}_y(y,t) \\ &\quad +\mathrm{div}_{\partial\Omega}[(1+tK(y))^{-2}\tilde{u}_y(y,t)]. \qquad (14)\end{aligned}$$

where $K = DN$ is the (degenerate) curvature matrix, and $\mathrm{div}_{\partial\Omega}$ is the divergent operator in $\partial\Omega$ (see [1] for details). We don't always distinguish $\tilde{u}$ from u and sometimes write $\frac{\partial u}{\partial N}$ for $\tilde{u}_t$ and $\nabla_{\partial\Omega}u$ for $\tilde{u}_t$.

Writing $\tilde{S}(t,y) = S(x(y,t)) = S(y+tN(y)) = \sum_{k\geq 0}\frac{S_k(y)t^k}{k!}$ we have, in a neighborhood of $\partial\Omega$ $\tilde{S}(t,0) = S(x(y,0)) = S_0(y) = i\theta(y)$ with $Re\Big(\frac{\partial\tilde{S}}{\partial t}(0,y)\Big) = Re\Big(\frac{\partial S}{\partial N}(x(y,0))\Big) > 0$.

Observe that some condition must be imposed on S in order to determine the coefficients $S_k(y)$. The condition (12) chosen above has the advantage of simplifying the computations.

We then have

$$\begin{aligned}\nabla S(x(y,t)) &= (\nabla S)(y+tN(y)) \\ &= \tilde{S}_t(y,t))N(y) + (1+tK(y))^{-1}\tilde{S}_y(y,t)\end{aligned}$$

and

$$(1+tK(y))^{-1} = 1 - tK(y) + t^2K^2(y) - t^3K^3(y) + ...$$

from which we obtain

$$\begin{aligned}0 &= \Big((\nabla S)(y+tN(y))\Big)^2 \\ &= (\nabla_{\partial\Omega}S_0(y))^2 + (S_1(y))^2 \\ &\quad +t\Big(2S_1(y)S_2(y) + 2\nabla_{\partial\Omega}S_0(y)\cdot\nabla_{\partial\Omega}S_1(y) + 2\nabla_{\partial\Omega}S_0(y)\cdot K(y)\nabla_{\partial\Omega}S_0(y)\Big) + ...\end{aligned}$$

Choosing $|\nabla_{\partial\Omega}\theta(y)| \equiv 1$, in the region of interest, we obtain recursively

$$S_1(y) = 1,$$

$$S_2(y) = -\nabla_{\partial\Omega}\theta(y)\cdot K(y)\nabla_{\partial\Omega}\theta(y),$$

and we can compute as many terms as needed. In this way, we obtain

$$S(y+tN(x)) \;=\; i\theta(y)+t-\frac{t^2}{2}q(y)+\frac{t^3}{3!}S_3(y)+\frac{t^4}{4!}S_4(y)+... \tag{15}$$

$$\begin{aligned}\nabla S(y+tN(y)) \;=\;& N+i\nabla_{\partial\Omega}\theta-t\Big(iK\nabla_{\partial\Omega}\theta+qN\Big)\\ &+\frac{t^2}{2}\Big(S_3N-\nabla_{\partial\Omega}q+2iK^2\nabla_{\partial\Omega}\theta\Big)\\ &+\frac{t^3}{3!}\Big(S_4N+\nabla_{\partial\Omega}S_3+3K\nabla_{\partial\Omega}q-6iK^3\nabla_{\partial\Omega}\theta\Big)+O(t^4)\end{aligned} \tag{16}$$

$$\begin{aligned}\nabla S(y+tN(y))\cdot\nabla \;=\;& i\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega}+\frac{\partial}{\partial t}+t\Big(-2iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega}-q\frac{\partial}{\partial t}\Big)\\ &+\frac{t^2}{2!}\Big(6iK^2\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega}-\nabla_{\partial\Omega}q\cdot\nabla_{\partial\Omega}+S_3\frac{\partial}{\partial t}\Big)\\ &+\frac{t^3}{3!}\Big(\nabla_{\partial\Omega}S_3\cdot\nabla_{\partial\Omega}+6K\nabla_{\partial\Omega}q\cdot\nabla_{\partial\Omega}-24iK^3\nabla_{\partial\Omega}\theta\\ &\quad\cdot\nabla_{\partial\Omega}+S_4\frac{\partial}{\partial t}\Big)\\ &+O(t^4)\end{aligned} \tag{17}$$

$$\Delta S(y+tN(y)) \;=\; \alpha(y)+t\beta(y)+\frac{t^2}{2}\rho(y)+\frac{t^3}{3!}\sigma(y)+O(t^4) \tag{18}$$

$$\begin{aligned}\Delta^2 S(y+tN(y)) \;=\;& \rho(y)+H_1(y)\beta(y)+\Delta_{\partial\Omega}\alpha(y)\\ &+t\Big(\sigma(y)+H_1(y)\rho(y)-H_2(y)\beta(y)+\nabla_{\partial\Omega}H_1(y)\cdot\nabla_{\partial\Omega}\alpha(y)\\ &+\Delta_{\partial\Omega}\beta(y)-2\ \mathrm{div}_{\partial\Omega}(K(y)\nabla_{\partial\Omega}\alpha(y))\Big)+O(t^2)\end{aligned} \tag{19}$$

