GLOBAL IN TIME SPATIAL ANALYTICITY OF SOLUTIONS TO FRACTIONAL BURGERS' EQUATIONS

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In Memory of Hebe A. Biagioni

Abstract

We investigate the asymptotic behavior for $t \to +\infty$ of the radius $\rho_{[u]}(t)$ of the spatial uniform analyticity of solutions $u(t,x)$ of the initial value problem for fractional Burgers' type equations with source term, $x \in \Omega$, $\Omega = \mathbb{R}$ or $\Omega = \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Two different estimates are proved: for $x \in \mathbb{R}$ and in the periodic case $x \in \mathbb{T}$ under suitable decay of $L^p(\Omega)$ norms of u when $t \to +\infty$. We also exhibit explicit solutions to Burgers' equation which show that our asymptotic estimates for $\rho_{[u]}(t)$ as $t \to +\infty$ are sharp in case $\Omega = \mathbb{R}$.

1 Introduction

We consider the IVP for inhomogeneous evolution equations of parabolic type with conservative quadratic term

$$\partial_t u + |D|^m u + \partial_x(u^2/2) = f(t,x), \qquad t > 0, x \in \Omega \tag{1.1}$$

$$u|_{t=0} = u^0, \tag{1.2}$$

where $\Omega = \mathbb{R}$ or, if we consider 2π periodic data, $\Omega = \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $m > 1$, and $|D|^m = |D_x|^m$ is the (nonlocal if $m \notin 2\mathbb{N}$) operator

$$|D|^m w(x) = \int_{\mathbb{R}} e^{ix\xi}|\xi|^m \hat{w}(\xi)\, đ\xi, \qquad đ\xi := (2\pi)^{-1}d\xi \tag{1.3}$$

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(respectively,

$$|D|^m w(x) \quad = \quad \frac{1}{2\pi}\sum_{\xi\in\mathbb{Z}} e^{ix\xi}|\xi|^m \hat{w}(\xi)) \tag{1.4}$$

if $\Omega = \mathbb{R}$ (respectively, $\Omega = \mathbb{T}$). Here

$$\hat{w}(\xi) = \mathcal{F}_{x\to\xi} w = \int_{\mathbb{R}} e^{-ix\xi} w(x)\, dx$$

(respectively,

$$\hat{w}(\xi) = \mathcal{F}_{x\to\xi} w = \int_0^{2\pi} e^{-ix\xi} w(x)\, dx)$$

stands for the continuous (respectively, discrete) Fourier transform if $\Omega = \mathbb{R}$ (respectively, $\Omega = \mathbb{T}$). In the periodic case we suppose that $u^0(x)$ and $f(t,x)$ have mean values zero, i.e.,

$$\int_0^{2\pi} u^0(x)\, dx \quad = \quad 0, \tag{1.5}$$

$$\int_0^{2\pi} f(t,x)\, dx \quad = \quad 0, \qquad t > 0. \tag{1.6}$$

The initial data u^0 can be singular, e.g., in some Sobolev $L^p(\Omega)$ based spaces or in homogeneous Besov spaces. If $m = 2$ we recover Burgers' equation, while for general m such evolution equations are related to physical models, see [6], [7], [8], [23] and the references therein.

It is well known (cf. [14], [28], [13], [3], [4], [24], [16], [17], [11], [27], [10] and the references therein) that, broadly speaking, solutions $u(t,x)$ to semi linear parabolic equations with analytic nonlinearities become uniformly analytic with respect to the spatial variables x for $t \in]0, t_0[$, for some $t_0 > 0$. One is naturally led to the following definition: given $u \in C(]0,T[: L^1_{loc}(\mathbb{R}))$ and $t \in]0,T[$, we define

$$\rho_{[u]}(t) = \sup\{\rho > 0 : u(t,\cdot) \in \mathcal{O}(\Omega_\rho)\} \tag{1.7}$$

with $\rho_{[u]}(t) := 0$ if it cannot be extended to a function in $\mathcal{O}(\Omega_\rho)$ for any $\rho > 0$. Here

$$\Omega_\rho = \{z \in \mathbb{C}^n : |\mathrm{Im} z| < \rho\}, \qquad \rho > 0 \tag{1.8}$$

while $\mathcal{O}(\Gamma)$ stands for the space of all holomorphic functions in an open set $\Gamma \subset \mathbb{C}^n$.

We are interested in finding conditions on weak global in time solutions of (1.1), (1.2) leading to

$$\lim_{t\to+\infty} \rho_{[u]}(t) = +\infty. \tag{1.9}$$

The main novelty of the present paper is the thorough analysis of the asymptotic behavior of $\rho_{[u]}(t)$ for $t \to +\infty$.

We point out that in general (1.9) does not hold unless we require a priori decay of $\|u(t,\cdot)\|_{L^p}$ as $t \to +\infty$ for some $p > 1$. Indeed, the results on the uniform analyticity and explicit examples of solitary wave solutions $u(t,x) = v(x+ct)$, for some $c \in \mathbb{R}$ (cf. [18], [9], [21], [20], [5]) show that it may happen that $\rho_{[u]}(t) = const$ for $t > 0$.

On the other hand, the results in [3] on analyticity for self-similar solutions of Navier-Stokes and Cahn-Hillard type equations in $\mathbb{R}^n$ suggest an estimate of the type $\rho_{[u]}(t) \geq ct^{1/m}$, $t \to +\infty$. Roughly speaking, we will show that under suitable decay conditions on a solution u and suitable uniform analytic–Gevrey estimates on the source term f we can always find $c > 0$ such that

$$\rho_{[u]}(t) \geq ct^{1/m}, \qquad t > 0 \tag{1.10}$$

In the case of periodic data we are able to improve the estimate for large t, namely

$$\rho_{[u]}(t) \geq ct, \qquad t \geq 1 \tag{1.11}$$

We will present explicit solutions $u(t,x)$ of (1.1) for $m = 2$, $\Omega = \mathbb{R}$ (Burgers' equation) with initial data Dirac delta functions such that (1.10) is sharp. Moreover, every such $u(t,x)$ extends to a meromorphic function in $x \in \mathbb{C}$ with simple poles for every $t > 0$.

The paper is organized as follows: in Section 1 we define scales of Banach spaces of Gevrey functions and state the main results. Sections 3 and 4 deal with the fundamental solution of $\partial_t + |D|^m$ in the framework of Gevrey spaces for $\Omega = \mathbb{R}$ and $\Omega = \mathbb{T}$, respectively. The key of the proof, a suitable decomposition of the Green function, is given in Section 5. The proofs of the main results are given in Section 6. We derive sharp estimates for $\rho_{[u]}(t)$ for some explicit solutions to Burgers' equation in the last section.

2 Gevrey spaces of uniformly analytic functions and statement of the main results

First we introduce $L^p(\mathbb{R}^n)$ based Banach spaces of uniformly Gevrey functions $G^\sigma_{un}(\mathbb{R}^n)$, $\sigma > 0$. Here $f \in G^\sigma_{un}(\mathbb{R}^n : L^p(\mathbb{R}^n))$ means that for some $C > 0$

$$\sup_{\alpha \in \mathbb{Z}^n_+} \left(\frac{C^{|\alpha|}}{(\alpha!)^\sigma} \sup_{x \in \mathbb{R}^n} |\partial^\alpha_x f(x)| \right) < +\infty, \tag{2.12}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+.$, if $\sigma = 1$, we obtain that every $f \in G^1_{un}(\mathbb{R}^n)$ is extended to a holomorphic function in $\{z \in \mathbb{C}^n;\ |Im\, z| < C^{-1}\}$. If $0 < \sigma < 1$ we get that $G^\sigma(\mathbb{R}^n : L^p)$ is a subspace of the set of all entire functions in $\mathbb{C}$ of exponential type $1/(1-\sigma)$ (see [26] for more details).

