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GENERALIZED HOPF DIFFERENTIALS

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1 Introduction

A basic tool in the theory of constant mean curvature (cmc) surfaces in space forms is the holomorphic quadratic differential discovered by Heinz Hopf. However, for more general target spaces the $(2,0)$–part of the second fundamental form of a cmc surface fails to be holomorphic.

The basic new result in [2] is that for cmc surfaces in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ holomorphicity can be restored with the help of explicit, geometrically defined correction terms.

Our generalized holomorphic quadratic differential is good enough to proceed along the lines of Hopf and prove that an immersed cmc sphere S^2 in such a product space must in fact be one of the embedded, rotationally-invariant surfaces described in the work of W. Y. and W. T. Hsiang [10] and R. Pedrosa [15], which are the simplest cmc surfaces in the product spaces. The distance spheres do not have constant mean curvature anymore.

The next step is to investigate the scope of the new construction [3]. More precisely, we ask for which class of (oriented) Riemannian 3–manifolds (M^3, g) there exists a correction field L that induces a holomorphic quadratic differential on any immersed cmc surface $\Sigma^2 \looparrowright (M^3, g)$. There is an amazingly simple necessary and sufficient condition, namely, L must satisfy a certain explicit inhomogeneous ODE-system.

Integrability for this ODE-system is by no means automatic; it rather imposes serious restrictions on the geometry of the 3–manifold. A tedious classification reveals that solutions exist if and only if (M^3, g) is a homogeneous bundle over a surface with totally-geodesic fibers. Again an analogue of Hopf's theorem can be established.

Further applications of our generalized holomorphic quadratic differential are conceivable. In particular, we expect that the theory of cmc tori can be extended to the homogeneous target spaces under consideration [1, 5, 6].

The preceding results suggest that homogeneous 3–manifolds with at least 4–dimensional isometry groups are an appropriate setting for global results about minimal surfaces and cmc surfaces. In order to test this thesis, we discuss some global properties of minimal surfaces in the Heisenberg group.

2 Classical Results for Cmc Surfaces in Space Forms

In this section we review the two most prominent classical results about surfaces with constant mean curvature. These results have been obtained in the 1950ies by A. D. Alexandrov and H. Hopf, respectively [4, 9]. Their proofs provide two very different approaches to the subject. In fact, even today it is fair to say that they represent the key ideas of the entire subject.

2.1 Alexandrov's Result.

Theorem. *Let Σ^2 be a closed embedded cmc surface in $\mathbb{R}^3$, in $\mathbb{H}^3$, or in a hemisphere $\mathbb{S}^3_+$. Then Σ^2 is a standard distance sphere.*

In other words, soap bubbles in $\mathbb{R}^3$ and $\mathbb{H}^3$ are always distance spheres. In $\mathbb{S}^3$, the same holds provided one restricts oneself to soap bubbles that are contained in a hemisphere.

Idea of the Proof. Pick a totally-geodesic (hyper-) plane that does not intersect the cmc surface Σ^2 and sweep it across that surface. In the given target spaces each of these planes H_t gives raise to an isometry, namely the reflection ϱ_t through H_t.

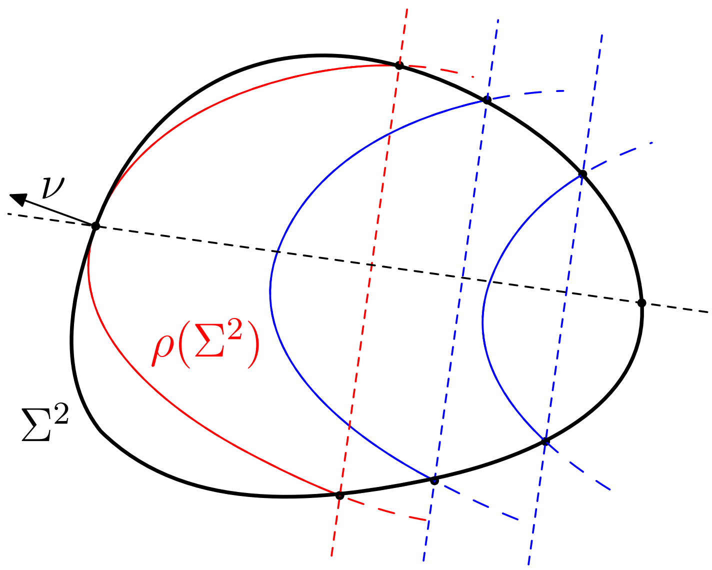

Figure 1: Alexandrov's moving planes argument.