where

- $q(y)=\nabla_{\partial\Omega}\theta(y)\cdot K(y)\nabla_{\partial\Omega}\theta(y)$;
- $\frac{\partial}{\partial\theta}=\nabla_{\partial\Omega}\theta(y)\cdot\nabla_{\partial\Omega}$;
- $S_3(y)=3\nabla_{\partial\Omega}\theta(y)\cdot K^2(y)\nabla_{\partial\Omega}\theta(y)-q^2(y)+i\frac{\partial q}{\partial\theta}(y)$;
- $S_4(y)=3q(y)S_3(y)-i\frac{\partial S_3}{\partial\theta}(y)-12\nabla_{\partial\Omega}\theta(y)\cdot K^3(y)\nabla_{\partial\Omega}\theta(y)-6i\nabla_{\partial\Omega}q(y)\cdot K(y)\nabla_{\partial\Omega}\theta(y)$;

- $H_m(y) =$ trace $K^m(y)$;
- $\alpha(y) = H_1(y) - q(y) + i\Delta_{\partial\Omega}\theta(y)$;
- $\beta(y) = S_3(y) - H_1(y)q(y) + i\frac{\partial H_1}{\partial\theta}(y) - 2i\ \mathrm{div}_{\partial\Omega}(K(y)\nabla_{\partial\Omega}\theta(y)) - H_2(y)$;
- $\rho(y) = S_4(y) + H_1(y)S_3(y) + 2H_2(y)q(y) - 4iK(y)\nabla_{\partial\Omega}\theta(y)\cdot\nabla_{\partial\Omega}H_1(y) - i\frac{\partial H_2}{\partial\theta}(y) + 3i\ \mathrm{div}_{\partial\Omega}(K^2(y)\nabla_{\partial\Omega}\theta(y)) - \frac{1}{2}\Delta_{\partial\Omega}q(y) + 2H_3(y)$;
- $\lambda(t,y) = \ln[\det(1+tK(y))] = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}t^m H_m(y)$ for t sufficiently small;
- $\sigma(y) = S_5(y) - 6H_4(y) - 6H_3(y)q(y) - 3H_2(y)S_3(y) + H_1(y)S_4(y) - 3\nabla_{\partial\Omega}H_1(y)\cdot\nabla_{\partial\Omega}q(y) + 18iK^2(y)\nabla_{\partial\Omega}H_1(y)\cdot\nabla_{\partial\Omega}\theta + \Delta_{\partial\Omega}S_3(y) + 6\ \mathrm{div}_{\partial\Omega}(K(y)\nabla_{\partial\Omega}q) - 24i\ \mathrm{div}_{\partial\Omega}(K^3(y)\nabla_{\partial\Omega}\theta(y)) + 2i\nabla_{\partial\Omega}H_3(y)\cdot\nabla_{\partial\Omega}\theta(y) + 6iK(y)\nabla_{\partial\Omega}H_2(y)\cdot\nabla_{\partial\Omega}\theta(y)$.

Writing now

$$\begin{aligned}
a(y+tN(y)) &= a_0(y) + a_1(y)t + a_2(y)\frac{t^2}{2} + ...\\
b(y+tN(y)) &= b_0(y) + b_1(y)t + b_2(y)\frac{t^2}{2} + ...\\
U_k(y+tN(y)) &= tU_k^1(y) + \frac{t^2}{2}U_k^2(y) + \frac{t^3}{3!}U_k^3(y) + ...\\
F_k(y+tN(y)) &= F_k^0(y) + tF_k^1(y) + \frac{t^2}{2}F_k^2(y) + ...
\end{aligned}$$

and using that

$$\begin{aligned}
(1+tK)^{-2} &= (1+tK)^{-1}(1+tK)^{-1}\\
&= (1-tK+t^2K^2-...)(1-tK+t^2K^2-...)\\
&= 1-2tK+3t^2K^2-4t^3K^3+O(t^4)
\end{aligned}$$

we obtain

$$\begin{aligned}
U_k^1|_{\partial\Omega} &= G_k\\
\Lambda U_k &= \Big(\alpha - q + 2i\frac{\partial}{\partial\theta}\Big)U_k^1 + U_k^2\\
&\quad + t\left\{\begin{array}{l} U_k^1\Big(\frac{1}{4}\alpha^2 + \frac{i}{2}\frac{\partial\alpha}{\partial\theta} + \frac{3}{2}\beta + i\alpha\frac{\partial}{\partial\theta} - 6iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} - 2iq\frac{\partial}{\partial\theta}\\ +S_3 + q^2 - \alpha q - i\frac{\partial q}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\Big) + \Big(\alpha + 2i\frac{\partial}{\partial\theta} - 3q\Big)U_k^2 + U_k^3\end{array}\right\} + O(t^2)
\end{aligned}$$