We recall the spaces of time dependent uniformly analytic functions in $\mathbb{R}$ (cf. [3], [4]). Let $q \in [1, +\infty]$, $\delta > 0$, $\theta \geq 0$. We define the global in time spatial Gevrey type Banach space $A^\theta(\delta; L^q(\Omega))$ as follows

$$A^\theta(\delta; L^q(\Omega)) \quad = \quad \{u \in C(]0, \infty[: H^{-\infty}_q(\Omega)) : \|u\|_{\theta,q;\delta} < +\infty\} \tag{2.13}$$

where

$$\|u\|_{\theta,q;\delta} \quad := \quad \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\delta^{|\alpha|}}{\alpha!} \sup_{t>0} \left(t^{\frac{|\alpha|}{m} + \theta} \|\partial^\alpha u(t, \cdot)\|_{L^q} \right). \tag{2.14}$$

Here $H^{-\infty}_q(\Omega) = \bigcap_{s \in \mathbb{R}} H^s_q(\Omega) \subset C^\infty(\Omega)$, i.e., $H^{-\infty}_q(\mathbb{R})$ is the set of all $u \in C^\infty(\mathbb{R})$ such that $\partial^\alpha_x u \in L^q(\mathbb{R})$ for all $\alpha \in \mathbb{Z}_+$ while in the periodic case we have $H^{-\infty}_q(\mathbb{T}) = C^\infty(\mathbb{T}) \cong C^\infty_{2\pi}(\mathbb{R})$. Next, we write $\mathcal{S}'(\mathbb{R})$ (respectively $\mathcal{S}'(\mathbb{T}) = \mathcal{D}'(\mathbb{T})$) for the space of all tempered (respectively periodic) distributions in $\mathbb{R}$ (respectively $\mathbb{T}$) while $\|f\|_{L^q}$ stands for the $L^q(\Omega)$ norm of f. We observe that if $\delta = 0$, $\mu = 2$, with the convention $0^0 = 1$, $A^\theta(0; L^q(\Omega))$ coincides with the global Kato–Fujita weighted space $C_\theta(L^q(\Omega))$ with norm $\|u\|_{\theta,q} := \|u\|_{\theta,q;0}$, used in the study of Navier–Stokes equations and, more generally, semilinear parabolic equations with singular initial data, e.g., see [1], [3], [4], [25] and the references therein. We introduce also the (semi)-norm

$$
\begin{aligned}
|\!|\!| u |\!|\!|_{\theta,q;\delta} &:= \|u\|_{\theta,q;\delta} - \|u\|_{\theta,q} \\
&= \sum_{\alpha\in\mathbb{Z}^n_+\setminus 0} \frac{\delta^{|\alpha|}}{\alpha!} \sup_{t>0} \left(t^{\frac{|\alpha|}{m}+\theta} \|\partial^\alpha u(t,\cdot)\|_{L^q} \right)
\end{aligned} \tag{2.15}
$$

and the corresponding partial sums

$$
S_N[u;\theta,q,\delta] = \sum_{\alpha\in\mathbb{Z}^n_+\setminus 0, |\alpha|\le N} \frac{\delta^{|\alpha|}}{\alpha!} \sup_{t>0} \left(t^{\frac{|\alpha|}{m}+\theta} \|\partial^\alpha u(t,\cdot)\|_{L^q} \right) \tag{2.16}
$$

for all $N \in \mathbb{N}$. We note that $A^\theta(\delta; L^q(\Omega)) \hookrightarrow A^\theta(\delta'; L^q(\Omega))$ provided $0 < \delta' < \delta$ and if $u \in A^\theta(\delta_0; L^q(\Omega))$ for some fixed $\delta_0 > 0$ then

$$
|\!|\!| u |\!|\!|_{\theta,q;\delta} \le \frac{\delta}{\delta_0} |\!|\!| u |\!|\!|_{\theta,q;\delta_0}, \qquad 0 < \delta \le \delta_0. \tag{2.17}
$$

Next, we denote by $\mathcal{S}'_0(\mathbb{T})$ the space of all 2π periodic distributions with mean value zero. Let $q \in [1, +\infty]$, $\delta > 0$, $\tau > 0$. We define the global in time L^q based spatial Gevrey Banach space $A^\tau_{\mathbb{T}}(\delta; L^q)$ as follows

$$
A^\tau_{\mathbb{T}}(\delta; L^q) = \{u \in C(]0,\infty[: \mathcal{S}'_0(\mathbb{T}) \bigcap (H^{-\infty}_q(\mathbb{T})) : \|u\|^{\exp}_{\tau,q;\delta} < +\infty\}, \tag{2.18}
$$

where

$$
\|u\|^{\exp}_{\theta,q;\delta} := \sum_{\alpha\in\mathbb{Z}^n_+} \sup_{t>0} \left(e^{\tau t} \frac{(\delta t)^{|\alpha|}}{\alpha!} \|\partial^\alpha u(t,\cdot)\|_{L^q(\mathbb{T})} \right). \tag{2.19}
$$

We introduce also the (semi)-norm

$$
\begin{aligned}
|\!|\!| u |\!|\!|^{\exp}_{\tau,q;\delta} &:= \|u\|^{\exp}_{\tau,q;\delta} - \|u\|^{\exp}_{\tau,q} \\
&= \sum_{\alpha\in\mathbb{Z}^n_+\setminus 0} \sup_{t>0} \left(e^{\tau t} \frac{(\delta t)^{|\alpha|}}{\alpha!} \|\partial^\alpha u(t,\cdot)\|_{L^q(\mathbb{T})} \right),
\end{aligned} \tag{2.20}
$$

where

$$
\|u\|^{\exp}_{\tau,q} := \|u\|^{\exp}_{\tau,q;0} = \sup_{t>0} \left(e^{\tau t} \|u(t,\cdot)\|_{L^q(\mathbb{T})} \right). \tag{2.21}
$$

Clearly $A^\tau_{\mathbb{T}}(\delta; L^q) \hookrightarrow A^\tau_{\mathbb{T}}(\delta'; L^q)$ provided $0 < \delta' < \delta$ and if $u \in A^\theta_{\mathbb{T}}(\delta_0; L^q)$ for some fixed $\delta_0 > 0$ then

$$|\!|\!| u |\!|\!|^{\exp}_{\tau,q;\delta} \;\leq\; \frac{\delta}{\delta_0} |\!|\!| u |\!|\!|^{\exp}_{\tau,q;\delta_0}, \qquad 0 < \delta \leq \delta_0. \tag{2.22}$$

As we will use mainly L^2 based Gevrey norms on the torus $\mathbb{T}$, we introduce simplified notations $\|u\|^{\exp}_{\tau} := \|u\|^{\exp}_{\tau,2}$, $|\!|\!| u |\!|\!|^{\exp}_{\tau,\delta} := |\!|\!| u |\!|\!|^{\exp}_{\tau,2;\delta}$, and so on.

We will use the partial sums

$$S^{exp}_N[u; \tau, \delta] \;=\; \sum_{\alpha\in\mathbb{Z}^n_+\setminus 0, |\alpha|\leq N} \sup_{t>0} \left(e^{\tau t} \frac{(\delta t)^{|\alpha|}}{\alpha!} \|\partial^\alpha u(t,\cdot)\| \right), \qquad N \in \mathbb{N} \tag{2.23}$$

Sobolev embedding theorems and the Cauchy integral formula for the radius of convergence of power series imply that if $u \in A^\theta(\delta; L^q(\Omega))$ (respectively, $u \in A^\theta_{\mathbb{T}}(\delta; L^q)$) then $u(t,x)$ is holomorphic in the strip $\{x \in \mathbb{C} : |\mathrm{Im} x| < \delta t^{1/m}\}$ (respectively, $\{x \in \mathbb{C} : |\mathrm{Im} x| < \delta t\}$).

We state the first main result of the paper.