When the plane enters the domain Ω bounded by Σ^2, one considers in addition the image $\varrho_t(\Sigma_t^2)$ of the part Σ_t^2 of the cmc surface that the planes have already swept across. Initially $\varrho_t(\Sigma_t^2)$ lies in the interior of Ω; however, it cannot stay there, since otherwise the planes H_t could not leave the compact surface Σ^2 which is absurd. So there is a first point of contact. Let p and t_0 denote this point and the corresponding parameter value in the planar sweep respectively.

Generically, the point p is contained in the interior of the mirrored part $\varrho_{t_0}(\Sigma_{t_0}^2)$, and thus the two surfaces are tangential at p with matching orientations. In a neighborhood of p, it is therefore possible to write the reflected piece as a graph over $\Sigma^2 = \partial\Omega$. Since $\varrho_{t_0}(\Sigma_{t_0}^2)$ is still contained in $\bar{\Omega}$, the underlying function u cannot change sign in a small open neighborhood of p, and so $u \equiv 0$ by the Hopf maximum principle. In other words, $\varrho_{t_0}(\Sigma_{t_0}^2)$ is itself a piece of the original surface Σ^2, and so one finds that Σ^2 is invariant under the reflection ϱ_{t_0}.

In the borderline case, where the first point of contact lies on the plane H_{t_0}, one can resort to a refined version of the maximum principle to prove that Σ^2

is invariant under ϱ_{t_0} in this case, too.

Varying the family of planes H_t, the preceding argument shows that Σ^2 is in fact invariant under the reflection through any plane through its center of mass. Hence it must consist of orbits of the orthogonal group $\mathsf{O}(3)$ generated by these reflections. But all these orbits, except for the one through the center of mass itself and possibly also the one through its antipodal point, are closed 2–manifolds.

□

The description and the figure make it amply clear why this argument is customarily referred to as Alexandrov's *moving planes argument*.

It is an extremely flexible argument that has been applied in many other contexts since. It immediately applies in the n–dimensional case, and, instead of assuming that the hypersurface has constant mean curvature, one may work with any other *elliptic* curvature function, i.e., with any curvature function that leads to a local equation satisfying the maximum principle. Examples of such curvature functions are the scalar curvature, the Gauss-Kronecker curvature, or the curvature functions defining elliptic Weingarten hypersurfaces.

In fact, the moving planes argument has even turned out to be fruitful for studying a certain kind of nonlinear elliptic equations [8].

Remark. Yet, when working in $\mathbb{S}^3$, the theorem requires the *additional hypothesis* that the cmc surface Σ^2 should be contained in a hemisphere. To understand the meaning of this additional hypothesis observe that

- each distance sphere $S^2 \subset \mathbb{S}^3$ is actually contained in a closed hemisphere $\mathbb{S}^3_+$, and
- in $\mathbb{S}^3$ itself there exist Clifford tori, i.e., cmc surfaces of genus 1. Even worse, following the ideas of Kapouleas [11, 12] one can construct cmc surfaces with arbitrarily large genus. Of course, none of these surfaces can be contained in a hemisphere, but it is possible to construct such examples in an arbitrarily small neighborhood of an equator.

2.2 Hopf's Result.

Theorem. *Let S^2 be an immersed sphere in $\mathbb{R}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3$ with constant mean curvature. Then S^2 is a standard distance sphere.*

This theorem differs from Alexandrov's theorem in two ways. First of all it is about *immersed spheres* rather than *closed embedded surfaces*. Secondly, in the case that the target space is the 3–sphere, there is no additional hypothesis requiring the surface to lie in a closed hemisphere.

Remark. For many years it had been an open question whether — at least for surfaces in euclidean space — the results of Alexandrov and Hopf might be special cases of a more general theorem. However, in 1984 H. W. Wente [18] showed that there actually exist immersed cmc tori in $\mathbb{R}^3$.

Ingredients in the Proof. The key step in Hopf's approach is to realize that for any immersed cmc surface Σ^2 the *Codazzi equations* imply that the $(2,0)$–part $Q := \pi_{2,0}(h_\Sigma)$ of the second fundamental form $h_\Sigma = \langle\, .\, , A\, .\,\rangle$ is a *holomorphic* quadratic differential on the surface.

On the other hand, it is a standard fact that a holomorphic quadratic differential on $\mathbb{S}^2 = \mathbb{CP}^1$ vanishes.