$$
\begin{aligned}
\Gamma U_{k-1} = &\begin{pmatrix} \alpha H_1 + \beta - H_2 + 2\Delta_{\partial\Omega} + 2iH_1 \frac{\partial}{\partial\theta} \\ +S_3 - 4iK\nabla_{\partial\Omega}\theta \cdot \nabla_{\partial\Omega} + i\frac{\partial H_1}{\partial\theta} - H_1 q + a_0 \end{pmatrix} U^1_{k-1} \\
&+\Big(\alpha + 2H_1 + 2i\frac{\partial}{\partial\theta} - 2q\Big) U^2_{k-1} + 2U^3_{k-1} \\
&+t \left\{ \begin{array}{l}
\left[\begin{array}{l}
\frac{3}{2}\rho + \frac{3}{2}H_1\beta + \frac{1}{2}\Delta_{\partial\Omega}\alpha - \alpha H_2 + \alpha\Delta_{\partial\Omega} + \nabla_{\partial\Omega}\alpha \cdot \nabla_{\partial\Omega} \\
+i\frac{\partial}{\partial\theta}\Delta_{\partial\Omega} + 2H_2 q - i\frac{\partial H_2}{\partial\theta} - 2iH_2\frac{\partial}{\partial\theta} + 3\nabla_{\partial\Omega}H_1 \cdot \nabla_{\partial\Omega} + 2H_3 + S_4 \\
+H_1 S_3 - 6 \operatorname{div}_{\partial\Omega}(K\nabla_{\partial\Omega}(\cdot)) - 2iK\nabla_{\partial\Omega}\theta \cdot \nabla_{\partial\Omega}H_1 \\
+i\Delta_{\partial\Omega}\frac{\partial}{\partial\theta} - 5\nabla_{\partial\Omega}q \cdot \nabla_{\partial\Omega} - \Delta_{\partial\Omega}q - 2q\Delta_{\partial\Omega} \\
+18iK^2\nabla_{\partial\Omega}\theta \cdot \nabla_{\partial\Omega} - 6iH_1K\nabla_{\partial\Omega}\theta \cdot \nabla_{\partial\Omega} \\
+a_0(\frac{1}{2}\alpha - q + i\frac{\partial}{\partial\theta}) + a_1 + \frac{1}{2}b_0 \cdot N + \frac{i}{2}b_0 \cdot \nabla_{\partial\Omega}\theta
\end{array} \right] U^1_{k-1} \\
+\Big(2\beta + \alpha H_1 + 2iH_1\frac{\partial}{\partial\theta} + i\frac{\partial H_1}{\partial\theta} - 3H_2 \\
+2\Delta_{\partial\Omega} - 3qH_1 - 8iK\nabla_{\partial\Omega}\theta \cdot \nabla_{\partial\Omega} + 3S_3 + a_0\Big) U^2_{k-1} \\
+\Big(\alpha - 4q + 2i\frac{\partial}{\partial\theta} + 2H_1\Big) U^3_{k-1} + 2U^4_{k-1}
\end{array} \right\} \\
&+O(t^2)
\end{aligned}
$$

$$
\begin{aligned}
LU_{k-2} = &\begin{pmatrix} 2H_3 + \Delta_{\partial\Omega}H_1 - 4 \operatorname{div}_{\partial\Omega}(K\nabla_{\partial\Omega}(\cdot)) - H_1H_2 \\ +2H_1\Delta_{\partial\Omega} + 4\nabla_{\partial\Omega}H_1 \cdot \nabla_{\partial\Omega} + a_0H_1 + b_0 \cdot N \end{pmatrix} U^1_{k-2} \\
&+\Big(2\Delta_{\partial\Omega} - 2H_2 + H_1^2 + a_0\Big) U^2_{k-2} + 2H_1U^3_{k-2} + U^4_{k-2} \\
&+t \left\{ \begin{array}{l}
\left[\begin{array}{l}
H_2^2 - 12K\nabla_{\partial\Omega}H_1 \cdot \nabla_{\partial\Omega} + 18 \operatorname{div}_{\partial\Omega}(K^2\nabla_{\partial\Omega}(\cdot)) - 2H_2\Delta_{\partial\Omega} \\
+2H_1H_3 - 4H_1 \operatorname{div}_{\partial\Omega}(K\nabla_{\partial\Omega}(\cdot)) + \nabla_{\partial\Omega}H_1 \cdot \nabla_{\partial\Omega}H_1 \\
-5\nabla_{\partial\Omega}H_2 \cdot \nabla_{\partial\Omega} - \Delta_{\partial\Omega}H_2 + \Delta^2_{\partial\Omega} - 6H_4 - 2 \operatorname{div}_{\partial\Omega}(K\nabla_{\partial\Omega}H_1(\cdot)) \\
-2 \operatorname{div}_{\partial\Omega}(K\nabla_{\partial\Omega}H_1) + 3H_1\nabla_{\partial\Omega}H_1 \cdot \nabla_{\partial\Omega} \\
+b_0 \cdot \nabla_{\partial\Omega} + b_1 \cdot N - a_0H_2 + a_0\Delta_{\partial\Omega} + a_1H_1 + c
\end{array} \right] U^1_{k-2} \\
+\begin{pmatrix} 6\nabla_{\partial\Omega}H_1 \cdot \nabla_{\partial\Omega} + 6H_3 - 8 \operatorname{div}_{\partial\Omega}(K\nabla_{\partial\Omega}(\cdot)) + a_0H_1 \\ +a_1 + b_0 \cdot N - 3H_2H_1 + 2H_1\Delta_{\partial\Omega} + \Delta_{\partial\Omega}H_1 \end{pmatrix} U^2_{k-2} \\
\Big(2\Delta_{\partial\Omega} - 4H_2 + H_1^2 + a_0\Big) U^3_{k-2} + 2H_1U^4_{k-2} + U^5_{k-2}
\end{array} \right\} \\
&+O(t^2).
\end{aligned}
$$