Theorem 2.1 *Let $1 < m < 3$, $2 \leq p \leq +\infty$, $p > 1/(m-1)$ and $p < 2/(m-2)$ in case $m \geq 2$ with the convention $p < +\infty$ if $m = 2$. Set*

$$\theta = \theta(m,p) = 1 - \frac{1}{m}(1 + \frac{1}{p}) \tag{2.24}$$

Suppose that for some $\delta_0 > 0$ the source term

$$f \in A^{2\theta}(\delta_0; L^1(\mathbb{R})). \tag{2.25}$$

Then if

$$u \in C([0,\infty[: \mathcal{S}'(\mathbb{R})) \bigcap C_\theta(L^p(\mathbb{R})) \tag{2.26}$$

is a weak solution to (1.1), then there exists $\delta = \delta(m, \|u\|_{\theta,L^p}, \|f\|_{2\theta,1;\delta_0}) \in]0, \delta_0]$ such that

$$u \in A^\theta(\delta; L^p(\mathbb{R})). \tag{2.27}$$

(2.26) holds if $u^0 \in |D|^{\theta m}(L^p(\mathbb{R})) =: \dot{H}^{-\theta m}_p(\mathbb{R})$.

Next, we deal with the periodic case.

Theorem 2.2 *Suppose that for some* $\delta_0 > 0$, $\tau_0 > 0$ *the source term*

$$f \in A_{\mathbb{T}}^{\tau_0}(\delta_0 : L^1(\mathbb{R})). \tag{2.28}$$

Then if for some $\theta_0 > 0$

$$u \in C([0, \infty[: \mathcal{S}'(\mathbb{T})) \bigcap C_{\theta_0, L^2}^{\mathbb{T}} \tag{2.29}$$

is a weak solution to (1.1), (1.2) then there exist $\delta = \delta(m, \|u\|_\theta^{\exp}, \|f\|_{\tau,1;\delta_0}^{\exp}) \in]0, \delta_0]$ *and* $\theta > 0$, $\theta < \min\{\tau, \theta_0\}$ *such that*

$$u \in A_{\mathbb{T}}^{\theta}(\delta : L^2). \tag{2.30}$$

In particular, (2.29) holds if $u^0 \in \dot{H}_2^{-\theta_0 m}(\mathbb{T})$.

3 Estimates for the fundamental solution in $\mathbb{R}$

We denote by $E_m(t, \cdot) = E_m(t, x)$ the fundamental solution of the operator $\partial_t + |D_x|^m$

$$E_m(t, x) = \int_{\mathbb{R}} e^{ix\xi - t|\xi|^m} \,đ\xi = \frac{1}{t^{1/m}} \mathcal{E}_m\left(\frac{x}{t^{1/m}}\right) \tag{3.31}$$

where $\mathcal{E}_m(z) = \int e^{iz\xi - |\xi|^m} \,đ\xi$.

Lemma 3.1 *Let* $\sigma \geq 1/m$. *We have*

$$\mathcal{E}_m \in G_{un}^{\sigma}(\mathbb{R} : L^p) \tag{3.32}$$

for all $1 \leq p \leq \infty$. *In particular, if* $m \in 2\mathbb{N}$, *then* $\mathcal{E}_m$ *belongs to the Gelfand–Shilov space* $S_{(m-1)/m}^{1/m}(\mathbb{R})$ *cf. [15], namely there exist* $a, b > 0$ *such that*

$$|\partial^k f(x)| \leq a^{k+1} (k!)^{(m-1)/m} e^{-b|x|^{1/m}}, \qquad k \in \mathbb{Z}_+,\ x \in \mathbb{R}. \tag{3.33}$$

Proof: We have

$$\begin{aligned} |D_z^\alpha \mathcal{E}_m|_\infty &\leq \int_{\mathbb{R}} |\xi|^\alpha e^{-|\xi|^m} \,đ\xi \\ &= \frac{1}{m\pi} \int_0^{+\infty} \eta^{(\alpha+1)/m - 1} e^{-\eta} \, d\eta = \frac{1}{m\pi} \Gamma((\alpha + 1)/m), \end{aligned} \tag{3.34}$$

for all $\alpha \in \mathbb{Z}_+$, with $\Gamma(z)$ standing for the Gamma function. The Stirling formula leads to (3.32) for $p = \infty$.

In order to derive $L^1(\mathbb{R}^n)$ estimates we need somewhat more subtle arguments. Let $|z| \geq 1$. Suppose that $\alpha \in \mathbb{N}$, $\alpha \geq 2$. In view of the identity

$$\frac{1}{iz}\partial_z(e^{iz\xi}) = e^{iz\xi}, \tag{3.35}$$

for $z \neq 0$, and the fact that $\xi^\alpha e^{-|\xi|^m} \in C^2(\mathbb{R})$ for $\alpha \geq 2$, we can integrate by parts twice and obtain

$$\begin{aligned} D_z^\alpha \mathcal{E}_m(z) &= -\frac{1}{z^2}\int_{\mathbb{R}} e^{iz\xi}\partial_\xi^2(\xi^\alpha e^{-|\xi|^m}) \, đ\xi \\ &= -\frac{1}{z^2}\int_{\mathbb{R}} e^{iz\xi} Q_\alpha^m(\xi) e^{-|\xi|^m} \, đ\xi \,, \end{aligned} \tag{3.36}$$

where $Q_\alpha^m(\xi) = \alpha\xi^{\alpha-2}((\alpha-1) - 2|\xi|^m) + m\xi^\alpha|\xi|^{m-2}(m|\xi|^m - (m-1))$. The integral in (3.36) is convergent near $\xi = 0$ since $Q_\alpha(\xi)$ is bounded by $C|\xi|^{-1+m}$ for $|\xi| \leq 1$ provided $\alpha \geq 1$ or $\alpha = 0$, $m > 1$. Therefore

$$|\partial_z^\alpha \mathcal{E}_m(z)| \;\leq\; \frac{1}{|z|^2}\sum_{j=1}^{4} \mathcal{L}_j^\alpha \tag{3.37}$$

with

$$\mathcal{L}_1^\alpha = \frac{\alpha(\alpha-1)}{m\pi}\Gamma((\alpha-1)/m) \tag{3.38}$$

$$\mathcal{L}_2^\alpha = \frac{2\alpha}{\pi}\Gamma((\alpha+m-1)/m) \tag{3.39}$$

$$\mathcal{L}_3^\alpha = \frac{2m}{\pi}\Gamma((\alpha+2m-1)/m) \tag{3.40}$$

$$\mathcal{L}_4^\alpha = \frac{2(m-1)}{\pi}\Gamma((\alpha+m-1)/m)\,. \tag{3.41}$$

It remains to show that $\mathcal{E}_m \in L^1(\mathbb{R})$ if $0 < m \leq 1$. We will use oscillatory integrals and the Fourier transform of homogeneous distributions (cf. [19]). Let $\varphi \in C_0^\infty(\mathbb{R})$, $0 \leq \varphi(\xi) \leq 1$ for $\xi \in \mathbb{R}$; $\varphi(\xi) = 1$ if $|\xi| \leq 1$; $\operatorname{supp}\varphi \subset [-2, 2]$. Set

$$k = k(m) = \min\{\ell \in \mathbb{N} : \; \ell m > 1\}. \tag{3.42}$$

Evidently $k > 1$ if $m \geq 1$. By the Taylor formula we get

$$
\begin{aligned}
e^{-|\xi|^m} &= \sum_{\ell=0}^{k-1} \frac{(-1)^\ell}{\ell!}|\xi|^{m\ell} + r_m^k(\xi) \\
r_m^k(\xi) &= \frac{(-1)^k}{k!}|\xi|^{mk}\int_0^1 (1-t)^{k-1}e^{-t|\xi|^m}\,dt
\end{aligned}
$$

which yields the following decomposition

$$
\begin{aligned}
\mathcal{E}_m(z) &= \mathcal{E}_m^1(z) + \mathcal{E}_m^2(z) + \sum_{\ell=0}^{k-1}\mathcal{H}_m^\ell(z) && (3.43)\\
\mathcal{E}_m^1(z) &= \int_{\mathbb{R}} e^{iz\xi}(1-\varphi(\xi))e^{-|\xi|^m}\,đ\xi && (3.44)\\
\mathcal{E}_m^2(z) &= \int_{\mathbb{R}} e^{iz\xi}\varphi(\xi)r_m^k(z)\,đ\xi && (3.45)\\
\mathcal{H}_m^\ell(z) &= \frac{(-1)^\ell}{\ell!}\int_{\mathbb{R}} e^{iz\xi}\varphi(\xi)|\xi|^\ell\,đ\xi, \qquad \ell = 1,\ldots,k-1. && (3.46)
\end{aligned}
$$