The upshot is that Q must vanish on any immersed cmc sphere in a space of constant curvature. Expanding the definition of Q, one finds that

$$\begin{aligned} Q(Y_1, Y_2) \;&=\; \tfrac{1}{4}\cdot\big(h_\Sigma(Y_1,Y_2) - h_\Sigma(J\,Y_1, J\,Y_2)\big) \\ &\qquad -\tfrac{1}{4} i\cdot\big(h_\Sigma(J\,Y_1, Y_2) + h_\Sigma(Y_1, J\,Y_2)\big)\;, \end{aligned}$$

and thus the identity $Q = 0$ is equivalent to saying that the traceless part of h_Σ vanishes. And complete, *totally-umbilical* surfaces Σ^2 in space forms like $\mathbb{R}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3$ are known to be distance spheres.

□

Remark. The preceding proof is very different from the proof of Alexandrov's theorem. In fact, because of the identity $\bar\partial\, Q = 0$ one may regard the quadratic differential Q as a *family of first integrals* for the cmc equation.

3 Straightforward Generalizations

The purpose of this section is to explain which of the facts about cmc surfaces in space forms can be extended in a straightforward manner to cmc surfaces in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. It will be helpful to unify notation and write M^2_κ as a shorthand for the simply-connected surface of constant curvature κ. Writing formulas for surfaces in $M^2_\kappa \times \mathbb{R}$ is not only shorter than listing the two cases separately; it also helps in understanding how things scale and what kind of limits can possibly occur.

First, in these spaces distance spheres do not have constant mean curvature anymore. However, there still exist rotationally-invariant cmc spheres, and so they are used as *standard comparison objects* instead. Their properties are described in the first subsection. In the second subsection we then explain how Alexandrov's result can be extended to cmc surfaces in these product spaces.

3.1 Rotationally-Invariant Cmc Spheres S^2_H in the Product Spaces $M^2_\kappa \times \mathbb{R}$.

The *meridian curve* $c(s) = (r(s), \xi(s))$ of any rotationally-invariant cmc surface can be obtained solving the following ODE-system:

$$\begin{aligned} \tfrac{\partial}{\partial s}\, r &= -\sin(\theta) \\ \tfrac{\partial}{\partial s}\, \xi &= \cos(\theta) \\ \tfrac{\partial}{\partial s}\, \theta &= 2H - \cos(\theta)\cdot \mathrm{ct}_\kappa(r) \end{aligned} \tag{1}$$

Here the pair (r, ξ) denotes the standard coordinate functions on the orbit space, which is either $[0, \pi/\sqrt{\kappa}] \times \mathbb{R}$ or $[0, \infty) \times \mathbb{R}$ depending on whether the first factor of the target space is a sphere or a hyperbolic plane. The θ–variable can be interpreted as a partial Gauss map; the vector field $(\cos\theta, \sin\theta)$ is precisely the unit normal vector field of the meridian curve.

The function ct_κ that appears in the expression for $\frac{\partial}{\partial s}\theta$ is the *generalized cotangent function*, i.e., the solution of the Riccati equation $\mathrm{ct}'_\kappa = -\kappa - \mathrm{ct}^2_\kappa$ that has a pole at $s = 0$.

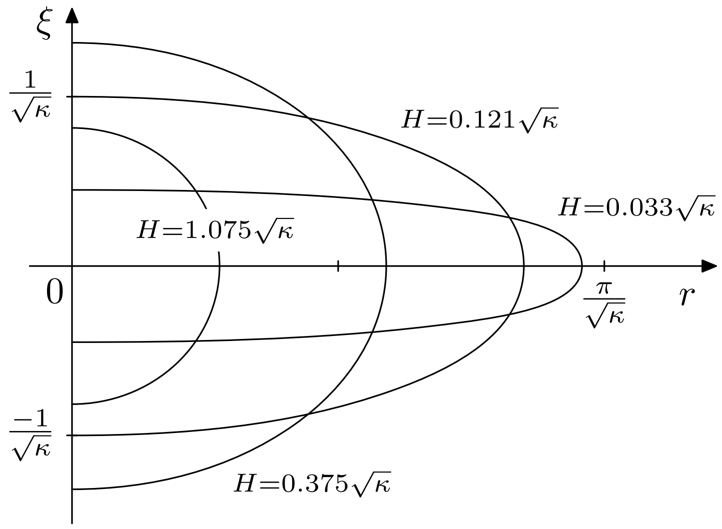

Figure 2: Meridian curves for cmc spheres in $\mathbb{S}^2 \times \mathbb{R}$

A first integral.