The coefficients of U_k for $k \geq 0$ can now be obtained by substituting the above expressions in (13) and comparing coefficients. For $k = 0$, we have

$$
\left\{ \begin{array}{ll}
\Lambda U_0 & = F_0 \\
\frac{\partial U_0}{\partial N}\Big|_{\partial\Omega} & = G_0 \\
U_0\Big|_{\partial\Omega} & = 0
\end{array} \right.
$$

and we obtain

$$\begin{aligned}
U_0^1 &= G_0 \\
U_0^2 &= F_0^0 - \Big(\alpha - q + 2i\frac{\partial}{\partial\theta}\Big)U_0^1 \\
U_0^3 &= F_0^1 - \Big(\alpha - 3q + 2i\frac{\partial}{\partial\theta}\Big)U_0^2 \\
&\quad - \left(\begin{array}{c} \frac{1}{4}\alpha^2 + \frac{i}{2}\frac{\partial\alpha}{\partial\theta} + \frac{3}{2}\beta + i\alpha\frac{\partial}{\partial\theta} + S_3 + q^2 \\ -6iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} - 2iq\frac{\partial}{\partial\theta} - \alpha q - i\frac{\partial q}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \end{array}\right)U_0^1.
\end{aligned} \tag{20}$$

For $k = 1$

$$\left\{\begin{array}{ll} \Lambda U_1 + \Gamma U_0 & = F_1 \\ \frac{\partial U_1}{\partial N}\Big|_{\partial\Omega} & = G_1 \\ U_1\Big|_{\partial\Omega} & = 0 \end{array}\right.$$

from which it follows that

$$\begin{aligned}
U_1^1 &= G_1 \\
U_1^2 &= F_1^0 - \Big(\alpha - q + 2i\frac{\partial}{\partial\theta}\Big)U_1^1 \\
&\quad - \left(\begin{array}{c} \alpha H_1 + \beta - H_2 + 2\Delta_{\partial\Omega} + 2iH_1\frac{\partial}{\partial\theta} \\ +S_3 - 4iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} + i\frac{\partial H_1}{\partial\theta} - H_1 q \end{array}\right)U_0^1 \\
U_1^3 &= F_1^1 - \left(\begin{array}{c} \frac{1}{4}\alpha^2 + \frac{i}{2}\frac{\partial\alpha}{\partial\theta} + \frac{3}{2}\beta + i\alpha\frac{\partial}{\partial\theta} + S_3 + q^2 \\ -6iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} - 2iq\frac{\partial}{\partial\theta} - \alpha q - i\frac{\partial q}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} \end{array}\right)U_1^1 \\
&\quad -\Big(\alpha - 3q + 2i\frac{\partial}{\partial\theta}\Big)U_1^2 - 2U_0^4 - \Big(\alpha - 4q + 2i\frac{\partial}{\partial\theta} + 2H_1\Big)U_0^3 \\
&\quad - \left(\begin{array}{c} 2\beta + \alpha H_1 - 3qH_1 - 8iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} \\ +2iH_1\frac{\partial}{\partial\theta} + i\frac{\partial H_1}{\partial\theta} - 3H_2 + 2\Delta_{\partial\Omega} + 3S_3 \end{array}\right)U_0^2 \\
&\quad - \left(\begin{array}{l} \frac{3}{2}\rho + \frac{3}{2}H_1\beta + \frac{1}{2}\Delta_{\partial\Omega}\alpha - \alpha H_2 + \alpha\Delta_{\partial\Omega} + \nabla_{\partial\Omega}\alpha\cdot\nabla_{\partial\Omega} \\ +i\frac{\partial}{\partial\theta}\Delta_{\partial\Omega} + 2H_2 q - i\frac{\partial H_2}{\partial\theta} - 2iH_2\frac{\partial}{\partial\theta} + 3\nabla_{\partial\Omega}H_1\cdot\nabla_{\partial\Omega} + 2H_3 + S_4 \\ +H_1S_3 - 6\ \mathrm{div}_{\partial\Omega}(K\nabla_{\partial\Omega}(\cdot)) - 2iK\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega}H_1 \\ +i\Delta_{\partial\Omega}\frac{\partial}{\partial\theta} - 5\nabla_{\partial\Omega}q\cdot\nabla_{\partial\Omega} - \Delta_{\partial\Omega}q - 2q\Delta_{\partial\Omega} \\ +18iK^2\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} - 6iH_1K\nabla_{\partial\Omega}\theta\cdot\nabla_{\partial\Omega} \end{array}\right)U_0^1.
\end{aligned}$$

In this way, one can compute as many coefficients as wished of the formal solution u of (11).