We can integrate by parts twice in the integrals in the RHS of (3.44) and (3.45) and obtain that $\mathcal{E}_m^j(z) = O(|z|^{-2})$ as $z \to \infty$, for $j = 1, 2$. The integration by parts is not possible in the integrals defining $\mathcal{H}_m^\ell(z)$ since non integrable singularities of the type $O(|\xi|^{-1-\mu})$, $\mu > 0$ appear near $\xi = 0$. We will represent the convergent integrals in the RHS of $\mathcal{H}_m^\ell(z)$ as sums of two oscillatory integrals (tempered distributions) with singularities at $z = 0$. We can write

$$
\int_{\mathbb{R}} e^{iz\xi}\varphi(\xi))|\xi|^{m\ell}\,đ\xi = \int_{\mathbb{R}} e^{iz\xi}|\xi|^{m\ell}\,đ\xi + \int_{\mathbb{R}} e^{iz\xi}(1-\varphi(\xi))|\xi|^{m\ell}\,đ\xi \quad (3.47)
$$

The first oscillatory integral in the RHS of (3.47) is the inverse Fourier transform of the homogeneous Schwartz distribution $|\xi|^{m\ell}$, and it is homogeneous of order $-1 - m\ell$, hence

$$
\int_{\mathbb{R}} e^{iz\xi}|\xi|^{m\ell}\,đ\xi = c_\pm^{m\ell}|z|^{-1-m\ell}, \qquad z \in \mathbb{R}\setminus 0 \quad (3.48)
$$

for some $c_\pm^{m\ell} \in \mathbb{C}$, with $c_\pm^{m\ell} = 0$ if $m\ell \in 2\mathbb{Z}_+$ since in that case the LHS equals (modulo a multiplicative constant) the $m\ell$–th derivative of the Dirac delta function (cf. [19]). The outcome of (3.48) is that the LHS belongs to

$L^1(|z| \geq 1)$ for all $\ell \in \mathbb{Z}_+$, $m > 0$. As to the second oscillatory integral in the RHS of (3.48), as $(1-\varphi(\xi))|\xi|^\ell$ is smooth, we can integrate by parts and write

$$\int_{\mathbb{R}} e^{iz\xi}(1-\varphi(\xi))|\xi|^{m\ell}\, d\!\!{}^-\xi \;=\; \frac{i^N}{z^N}\int_{\mathbb{R}} e^{iz\xi}\partial_\xi^N\left((1-\varphi(\xi))|\xi|^{m\ell}\right)\, d\!\!{}^-\xi \quad (3.49)$$

for all $z \in \mathbb{R}\setminus 0$, $N \in \mathbb{N}$. Since

$$\sup_{\xi\in\mathbb{R}}(1+|\xi|)^{N-m\ell}|\partial_\xi^N\left((1-\varphi(\xi))|\xi|^{m\ell}\right)| \;=:\; M_N < +\infty \quad (3.50)$$

if $N \geq [m\ell]+2$ ($[r]$ standing for the integer part of r) we get by (3.49) that

$$\left|\int_{\mathbb{R}} e^{iz\xi}(1-\varphi(\xi))|\xi|^{m\ell}\, d\!\!{}^-\xi\right| \;\leq\; \frac{C_N}{|z|^N} \leq \frac{C_N}{z^2}, \qquad |z| \geq 1 \quad (3.51)$$

with $C_N = M_N \displaystyle\int_{\mathbb{R}} \frac{1}{(1+|\xi|)^{N-m\ell}}\, d\!\!{}^-\xi < +\infty$. This yields the validity of the $L^1(\mathbb{R})$ estimates. Standard interpolation arguments conclude the proof of (3.32).

Suppose now that $m = 2k$, $k \in \mathbb{N}$. Then $e^{-|\xi|^m} = e^{-\xi^{2k}} \in S_{1/m}^{(m-1)/m}(\mathbb{R})$ by properties of the Fourier transform in the Gelfand–Shilov spaces (cf. [15], see also the direct estimates in [4]). The proof is complete.

□

The next theorem derives Gevrey type estimates for the fundamental solution E_m for all $m \geq 1$ which might be of independent interest.

Theorem 3.2 *Let $m \geq 1$. Then*

$$E_m \;\in\; A^{\frac{1}{m}(1-\frac{1}{q})}(\delta; L^q) \quad (3.52)$$

$$\partial E_m \;\in\; A^{1/m+\frac{1}{m}(1-\frac{1}{q})}(\delta; L^q) \quad (3.53)$$

for every $q \in [1, +\infty]$, and we claim that there exist two positive continuous bounded functions $C(q)$ and $D(q)$, $1 \leq r \leq \infty$, such that

$$\|E_m\|_{1/m(1-1/q,q;\delta} \;\leq\; C(q)Exp_{(m-1)/m}(C(q)\delta), \qquad \delta \geq 0; \quad (3.54)$$

$$|\!|\!|E_m|\!|\!|_{1/m(1-1/q),q;\delta} \;\leq\; C(q)\delta Exp_{(m-1)/m}(C(q)\delta), \qquad \delta \geq 0; \quad (3.55)$$

$$\|\partial E_m\|_{1/m+1/m(1-1/q),q;\delta} \;\leq\; D(q)Exp_{(m-1)/m}(D(q)\delta), \qquad \delta \geq 0; \quad (3.56)$$

$$|\!|\!|\partial E_m|\!|\!|_{1/m+1/m(1-1/q),q;\delta} \;\leq\; D(q)\delta Exp_{(m-1)/m}(D(q)\delta), \qquad \delta \geq 0, \quad (3.57)$$

where

$$Exp_{\sigma}(z) := \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k!)^{\sigma}}, \qquad z \in \mathbb{R} \tag{3.58}$$

for $\sigma > 0$ *while*

$$Exp_{0}(z) := \sum_{k=1}^{\infty} z^{k-1} = \frac{1}{1-z}, \qquad |z| < 1. \tag{3.59}$$

Proof: One observes that

$$\partial^k E_m = \partial_x^k E_m(t,x) = \frac{1}{t^{k/m}} \mathcal{E}_m^{(k)}(\frac{x}{t^{1/m}}), \qquad t > 0, x \in \mathbb{R} \tag{3.60}$$

for all $k \in \mathbb{Z}_+$. Lemma 3.1 and the Stirling formula imply that there exist continuous positive functions $C_j(p)$, $j = 0, 1$, $1 \leq p \leq +\infty$ such that

$$\|\mathcal{E}_m^{(k+j)}\|_{L^p(\mathbb{R}^n)} \leq (C_j(p))^{k+1} k!^{1/m}, \qquad k \in \mathbb{N},\ j = 0, 1. \tag{3.61}$$

Hence, combining (3.60), (3.61), we get

$$\|\partial^j E_m\|_{1/m(1-1/q),q;\delta} \leq C_j(q) \sum_{k=0}^{\infty} \frac{(C_j(q)\delta)^k}{(k!)^{(m-1)/m}} \tag{3.62}$$

which yields the proof of (3.54) and (3.56) in view of (3.59) and (3.59). Similar arguments lead to (3.55) and (3.57). The proof is complete.

□

4 Estimates for the fundamental solution in the periodic case

Let us consider the fundamental solution $E_m^{per}(t, \cdot)$ in the case $\Omega = \mathbb{T}$. We recall an expression for $E_m^{per}(t,x)$ via its Fourier series, namely

$$E_m^{per}(t,x) = \mathcal{F}^{-1}_{\xi \to x} \widehat{E_m^{\mathbb{T}}}(t,\xi) = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} e^{ix\xi - t|\xi|^m}. \tag{4.63}$$

In view of the mean value conditions (1.5) and (1.6) one introduces

$$E_m^{\mathbb{T}}(t,x) = \mathcal{F}^{-1}_{\xi \to x} (\widehat{E_m^{\mathbb{T}}}(t,\xi) - \frac{1}{2\pi}) = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z} \backslash 0} e^{ix\xi - t|\xi|^m}. \tag{4.64}$$

Clearly

$$\partial^k E_m^{\mathbb{T}}(t,x) = \frac{1}{2\pi}\sum_{\xi\in\mathbb{Z}\setminus 0} e^{ix\xi - t|\xi|^m}(i\xi)^k \tag{4.65}$$

and the Parseval identity yields

$$\|\partial^k E_m^{\mathbb{T}}(t,\cdot)\|^2 = \sum_{\xi\in\mathbb{Z}\setminus 0} \xi^{2k} e^{-2t|\xi|^m}, \qquad k\in\mathbb{N}. \tag{4.66}$$

We show a discrete analogue to Lemma 3.1.