The ODE-system for the meridian curves of rotationally-invariant cmc surfaces given above is invariant w.r.t. translations in the ξ–direction. For this reason it has the following first integral:

$$I_{\kappa,H} := \cos(\theta) \cdot \mathrm{sn}_\kappa(r) - 4H \cdot \mathrm{sn}_\kappa(\tfrac{1}{2}r)^2$$

In fact, this expression was already known to the Hsiang brothers [10] when they did their work on soap bubbles in products of euclidean and hyperbolic spaces in 1989.

Note that the meridian curve $c(s)$ *intersects* the boundary of the orbit space, which is the projection of the fixed point set of the given 1–parameter group of rotations, if and only if the *first integral* $I_{\kappa,H}$ *vanishes* or, in case $\kappa > 0$, also if $I_{\kappa,H} = -4H/\kappa$.

Explicit solutions.

The ODE-system (1) actually has enough first integrals in order to describe the meridian curve as the level set of a function that resembles the standard quadric:

$$1 \;=\; (4H^2+\kappa)\;\cdot \mathrm{sn}_\kappa^2\big(\tfrac{1}{2}r\big) +\; 4H^2\;\cdot \mathrm{sn}_{-\kappa}^2\big(\tfrac{1}{2}\,\xi\,\sqrt{1+\tfrac{\kappa}{4H^2}}\,\big) \tag{2}$$

Here sn_κ denotes the generalized sine function, i.e., the solution of the differential equation $\mathrm{sn}_\kappa'' + \kappa\,\mathrm{sn}_\kappa = 0$ with initial data $\mathrm{sn}_\kappa(0) = 0$ and $\mathrm{sn}_\kappa'(0) = 1$.

To help with intuition, we specialize this equation to the case $\kappa = 1$ and rewrite all the occurrences of the generalized sine function in terms of its classical counterparts sin and sinh.

$$1 \;=\; (1+4H^2)\;\cdot \sin^2\big(\tfrac{1}{2}r\big) +\; 4H^2\;\cdot \sinh^2\big(\tfrac{1}{2}\,\xi\,\sqrt{1+\tfrac{1}{4H^2}}\,\big)$$

It is also easy to check that for $\kappa = 0$ equation (2) indeed boils down to the classical quadric $1 = H^2\cdot(r^2+\xi^2)$.

Observation. The cmc spheres S_H^2 with $0 < 4H^2 < \kappa$ are *not* contained in the product of a closed hemisphere and the real axis.

Principal curvatures.

In each product space $M_\kappa^2\times\mathbb{R}$ there still exists a totally-geodesic slice for the action of the 1–parameter group of rotations that preserves the cmc surface. Hence the principle directions are the tangent vectors to the meridians and the circles of latitude, respectively. W.r.t. this basis the second fundamental form is given by

$$h_\Sigma = \begin{pmatrix} H+\frac{\kappa}{4H}\cdot\cos^2(\theta) & 0 \\ 0 & H-\frac{\kappa}{4H}\cdot\cos^2(\theta)\end{pmatrix}$$

Thus the spheres S_H^2 in the product spaces are *not totally-umbilical.* The linear combinations $2H\cdot h_\Sigma - \kappa\cdot \mathrm{d}\xi^2$, however, are multiples of the induced metric $\iota^\star g$.

3.2 Alexandrov's Result for Cmc Surfaces in the Product Spaces $M_\kappa^2\times\mathbb{R}$.

The moving planes argument used in the proof of Alexandrov's theorem carries over verbatim to cmc surfaces embedded into the product spaces $M_\kappa^2\times\mathbb{R}$. The argument proves less though, since the product spaces admit fewer reflections; all planes of reflection are either vertical or horizontal.

Theorem. *Let* Σ^2 *be a closed embedded cmc surface in* $\mathbb{H}^2 \times \mathbb{R}$ *or* $\mathbb{S}^2_+ \times \mathbb{R}$*. Then* Σ^2 *is a rotationally-invariant vertical bi-graph.*

In other words, here the conclusion is that the surface Σ^2 is one of the rotationally-invariant cmc spheres S^2_H described in the preceding subsection.