3 Exact solutions

We now show that the approximate solutions obtained in the previous sections are close to real solutions.

Let L, $\mathcal{A}_L$ and $\mathcal{C}_L$ be the operators defined in (2) (3) (4)(5) (6) and (7). We first show that $\mathcal{A}_L$ and $\mathcal{C}_L$ are well defined. Using the same notations of the introduction, we observe that L, as a compact perturbation of the Bilaplacian, is a Fredholm operator of index 0, when considered as an operator from $W^{4,p} \cap W_0^{2,p}(\Omega, \mathbb{C})$ into $L^p(\Omega, \mathbb{C})$. Thus, we have $L^p(\Omega, \mathbb{C}) = \mathcal{R}(L) \oplus [w_1, ..., w_m]$. Given $f = f_1 + f_2 \in L^p(\Omega, \mathbb{C})$ with $f_1 \in \mathcal{R}(L)$ and $f_2 \in [w_1, ..., w_m]$ there exists a unique $v \in W^{4,p} \cap W_0^{1,p}(\Omega, \mathbb{C})$ such that $Lv = f_1$, $\frac{\partial v}{\partial N} = g$ on $\partial\Omega$ and $\int_\Omega v\bar{\tau}_i = 0$ for all $1 \leq i \leq m$. (The existence of v follows from results in [2] and the uniqueness follows from the conditions $\int_\Omega v\bar{\tau}_i = 0$ para todo $1 \leq i \leq m$).

Suppose now that Ω is a C^{5+N-k} regular region, $N \geq 0$ is an integer

$$F(x) = \Big(F_0 + \frac{F_1}{2\omega} + ... + \frac{F_N}{(2\omega)^N}\Big), \quad G(x) = \Big(G_0 + \frac{G_1}{2\omega} + ... + \frac{G_N}{(2\omega)^N}\Big),$$

with F_k C^{2+N-k} in Ω and G_k C^{3+N-k} on $\partial\Omega$, for $k = 1, 2, \cdots, N$. Suppose also θ is C^{5+N} in $\partial\Omega$ and the coefficients a, b and c of L are C^{N+2}, C^{N+1} and C^N respectively in Ω.

We can choose $S(y + tN(y))$ of class C^{5+N} such that

$$(\nabla S)^2 = O(t^{4+N}), \tag{21}$$

and U_k of class C^{4+N-k} $0 \leq k \leq N$ in Ω, with

$$\begin{cases} \Lambda U_k + \Gamma U_{k-1} + LU_{k-2} - F_k & = O(t^{2+N-k}) \quad k = 0, \cdots, N \\ \Gamma U_N + LU_{N-1} & = O(t) \\ U_k|_{\partial\Omega} = 0, & \frac{\partial U_k}{\partial N}|_{\partial\Omega} = G_k \end{cases} \tag{22}$$

uniformly in $-\delta \leq t =$ dist$(x, \partial\Omega) \leq \delta$, for some $\delta > 0$, $(U_{-2} = U_{-1} \equiv 0)$.

Finally we choose a compact supported C^∞ 'cutoff function' χ of class C^∞, $\chi \equiv 1$ for $-\delta \leq t =$ dist$(x, \partial\Omega) \leq \delta$ but χ supported near this set and let

$$u(x) = e^{\omega S(x)}\Big(U_0(x) + \frac{U_1(x)}{2\omega} + ... + \frac{U_N(x)}{(2\omega)^N}\Big) \tag{23}$$

with S and U_k as in (21) and (22).

Theorem 2 *Suppose u is given by (23), $v = \mathcal{A}_L(f) + \mathcal{C}_L(g)$, with*

$$\|f - \chi(2\omega)^2 e^{\omega S} \sum_{k=0}^{N} \frac{F_k}{(2\omega)^k}\|_{L^p(\Omega,\mathbb{C})} = O(\omega^{-N})$$

and

$$\|g - e^{i\omega\theta} \sum_{k=0}^{N} \frac{G_k}{(2\omega)^k}\|_{C^3(\partial\Omega,\mathbb{C})} = O(\omega^{-N}).$$

Then

$$\|\chi u - v\|_{W^{4,p} \cap W_0^{2,p}(\Omega,\mathbb{C})} = O(\omega^{-N}) \text{ as } \omega \to +\infty.$$