Lemma 4.1 *Let $m > 1$. Then we can find $C_0 = C_0(m)$ such that for every $\eta \in]0,1[$ we have*

$$\|\partial_x^{k+j} E_m^{per}(t,\cdot)\| \le \frac{C_0^{k+1}}{(1-\eta)^{k+1/2m+j/m}} e^{-\eta t}(k!)^{(m-1)/m} t^{-1/2m-j/m-k} \quad t>0,\ k\in\mathbb{Z}_+ \tag{4.67}$$

for $j = 0, 1$.

Proof. We will give the proof for $j = 1$ (the case $j = 0$ is easier to deal with). Since $|\xi| \ge 1$ implies $t^k|\xi|^k \le t^k|\xi|^{km}$ for all $t > 0$ we get

$$\begin{aligned}
\|\frac{t^{3/(2m)+k}}{k!}\partial^{k+1}E(t,\cdot)\|^2 &\le e^{-2\eta t}\frac{t^{1/m}}{2\pi}\sum_{\xi\in\mathbb{Z}\setminus 0} e^{-2t(1-\eta)|\xi|^m} t^{2/m+2k}\xi^{2k+2} \\
&\le e^{-2\eta t}\frac{t^{1/m}}{2\pi}\sum_{\xi\in\mathbb{Z}\setminus 0} e^{-2t(1-\eta)|\xi|^m}(t|\xi|^m)^{2k+2/m} \\
&= e^{-2\eta t}\frac{t^{1/m}}{\pi(2(1-\eta))^{2k+2/m}}\sum_{j=1}^{\infty} e^{-2t(1-\eta)j^m} \\
&\quad (2t(1-\eta)j^m)^{2k+2/m} \\
&\le e^{-2\eta t}\frac{t^{1/m}}{\pi(2(1-\eta))^{2k+2/m}} \\
&\times \Psi_m(2(1-\eta)t; 2k+2/m)
\end{aligned} \tag{4.68}$$

where

$$\Psi_m(r;\ell) = \sum_{j=1}^{\infty} e^{-rj^m}(rj^m)^\ell \tag{4.69}$$

We focus on estimating $\Psi_m(r;\ell)$ for $r > 0$, $\ell \in \mathbb{Z}_+$.

We need an auxiliary assertion.

Lemma 4.2 *Let $m > 0$, $\ell \geq 0$, $r > 0$. Then the function $g(z) := z^{m\ell}e^{-rz^m}$ admits a unique maximum at $z = (\ell/mr)^{1/m}$ and*

$$\sup_{z\geq 0} g(z) \quad = \quad g((\frac{\ell}{r})^{1/m}) = \left(\frac{\ell}{r}\right)^{\ell} e^{-\ell} \tag{4.70}$$

Proof: Straightforward calculations lead to

$$g'(z) = mz^{m\ell-1}e^{-rz^m}(\ell - rz^m), \qquad z > 0$$

and $g'(z) > 0$ for $z \in]0, (\frac{\ell}{r})^{1/m}[$ and $g'(z) < 0$ if $z \in](\frac{\ell}{r})^{1/m}, +\infty[$. The proof of the lemma is complete.

□

Next, we apply the lemma and readily obtain

$$\begin{aligned} \Psi_m(r;\ell) &\leq \ell^\ell e^{-\ell} \sum_{1\leq j\leq(\frac{\ell}{r})^{1/m}} 1 + \sum_{j\geq(\frac{\ell}{r})^{1/m}} e^{-rj^m}(rj^m)^\ell \\ &\leq r^{-1/m}\ell^{(\ell+1)/m}e^{-\ell} + \int_0^{+\infty} e^{-ry^m}(ry^m)^\ell \, dy \\ &= r^{-1/m}\left(\ell^{(\ell+1)/m}e^{-\ell/m} + \int_0^{+\infty} e^{-y^m}y^\ell \, dy\right) \\ &= r^{-1/m}\left(\ell^{(\ell+1)/m}e^{-\ell} + \frac{1}{m}\Gamma(\ell+1/m)\right). \end{aligned} \tag{4.71}$$

Finally, applying (4.71) with $r = (2(1-\eta)t$, $\ell = 2k + 2/m$ to (4.68) and using the Stirling formula, we complete the proof of (4.67).

□

5 Decomposition of the Green function

First we derive Gevrey estimates related to the contribution of the source term $f(t,x)$ in the case $\Omega = \mathbb{R}$. We define the Green function G_m as $G_m[f](t,\cdot) = \displaystyle\int_0^t E_m(t-s,\cdot) * f(s,\cdot)ds$.

Lemma 5.1 *There exists $C > 0$ depending only on p and m such that*

$$\begin{aligned} S_N[G_m[f];\theta,p,\delta] &\leq C\delta Exp_{(m-1)/m}(C\delta)\,\|f\|_{2\theta,p/2} \\ &+ CExp_{(m-1)/m}(C\delta)|\!|\!|f|\!|\!|_{2\theta,p/2;\delta} \end{aligned} \tag{5.72}$$

for all $f \in A^{2\theta}(\delta_0; L^{p/2})$, $N \in \mathbb{N}$, $\delta \in]0, \delta_0[$. *In particular, letting* $N \to +\infty$, *we get*

$$
\begin{aligned}
\|[G_m[f]\|_{\theta,p;\delta} &\leq C\delta Exp_{(m-1)/m}(C\delta)\, \|f\|_{2\theta,p/2} \\
&+ C Exp_{(m-1)/m}(C\delta) |\!|\!| f |\!|\!|_{2\tau,1;\delta}
\end{aligned}
\tag{5.73}
$$

and

$$
\lim_{\delta \to 0} \|[G_m[f]\|_{\theta,p;\delta} = 0. \tag{5.74}
$$

Proof. We recall the well known property for derivatives of the convolution

$$
\partial^k (f * g) = \partial^{k-\alpha} f * \partial^\alpha g, \qquad \alpha = 0, 1, \ldots, k.
$$

We have

$$
\begin{aligned}
\{G_m[f]\}_k^\delta(t,x) &:= \frac{\delta^k t^{k/m}}{k!} \int_0^t \partial^k (E_m(t-s,\cdot) * f(s,\cdot))\, ds, \\
&= \frac{\delta^k t^{k/m}}{k!} \int_0^t \partial^{k-1} (\partial E_m(t-s,\cdot) * f(s,\cdot))\, ds, \\
&= \int_0^t \frac{\delta^k t^{k/m}}{((t-s)^{1/m} + s^{1/m})^k s^\tau} \left(\frac{(t-s)^{k/m}}{k!} \partial^k E_m(t-s,\cdot) \right) \\
&\quad s^\tau f(s,\cdot)\, ds \\
&+ \sum_{j=1}^k \int_0^t \frac{\delta^k t^{k/m}}{((t-s)^{1/m} + s^{1/m})^k s^\tau} \\
&\times \frac{(t-s)^{(k-j)/m}}{(k-j)!} \partial^{k-j} E_m(t-s,\cdot) \\
&\quad * s^\tau \frac{\delta^j s^{j/m+\tau}}{j!} \partial^j f(s,\cdot)\, ds
\end{aligned}
\tag{5.75}
$$