Proof. From our hypotheses and the computations of the previous section, we have

$$Lu - (2\omega)^2 e^{\omega S} \sum_{k=1}^{N} \frac{F_k}{(2\omega)^k} =$$

$$= \frac{e^{\omega S}}{(2\omega)^N} \left\{ \begin{array}{l} \left[\begin{array}{l} \frac{1}{16}(2\omega t)^{4+N} \frac{((\nabla S)^2)^2}{t^{4+N}} + \frac{1}{2}(2\omega t)^{3+N} \Big(\frac{1}{2} \frac{(\nabla S)^2 \Delta S}{t^{3+N}} \\ + \frac{1}{2} \frac{\nabla S \cdot \nabla [(\nabla S)^2]}{t^{3+N}} + \frac{(\nabla S)^2 \nabla S \cdot \nabla}{t^{3+N}} \Big) \\ + \frac{1}{2}(2\omega t)^{2+N} \Big(\frac{\nabla[(\nabla S)^2] \cdot \nabla}{t^{2+N}} + \frac{(\nabla S)^2 \Delta}{t^{2+N}} + \frac{1}{2} \frac{\Delta [(\nabla S)^2]}{t^{2+N}} \Big) \end{array} \right] u \\ + \sum_{k=0}^{N} (2\omega t)^{2+N-k} \Big(\frac{\Lambda U_k}{t^{2+N-k}} + \frac{\Gamma U_{k-1}}{t^{2+N-k}} + \frac{L U_{k-2}}{t^{2+N-k}} - \frac{F_k}{t^{2+N-k}} \Big) \\ + (2\omega t) \Big(\frac{\Gamma U_N}{t} + \frac{L U_{N-1}}{t} \Big) + L U_N \end{array} \right\}.$$

Therefore

$$\Big| L[\chi(x)u(x)] - \chi(x)(2\omega)^2 \sum_{k=0}^{N} \frac{F_k(x)}{(2\omega)^k} \Big| \leq \frac{e^{\frac{\omega t}{4}}}{(2\omega)^N} \Big\{ C \sum_{k=0}^{N+4} |2\omega t|^k \Big\}$$

for some $C > 0$, since $ReS(x) < \frac{t}{2}$ in Ω near $\partial\Omega$. Thus

$$L\chi u - f = O(\omega^{-N}) \text{ as } \omega \to +\infty, \text{ uniformly in } \Omega \text{ and } \delta \leq t \leq 0. \tag{24}$$

Since $v = \mathcal{A}_L(f) + \mathcal{C}_L(g)$ there exist $\alpha_1, ..., \alpha_m \in \mathbb{C}$ such that

$$\begin{cases} Lv & = f + \sum_{i=1}^m \alpha_i w_i \text{ in } \Omega \\ \frac{\partial v}{\partial N} & = g \text{ on } \partial\Omega \\ v & = 0 \text{ on } \partial\Omega \end{cases} \tag{25}$$

with $\int_\Omega v\bar{\tau}_i = 0$ for any $1 \leq i \leq m$. We prove the α_i are uniquely determined. In fact, if $\{\sigma_1, ..., \sigma_m\}$ is a basis of $\mathcal{N}(L^*)$, we have for each $1 \leq j \leq m$

$$\begin{aligned} \sum_{i=1}^m \alpha_i \int_\Omega \bar{\sigma}_j w_i &= \int_\Omega \bar{\sigma}_j (Lv - f) \\ &= \int_{\partial\Omega} \Delta \bar{\sigma}_j g - \int_\Omega \bar{\sigma}_j f. \end{aligned}$$

It is then enough to show the matrix $\Big[\int_\Omega \bar{\sigma}_j w_i \Big]_{i,j=1}^m$ is nonsingular. Suppose $\gamma_1, ..., \gamma_m$ are scalars such that $\sum_{i=1}^m \gamma_i \int_\Omega \bar{\sigma}_j w_i = 0$ for $1 \leq j \leq m$. Then $\sum_{i=1}^m \gamma_i w_i \in \mathcal{N}(L^*)^\perp = [\sigma_1, ..., \sigma_m]^\perp = \mathcal{R}(L)$, from which we obtain $\gamma_1 = ... = \gamma_m = 0$ proving the claim.

Let then

$$z = \chi u - v - \sum_{i=1}^m \beta_i \phi_i$$

with $\beta_1, ..., \beta_m \in \mathbb{C}$ chosen in such a way that $\int_\Omega z\bar{\tau}_j = 0$ for all $1 \leq j \leq m$.

We can show, proceeding as above, that $\left[\int_\Omega \phi_i \bar{\tau}_j\right]_{i,j=1}^m$ is nonsingular. Furthermore, we have

$$\begin{aligned}
\frac{\partial z}{\partial N} &= \frac{\partial}{\partial N}(\chi u - v)\\
&= e^{i\omega\theta}\sum_{k=0}^{N}\frac{G_k}{(2\omega)^k} - g\\
&= O(\omega^{-N}), \quad \text{uniformly in } \partial\Omega \text{ as } \omega \to +\infty.
\end{aligned}$$

By the Riemann-Lebesgue lemma

$$\begin{aligned}
\sum_{k=1}^{m}\alpha_i \int_\Omega \bar{\sigma}_j w_i &= \int_{\partial\Omega}\Delta\bar{\sigma}_j g - \int_\Omega \bar{\sigma}_j f\\
&= \sum_{k=0}^{N}(2\omega)^{-k}\int_{\partial\Omega} e^{i\omega\theta}\Delta\bar{\sigma}_j G_k - \sum_{k=0}^{N}(2\omega)^{-k+2}\int_{\partial\Omega} e^{\omega S}\bar{\sigma}_j(\chi F_k)\\
&+ O(\omega^{-N})\\
&= O(\omega^{-N})
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\sum_{i=i}^{m}\beta_i \int_\Omega \phi_i\bar{\tau}_j &= \int_\Omega (\chi u)\bar{\tau}_j\\
&= \int_\Omega \Big(\chi e^{\omega S}\sum_{k=0}^{N}\frac{U_k}{(2\omega)^k}\Big)\bar{\tau}_j\\
&= O(\omega^{-N})
\end{aligned} \tag{27}$$

as $\omega \to +\infty$ since F_k, G_k and U_k are C^{2+N-k}, C^{3+N-k} and C^{4+N-k} respectively for $0 \le k \le N$, that is , $|\alpha_i| = O(\omega^{-N})$ and $|\beta_i| = O(\omega^{-N})$ for any $1 \le i \le m$ as $\omega \to +\infty$. Since $Lz = L(\chi u) - Lv$ it follows from (24), (25) and (26) that

$$\left\{\begin{array}{ll}
Lz & = O(\omega^{-N}) \text{ in } \Omega\\
\frac{\partial z}{\partial N} & = O(\omega^{-N}) \text{ on } \partial\Omega\\
z & = 0 \text{ on } \partial\Omega
\end{array}\right. \tag{28}$$

as $\omega \to +\infty$. Therefore, we obtain, from (27) and (28) that

$$\|\chi u - v\|_{W^{4,p}(\Omega,\mathbb{C})} = \|z - \sum_{i=1}^{m}\beta_i w_i\|_{W^{4,p}(\Omega,\mathbb{C})} = O(\omega^{-N})$$

as $\omega \to +\infty$. □

4 An application

Let Υ be the operator defined in (8). Using the method of the previous sections we prove the following

Theorem 3 *If Υ is of finite rank then*

$$\frac{\partial}{\partial N}(\Delta u \Delta v) \equiv 0 \ on\ \partial\Omega. \tag{29}$$

Proof. In view of (1), it is enough to show that

$$\Upsilon\Big(\cos(\omega\theta)\Big) = \cos(\omega\theta)\frac{\partial}{\partial N}(\Delta u\Delta v)\Big|_{\partial\Omega} + O(\omega^{-1}) \text{ as } \omega \to +\infty. \tag{30}$$

To obtain (30), we show that

$$\begin{aligned}&\Big\{\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(\cos(\omega\theta)\Delta u)\Big)\Big) - \mathcal{C}_{L^*(u)}(\cos(\omega\theta)\Delta v)\Big]\\ &-\Delta v\Delta\Big(\mathcal{C}_{L(u)}(\cos(\omega\theta)\Delta u)\Big)\Big\}\Big|_{\partial\Omega} = O(\omega^{-1}) \text{ as } \omega \to +\infty.\end{aligned} \tag{31}$$

Let $e^{\omega S}\sum_{k=0}^{N}\frac{U_k}{(2\omega)^k}$ be the approximate value of $\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)$ given by

$$\begin{cases}\Lambda U_k + \Gamma U_{k-1} + L(u)U_{k-2} = 0\\ \frac{\partial U_k}{\partial N}\Big|_{\partial\Omega} = \begin{cases}\Delta u|_{\partial\Omega} & k=0\\ 0 & 0<k\le N\end{cases}\\ U_k|_{\partial\Omega} = 0\end{cases} \tag{32}$$

and $e^{\omega S}\sum_{k=0}^{N}\frac{V_k}{(2\omega)^k}$ the approximate value of

$$\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big)\Big) - \mathcal{C}_{L^*(u)}(e^{\omega i\theta}\Delta v), \quad \text{given by}$$

$$\begin{cases}\Lambda V_k + \Gamma V_{k-1} + L^*(u)V_{k-2} = \Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)U_{k-2}\\ \frac{\partial V_k}{\partial N}\Big|_{\partial\Omega} = \begin{cases}-\Delta v|_{\partial\Omega} & k=0\\ 0 & 0<k\le N\end{cases}\\ V_k|_{\partial\Omega} = 0\end{cases} \tag{33}$$