Applying first the Young inequality for the $L^p(\mathbb{R})$ norm with $1+1/p = 1/q+2/p$, $1/q = 1-1/p$, then a summation from $k=1$ to N in (5.75) and changing the order of summation we get

$$
\begin{aligned}
S_N[G[f]; \theta, p, \delta] &\leq \left(t^\theta \int_0^t \frac{1}{(t-s)^{1/m(1+1/p)} s^{2\theta}}\, ds \right) C(p)\delta \\
&\times \ \mathrm{Exp}_{(m-1)/m}(C(p)\delta)\, \|f\|_{2\theta,p/2} \\
&+ \left(t^\theta \int_0^t \frac{1}{(t-s)^{1/m(1+1/p)} s^{2\theta}}\, ds \right) \\
&\times \ \mathrm{Exp}_{(m-1)/m}(C(p)\delta) |\!|\!| f |\!|\!|_{2\theta,p/2;\delta}
\end{aligned}
\tag{5.76}
$$

Since the restrictions on p and m in the case $\Omega = \mathbb{R}$ imply $2\theta < 1$, $1/m(1+1/p) < 1$ we have

$$
\begin{aligned}
t^{\theta} \int_0^t \frac{1}{(t-s)^{1/m(1+1/p)} s^{2\theta}} \, ds &= \int_0^1 \frac{1}{(1-s)^{1/m(1-1/p)} s^{2\theta}} \, ds \\
&= B(1 - 1/m(1+1/p), 1 - 2\theta), \quad (5.77)
\end{aligned}
$$

$B(\mu, \nu)$ being the Beta function. The proof of (5.72) is complete.

□

Next, we investigate the source term in the periodic case.

Lemma 5.2 *Let $\theta, \tau, \eta > 0$, $\theta < \tau < 1$, $\theta < \eta < 1$. Then there exists $C = C(\eta, \tau, \theta) > 0$ such that the following estimate holds*

$$
S_N^{exp}[G_m^{per}[f]; \theta, \delta] \leq \frac{C}{1 - C\delta} (\delta \, \|f\|_{\tau,1}^{exp} + [\![f]\!]_{\tau,1;\delta}^{exp}) \quad (5.78)
$$

for all $N \in \mathbb{N}$, $f \in A_{\mathbb{T}}^{\tau}(\delta_0)$, $\delta \in]0, \delta_0]$, $\delta < C^{-1}$. In particular, letting $N \to +\infty$, we obtain

$$
[\![G_m^{per}[f]]\!]_{\theta,\delta}^{exp} \leq \frac{C}{1 - C\delta} (\delta \, \|f\|_{\tau,1}^{exp} + [\![f]\!]_{\tau,1;\delta}^{exp}) \quad (5.79)
$$

and

$$
\lim_{\delta \to 0} [\![G_m^{per}[f]]\!]_{\theta,\delta}^{exp} = 0. \quad (5.80)
$$

Proof. We have

$$
\begin{aligned}
\{G[f]\}_k^{\delta}(t,x) &:= \frac{\delta^k t^k}{k!} \int_0^t \partial^k (E_m(t-s,\cdot) * f(s,\cdot)) \, ds, \\
&= \sum_{j=0}^{k} \int_0^t \frac{\delta^{k-j}(t-s)^{k-j}}{(k-j)!} \partial^{k-j} E_m^{\mathbb{T}}(t-s,\cdot) \\
&\quad * \frac{\delta^j s^j}{j!} \partial^j f(s,\cdot) \, ds. \quad (5.81)
\end{aligned}
$$

Choosing $\eta \in]\theta, 1[$ and applying (4.67) for $j = 0$ we get

$$\begin{aligned}
\|e^{\theta t}\{G[f]\}_k^\delta(t,\cdot)\| &\leq \sum_{j=0}^{k} \int_0^t e^{-(\eta-\theta)(t-s)} e^{-(\tau-\theta)s} \frac{C^{k-j+1}\delta^{k-j}}{(1-\eta)^{k-j+1/(2m)}} \\
&\times \|e^{\tau s} \frac{\delta^j s^j}{j!} \partial^j f(s,\cdot))\|_{L^1}\, ds \qquad (5.82) \\
&\leq B_m(\eta-\theta, \tau-\theta) \sum_{j=0}^{k} \frac{C^{k-j+1}\delta^{k-j}}{(1-\eta)^{k-j+1/(2m)}} \\
&\times \sup_{s>0} \left(\|e^{\tau s} \frac{\delta^j s^j}{j!} \partial^j f(s,\cdot))\|_{L^1} \right) \qquad (5.83)
\end{aligned}$$

where

$$B_m(\mu,\nu) = \sup_{t>0} \left(\int_0^t \frac{e^{-\mu(t-s)} e^{-\nu s}}{(t-s)^{3/2m)}} \right) < +\infty.$$

Then we conclude as in the previous lemma

□

Similarly to the arguments in Lemma 5.2 (applied for the L^2 norm) we get

$$\begin{aligned}
S_N^\theta[G[f];\delta] &\leq B_m(\eta-\theta, \tau-\theta) \frac{C\delta}{1-\eta-C\delta} \|f\|_{\tau,1}^{\exp} \\
&+ B_m(\theta,\tau) \frac{C}{1-\eta-C\delta} [\![f]\!]_{\tau,1;\delta}^{\exp}. \qquad (5.84)
\end{aligned}$$

The proof of the lemma is complete.

□

The next assertion is the first crucial step in deriving the analytic–Gevrey estimates for large t.

Lemma 5.3 *Let $\delta > 0$, $k \in \mathbb{N}$, $\varepsilon \in]0, 1[$. Then*

$$\frac{\delta^k t^{k/m}}{k!} \partial_x^k (G_m[uu_x](t,x)) = \mathcal{N}_k^{\varepsilon;\delta,t}[u](t,x) + \mathcal{R}_k^{\varepsilon;\delta,t}[u](t,x), \qquad (5.85)$$

where

$$\begin{aligned}
\mathcal{N}_k^{\varepsilon;\delta,t}[u](t,x) &= \int_0^{(1-\varepsilon)t}\int_{\mathbb{R}} \frac{\delta^k t^{k/m}}{k!(t-s)^{(k+1)/m}} \\
&\times \mathcal{E}_m^{(k+1)}(\frac{x-y}{(t-s)^{1/m}})u^2(s,y)\,dyds, \qquad (5.86)\\
\mathcal{R}_k^{\varepsilon;\delta}[u](t,x) &= \mathcal{R}_{k,0}^{\varepsilon;\delta}[u] + \mathcal{P}_k^{\varepsilon;\delta}[u](t,x) + \mathcal{Q}_k^{\varepsilon;\delta}[u](t,x), \qquad (5.87)\\
\mathcal{R}_{k,0}^{\varepsilon;\delta}[u](t,x) &= \int_{(1-\varepsilon)t}^{t}\int_{\mathbb{R}} \left(\frac{t^{1/m}}{(t-s)^{1/m}+s^{1/m}}\right)^k \frac{\delta^k}{k!(t-s)^{1)/m}} \\
&\times \mathcal{E}_m^{(k+1)}(\frac{x-y}{(t-s)^{1/m}})u^2(s,y)\,dyds, \qquad (5.88)\\
\mathcal{P}_k^{\varepsilon;\delta}[u](t,x) &= \sum_{\alpha=1}^{k}\int_{(1-\varepsilon)t}^{t}\int_{\mathbb{R}} \left(\frac{t^{1/m}}{(t-s)^{1/m}+s^{1/m}}\right)^k \frac{\delta^{k-\alpha}}{(k-\alpha)!(t-s)^{1)/m}} \\
&\times \mathcal{E}_m^{(k-\alpha+1)}(\frac{x-y}{(t-s)^{1/m}})u(s,y)\frac{\delta^\alpha s^{\alpha/m}}{\alpha!}\partial_y^\alpha u(s,y)\,dyds, \qquad (5.89)\\
\mathcal{Q}_k^{\varepsilon;\delta}[u](t,x) &= \sum_{\alpha=2}^{k}\int_{(1-\varepsilon)t}^{t}\int_{\mathbb{R}} \left(\frac{t^{1/m}}{(t-s)^{1/m}+s^{1/m}}\right)^k \frac{\delta^{k-\alpha}}{(k-\alpha)!(t-s)^{1)/m}} \\
&\times \mathcal{E}_m^{(k-\alpha+1)}(\frac{x-y}{(t-s)^{1/m}})\sum_{\alpha_1=1}^{\alpha-1}\frac{\delta^{\alpha_1}s^{\alpha_1/m}}{\alpha_1!}\partial_y^{\alpha_1}u(s,y) \\
&\times \frac{\delta^{\alpha-\alpha_1}s^{(\alpha-\alpha_1)/m}}{(\alpha-\alpha_1)!}\partial_y^{\alpha-\alpha_1}u(s,y)\,dyds, \qquad (5.90)
\end{aligned}$$

with the convention $\mathcal{Q}_k^{\varepsilon;\delta}[u]=0$ *if* $k=1$.