following the notation of section (2). From theorem (2), we obtain

$$\begin{aligned}\Delta\mathcal{C}_{L(u)}\Big(e^{\omega i\theta}\Delta u\Big)\Big|_{\partial\Omega} &= \Big(H\partial_t + \partial_{tt}\Big)\Big(e^{\omega S}\sum_{k=0}^{N}\frac{\sum_{i\ge1}\frac{t^i}{i!}U_k^i}{(2\omega)^k}\Big)\Big|_{t=0} + O(\omega^{-N})\\ &= e^{\omega i\theta}\Big(H\sum_{k=0}^{N}\frac{U_k^1}{(2\omega)^k} + 2\omega\sum_{k=0}^{N}\frac{U_k^1}{(2\omega)^k} + \sum_{k=0}^{N}\frac{U_k^2}{(2\omega)^k}\Big)\\ &\quad +O(\omega^{-N})\\ &= e^{\omega i\theta}\Big(\Delta u(2\omega) + \Big[H\Delta u + U_0^2\Big]\Big)\Big|_{\partial\Omega} + O(\omega^{-1})\end{aligned}$$

since

$$U_k^1 = \frac{\partial U_k}{\partial N}\Big|_{\partial\Omega} = \begin{cases} \Delta u|_{\partial\Omega} & k = 0 \\ 0 & k > 0. \end{cases}$$

Similarly, we obtain

$$\Delta\Big(e^{\omega S}\sum_{k\geq 0}\frac{V_k}{(2\omega)^k}\Big)\Big|_{\partial\Omega} = e^{\omega i\theta}\Big(- \Delta v(2\omega) + \Big[- H\Delta v + V_0^2\Big]\Big)\Big|_{\partial\Omega} + O(\omega^{-1})$$

since

$$V_k^1 = \frac{\partial V_k}{\partial N}\Big|_{\partial\Omega} = \begin{cases} -\Delta v|_{\partial\Omega} & k = 0 \\ 0 & k > 0. \end{cases}$$

Therefore, we have

$$\begin{aligned} &\Big\{\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\tfrac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big) - \mathcal{C}_{L^*(u)}(e^{\omega i\theta}\Delta v)\Big] \\ &-\Delta v\Delta\Big(\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big)\Big\}\Big|_{\partial\Omega} \\ &\qquad = e^{\omega i\theta}\Big[\Delta u V_0^2 - \Delta v U_0^2\Big]\Big|_{\partial\Omega} + O(\omega^{-1}) \end{aligned} \tag{34}$$

since $\Delta u\Delta v \equiv 0$ on $\partial\Omega$. From (20), it follows that

$$U_0^2 = -\Big(\alpha - q + 2i\frac{\partial}{\partial\theta}\Big)\Delta u\Big|_{\partial\Omega} \text{ and}$$

$$\begin{aligned} V_0^2 &= \Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)U_{-2}\Big|_{\partial\Omega} + \Big(\alpha - q + 2i\frac{\partial}{\partial\theta}\Big)\Delta v\Big|_{\partial\Omega} \\ &= \Big(\alpha - q + 2i\frac{\partial}{\partial\theta}\Big)\Delta v\Big|_{\partial\Omega} \end{aligned}$$

since $U_{-2} \equiv 0$ in a neighborhood of $\partial\Omega$. Therefore, we obtain

$$\begin{aligned} \Big(\Delta u V_0^2 - \Delta v U_0^2\Big)\Big|_{\partial\Omega} &= \Big\{(\alpha - q)(\Delta u\Delta v) + 2i\Delta u\frac{\partial}{\partial\theta}\Delta v \\ &\quad +(\alpha - q)(\Delta v\Delta u) + 2i\Delta v\frac{\partial}{\partial\theta}\Delta u\Big\}\Big|_{\partial\Omega} \\ &= 2i\frac{\partial}{\partial\theta}(\Delta u\Delta v)\Big|_{\partial\Omega} \\ &= 0 \text{ on } \partial\Omega \end{aligned}$$

since $\Delta u\Delta v = 0$ on $\partial\Omega$. Therefore

$$\begin{aligned} &\Big\{\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big) - \mathcal{C}_{L^*(u)}(e^{\omega i\theta}\Delta v)\Big] \\ &-\Delta v\Delta\Big(\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big)\Big\}\Big|_{\partial\Omega} = O(\omega^{-1}) \text{ as } \omega \to +\infty. \end{aligned} \tag{35}$$

Since

$$\Big\{\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(\cos(\omega\theta)\Delta u)\Big)\Big) - \mathcal{C}_{L^*(u)}(\cos(\omega\theta)\Delta v)\Big]$$
$$-\Delta v\Delta\Big(\mathcal{C}_{L(u)}(\cos(\omega\theta)\Delta u)\Big)\Big\}\Big|_{\partial\Omega}$$
$$= Re\Big\{\Big\{\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big) - \mathcal{C}_{L^*(u)}(e^{\omega i\theta}\Delta v)\Big]$$
$$-\Delta v\Delta\Big(\mathcal{C}_{L(u)}(e^{\omega i\theta}\Delta u)\Big)\Big\}\Big|_{\partial\Omega}\Big\}$$

we obtain (30) from (35) □

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