Proof: We have

$$\begin{aligned}
\{F[u]\}_k^\delta(t,x) &:= \frac{\delta^k t^{k/m}}{k!}\partial_x^k(G_P[uu_x](t,x)) \\
&= \{F[u]\}_k^{\delta;1}(t,x) + \{F[u]\}_{k;2}^\delta(t,x), \qquad (5.91)
\end{aligned}$$

where

$$\begin{aligned}
\{F[u]\}_k^{\delta;1}(t,x) &:= \frac{\delta^k t^{k/m}}{k!}\int_0^{(1-\varepsilon)t}\int_{\mathbb{R}} \partial_x^{k+1}E_m(t-s,x-y)u^2(s,y)\,dyds \\
&= \int_0^{(1-\varepsilon)t}\int_{\mathbb{R}} \frac{\delta^k t^{k/m}}{k!(t-s)^{(k+1)/m}}\mathcal{E}_m^{(k+1)}(\frac{x-y}{(t-s)^{1/m}})u^2(s,y)\,dyds \\
&= \mathcal{R}_k^{\varepsilon;\delta,t}[u](t,x) \qquad (5.92)
\end{aligned}$$

and

$$\begin{aligned}\{F[u]\}_k^{\delta;2}(t,x) &= \sum_{\alpha=0}^{k}\int_{(1-\varepsilon)t}^{t}\int_{\mathbb{R}}\left(\frac{t^{1/m}}{(t-s)^{1/m}+s^{1/m}}\right)^{k}\frac{\delta^{k-\alpha}}{(k-\alpha)!(t-s)^{1)/m}} \\ &\times \mathcal{E}_m^{(k-\alpha+1)}(\frac{x-y}{(t-s)^{1/m}})\frac{\delta^{\alpha}s^{\alpha/m}}{\alpha!}\partial_y^{\alpha}(u^2(s,y))\,dy\,ds. \qquad (5.93)\end{aligned}$$

The Leibnitz rule leads to

$$\frac{1}{\alpha!}\partial_x^{\alpha}(u^2) = 2u\frac{1}{\alpha!}\partial_x^{\alpha}u+\sum_{j=1}^{\alpha-1}\frac{1}{j!}\partial^{j}u\frac{1}{(\alpha-j)!}\partial_x^{\alpha-j}u \qquad (5.94)$$

for $\alpha\geq 2$. Combining (5.93) and (5.94), we get (5.87), (5.89), and (5.90).

□

We define $C_\theta^\infty(L^p)$ as the set of all $C_\theta(L^p)$ such that

$$\sup_{t>0}\left(t^{\frac{\alpha}{m}+\theta}\|\partial^{\alpha}u(t,\cdot)\|_{L^p}\right) < +\infty, \qquad \alpha\in\mathbb{Z}_+. \qquad (5.95)$$

The next assertion plays a crucial role in the proof of global in time uniform spatial analytic regularity of solutions to the IVP for $\Omega=\mathbb{R}$.

Proposition 5.4 *Let $u\in C_\theta^\infty(L^p)$, $p\geq 2$, $q=p/(p-1)$. Then*

$$\begin{aligned}S_N[G_m[uu_x],\theta,p,\delta] &\leq C^2(q)\delta\varepsilon^{-1}Exp_m(C(q)\delta\varepsilon^{-1})\|u\|_{\theta,p}^2 \\ &\times \int_0^{1-\varepsilon}\frac{1}{(1-s)^{1/m(1+1/p)}s^{2\theta}}\,ds \\ &+ C^2(q)\frac{\delta}{1-\varepsilon}Exp_m(C(q)\frac{\delta}{1-\varepsilon})\|u\|_{\theta,p;\delta}^2 \\ &\times \int_{1-\varepsilon}^{1}\frac{1}{(1-s)^{1/(mp)}s^{2\theta}}\,ds \\ &+ \left(C(q)\|u\|_{\theta,L^p}\int_{(1-\varepsilon)}^{1}\frac{1}{(t-s)^{1/m(1+1/p)}s^{2\theta}}\,ds\right) \\ &\quad S_N^{\delta}[u;\theta,L^p] \\ &+ \left(C^2(q)\delta Exp_m(C(q)\delta)\int_{(1-\varepsilon)}^{1}\frac{1}{(1-s)^{1/m(1+1/p)}s^{2\theta}}\,ds\right) \\ &\times (S_{N-1}^{\delta}[u;\delta,\theta,L^p])^2 \qquad (5.96)\end{aligned}$$

for all $\varepsilon \in [0,1[$, $N \in \mathbb{N}$, $\delta > 0$. *In particular, we can find* $\varepsilon_0 \in]0,1[$, *such that*

$$\kappa_\varepsilon := C^2(q)\|u\|_{\theta,L^p} \int_{(1-\varepsilon)}^{1} \frac{1}{(t-s)^{1/(mp)} s^{2\theta}}\, ds \le 1/2, \qquad \varepsilon \in [0,\varepsilon_0]. \quad (5.97)$$

Proof: First, we observe that $u \in C_\theta^\infty(L^p)$ iff $S_N[u;\theta,p,\delta] < +\infty$ for all $\delta > 0$, $n \in N$. Next, we estimate the L^p norm of the LHS by the L^p norms of (5.86), (5.88), (5.89), (5.90). The summation from $k = 1$ to N and standard combinatorial arguments lead to proof of (5.96).

□

In the periodic case we do not need the decomposition of $G_m[uu_x]$ in view of the exponential decay property. We have

Proposition 5.5 *Let* $\eta \in]0,1[$, $\theta \in]0,\eta[$. *Then there exists* $C_0 > 0$ *such that*

$$S_N^{exp}[G_m[uu_x];\theta,\delta] \;\le\; \frac{C_0\delta}{1-C_0\delta} M (S_{N-1}^{exp}[u;\theta,\delta])^2 \quad (5.98)$$

for all $\delta \in]0, C_0^{-1}[$, $N \in \mathbb{N}$, *where*

$$M \;=\; \sup_{t>0} \left(\int_0^t \frac{e^{-(\eta-\theta)(t-s)}}{(t-s)^{3/(2m)}} e^{-\theta s}\, ds \right) < +\infty. \quad (5.99)$$

Proof: We derive (5.98) and (5.99) by straightforward estimates of the $L^2(\mathbb{T})$ norm of the LHS by (5.86) for $\Omega = \mathbb{T}$, taking into account that $0 < \theta < \eta < 1$.

□

6 Proof of the main results

The estimates in section 5 allow us to propose simultaneous proofs of the two main theorems.

We reduce in a standard way the IVP to the integral equation

$$u = U^0(t,x) + G_m[uu_x], \quad (6.100)$$

where

$$U^0(t,x) \quad = \quad E_m(t,\cdot) * u^0(x) + G_m[f](t,x). \tag{6.101}$$

Let $\Omega = \mathbb{R}$. We fix $\varepsilon \in]0, \varepsilon_0[$ satisfying (5.97). Then in view of (6.100), the local in time regularity in x and (5.96) combined with bootstrap type arguments lead to the existence of $C > 0$ such that

$$\begin{aligned} S_N[u;\theta,p,\delta] \quad &\leq \quad \frac{C}{1-\kappa_\varepsilon}\|U^0\|_{\theta,p;\delta} + C\delta\|u\|^2_{\theta,p;\delta} \\ &+ \quad C(S_{N-1}[u;\theta,p,\delta])^2 \end{aligned} \tag{6.102}$$

for all $N \in \mathbb{N}$, $0 < \delta \leq 1$ with the convention

$$S_0[u;\theta,p,\delta] \quad = \quad \|U^0\|_{\theta,p,\delta} + C\delta\|u\|^2_{\theta,p;\delta}. \tag{6.103}$$

Similarly, if $\Omega = \mathbb{T}$ we obtain, using Proposition 5.5, that one can find $C > 0$ such that

$$S_N^{exp}[u;\theta,\delta] \quad \leq \quad \|U^0\|^{exp}_{\theta,\delta} + C(S^{exp}_{N-1}[u;\theta,\delta])^2 \tag{6.104}$$

for $N \in \mathbb{N}$, $\delta > 0$, where

$$S_0^{exp}[u;\theta,\delta] \quad = \quad \|U^0\|^{exp}_{\theta,\delta}. \tag{6.105}$$

We have the freedom to choose $\delta > 0$ small enough and we observe that for $0 < \delta \ll 1$ the Picard type iteration inequalities (6.102) (respectively, (6.105)) imply that

$$\sup_{N\in\mathbb{N}} S_N[u;\theta,p,\delta] \quad = \quad \|u\|_{\theta,p,\delta} < +\infty \tag{6.106}$$

(respectively,

$$\sup_{N\in\mathbb{N}} S_N^{exp}[u;\theta,\delta] \quad = \quad \|u\|^{exp}_{\theta,\delta} < +\infty). \tag{6.107}$$

Therefore Theorem 2.1 (respectively, Theorem 2.2) is proved.

□

Remark 6.1 *We can show global in time estimates on $\rho_{[u]}(t)$ for solutions to some evolution equations with conservative terms but with non homogeneous dissipative parts, like generalizations of Kuramoto–Sivashinski equations (see [2], [4] and the references therein). However the validity of (1.9) remains an open problem for such equations.*

7 Sharp estimates on the radius of the analyticity for Burgers' equation

The aim of this section is to investigate the spatial analyticity of some explicit solutions to Burgers' equation. We recall that by the Hopf-Cole formula the solution to the IVP

$$\begin{aligned} \partial_t u + \partial_x(u^2/2) - u_{xx} &= 0, \qquad x \in \mathbb{R},\, t > 0 \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R} \end{aligned} \tag{7.108}$$

is given by

$$u(t,x) = -2\partial_x(\ln v(t,x)) = -2\frac{\partial_x v(t,x)}{v(t,x)}, \qquad t > 0, x \in \mathbb{R} \tag{7.109}$$

$$v(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y)\, dy, \qquad t > 0, x \in \mathbb{R} \tag{7.110}$$

where

$$v_0(y) = e^{U_0(y)}, \qquad U_0'(y) = u_0(y) \tag{7.111}$$

satisfying $U_0(y) = o(y^2)$ as $y \to \infty$. We recall the well known fact that in that case $v(t,x) = e^{t\Delta}v_0$ extends to an entire function in $x \in \mathbb{C}$ for positive times. Hence, the question of the radius of the analyticity $\rho(t)$ is reduced to the study of the zero(s) of the entire function $v(t,x)$, $x \in \mathbb{C}$ for $t > 0$, namely

$$\rho(t) = \min\{|\mathrm{Im} z(t)|; z \in \mathbb{C},\, v(t,z) = 0\}, \qquad t > 0. \tag{7.112}$$

.

We choose u^0 to be a multiple of the Dirac function massed at the origin, i.e.,

$$u_0 = c\delta(x), \qquad c \in \mathbb{R} \setminus 0. \tag{7.113}$$

We introduce the function

$$D(z) = \frac{1}{\pi}\int_z^{+\infty} e^{-y^2}\,dy, \qquad z \in \mathbb{R}. \tag{7.114}$$

Straightforward arguments imply that $D(z)$ is an entire function in $\mathbb{C}$, $0 < D(\xi) < 1$, $\xi \in \mathbb{R}$, and for every $\mu \in]-\infty, -1[\bigcap]0, +\infty[$ one can find $\sigma = \sigma(\mu)$ such that

$$D(z) + \mu \neq 0, \qquad z \in \mathbb{C},\ |\mathrm{Im} z| < \sigma. \tag{7.115}$$

We have

Proposition 7.1 *Let $u(t,x)$ be the solution of the IVP (7.108) for Burgers' equation defined by (7.109) with initial data u_0 given by (7.113). Then there exists at most one exceptional value $c_0 \in \mathbb{R}$, $c_0 \neq 0$, such that the entire function $D(z) + (e^{c_0} - 1)^{-1} \neq 0$, $z \in \mathbb{C}$, i.e., the set N_c of all $z \in \mathbb{C}$ satisfying $D(z) + (e^c - 1)^{-1} = 0$ is empty iff $c = c_0$. Moreover, $u(t,x)$ extends to a holomorphic function in the strip $|Imx| < 2d_c\sqrt{\pi t}$ provided $N_c \neq \emptyset$, where*

$$d_c = \inf\{|Imz|;\ z \in N_c\}. \tag{7.116}$$

We note that $d_c > 0$ because of (7.115). Finally, if $N_c \neq \emptyset$ and $t > 0$, the function $u(t,x)$ extends as a meromorphic function with single poles at $x = 2\kappa_c\sqrt{\pi t}$, $\kappa_c \in N_c$.

Proof. We choose as a primitive function $U^0(y) = cH(y)$, where $H(y)$ stands for the Heaviside function. Then we have

$$\begin{aligned} v(t,x) &= \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} e^{cH(y)}\,dy \\ &= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}}\,dy \\ &+ e^c\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{(x-y)^2}{4t}} dy \end{aligned} \tag{7.117}$$

Combining (7.117) with the definition of $D(z)$ we rewrite v as follows:

$$\begin{aligned} v(t,x) &= 1+(e^c-1)\frac{1}{\sqrt{\pi}}\int_{-x/(2\sqrt{\pi t})}^{+\infty} e^{-y^2}\,dy \\ &= 1+(e^c-1)D(-\frac{x}{2\sqrt{\pi t}}) \end{aligned} \tag{7.118}$$

when $c\in\mathbb{R}$, $c\neq 0$. Thus $v(t,x)=0$ for $t>0$, $x\in\mathbb{C}$ iff

$$D(-\frac{x}{2\sqrt{\pi t}})+(e^c-1)^{-1} = 0. \tag{7.119}$$

Since $D(z)$ is not a polynomial the existence of at most one c_0 follows from the great Picard theorem in complex analysis (e.g., cf. [22]). We note that $(e^c-1)^{-1}>0$ if $c>0$ while $(e^c-1)^{-1}<-1$ when $c<0$. In view of (7.115), (7.118), (7.119), (7.111) and the definition of d_c the proof is complete.

□

Remark 7.2 *One can investigate the spatial analyticity of particular family of weak solutions (used in [12] for non–uniqueness in $H^s(\mathbb{R})$, $s<-1/2$) and obtain somewhat surprising different from (1.10) asymptotic behavior of $\rho_{[u]}(t)$ for $t\to+\infty$. More precisely, let*

$$\begin{aligned} u_c(t,x) &= -2\partial_x\ln(1+v_c(t,x)) = -2\frac{\partial_x v_c(t,x)}{1+v_c(t,x)}, \qquad t>0, x\in\mathbb{R} \\ v_\sigma(t,x) &= \frac{\sigma}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}, \qquad t>0, x\in\mathbb{R} \end{aligned}$$

where c is a positive constant. For each $c>0$ the function u_c solves Burgers' equation for $t>0$, $u\in C([0,+\infty[:\ H^{-1/2-\varepsilon}(\mathbb{R}))$, $\varepsilon>0$ and $u(0,\cdot)=0$ in a weak sense cf. [12]. Then we can prove that

$$\lim_{t\to+\infty}\frac{\rho_{[u_c]}(t)}{\sqrt{t}\ln t} = \rho_c > 0 \tag{7.120}$$

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