Remark. Closed embedded cmc surfaces $\Sigma^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}$ that do not project into some hemisphere $\mathbb{S}^2_+$ are *only* guaranteed to be *vertical bi-graphs.*

Caveat. In $\mathbb{S}^2 \times \mathbb{R}$, there again *exist* embedded cmc tori and embedded cmc surfaces of higher genus. In other words, the restriction to cmc surfaces that project into a hemisphere is again an essential hypothesis. However, not all of the rotationally-invariant cmc spheres $S^2_H \subset \mathbb{S}^2 \times \mathbb{R}$ do project into hemispheres, and so this restriction is *highly undesirable.*

4 New Results for Cmc Surfaces in the Product Spaces $M^2_\kappa \times \mathbb{R}$

The theorems presented in this section have been obtained in cooperation with Harold Rosenberg from Paris 7 [2]. Our principal contribution is to introduce a holomorphic quadratic differential along the lines of Hopf's work for cmc surfaces in these more general target spaces. Based on this result we then establish the analogue of Hopf's classification of immersed cmc spheres in the product spaces.

4.1 Obstacles for Generalizing the Holomorphic Quadratic Differential.

In fact, there are *two obstacles* that are commonly mentioned when it comes to extending Hopf's holomorphic quadratic differential to cmc surfaces in more general target spaces.

First, for target manifolds (M^3, g) other than space forms, the r.h.s. of the *Codazzi equations*

$$\langle \nabla_X A\cdot Y - \nabla_Y A\cdot X\,, Z\rangle = \langle R(X,Y)\,\nu\,, Z\rangle\ .$$

does *not vanish* anymore. Here ν and $A = D\nu$ denote the unit normal field and the Weingarten map of the cmc surface, respectively. ∇ and D denote the Levi-Civita connections of the surface and the 3–manifold, respectively. As usual, $\nabla_X Y = (\mathrm{D}_X Y)^{tan}$. Thus we find that $\bar\partial(\pi_{2,0}(h_\Sigma))$ does *not vanish* for all cmc surfaces in the products $M^2_\kappa \times \mathbb{R}$ anymore.

The second issue is that the rotationally-invariant cmc spheres S^2_H in the product spaces $M^2_\kappa \times \mathbb{R}$ are *not totally-umbilical* as explained in Subsection 3.1. In particular, $\pi_{2,0}(h_\Sigma)$ cannot be holomorphic on any of the spheres S^2_H, which puts the problems with the Codazzi equations into a somewhat different light.

4.2 Main Results.

Inspecting the formulas from Subsection 3.1 more closely, one discovers one *encouraging fact* though: For each sphere S^2_H the $(2,0)$-part Q of the field $q := 2H\,h_\Sigma - \kappa\,\iota^\star(\mathrm{d}\xi^2)$ vanishes. The important observation here is that q is a linear combination of h_Σ and $\iota^\star(\mathrm{d}\xi^2)$ with *constant* coefficients.

So there is hope that Q *may be holomorphic* on all cmc surfaces in the product spaces $M^2_\kappa \times \mathbb{R}$, and this indeed works out:

Theorem 1. *Let $(\kappa, H) \neq 0$, and let $L := \mathrm{d}\xi^2$ be the symmetric bilinear form corresponding to the vertical projectors in $M^2_\kappa \times \mathbb{R}$. Then the expression*

$$Q := 2H\cdot\pi_{2,0}(h_\Sigma) - \kappa\cdot\pi_{2,0}(\iota^\star L)\ .$$

defines a natural holomorphic quadratic differential on any immersed cmc surface $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ with mean curvature H.

The proof of this theorem is essentially a direct computation, though a much more elaborate one than in the case of constant curvature target spaces.

In Subsection 4.3 we explain in more detail what the basic ingredients are and give a structural argument why things actually work out.

As in H. Hopf's work, Theorem 1 is the *key to* classifying immersed cmc spheres:

Theorem 2. *Any immersed cmc sphere S^2 in a product space $M^2_\kappa \times \mathbb{R}$ is congruent to one of the embedded, rotationally-invariant cmc spheres S^2_H described in Subsection 3.1.*

The proof of this theorem closely follows the argument in the classical case that has been described in Subsection 2.2. Again the starting point is to combine Theorem 1 with the fact that the space of holomorphic quadratic differentials on $\mathbb{S}^2 = \mathbb{CP}^1$ is trivial. In order to finish the argument, it suffices to classify cmc surfaces with $Q \equiv 0$.

Theorem 3. *Let $(\kappa, H) \neq 0$, and let $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ be a complete surface with constant mean curvature H and vanishing holomorphic quadratic differential Q. Furthermore, let $\theta := \arcsin(\mathrm{d}\xi \cdot \nu)$. Then the following holds*:

- *if $\kappa + 4H^2 > 0$, then the surface Σ^2 is congruent to one of the embedded, rotationally-invariant cmc spheres S^2_H described in Subsection 3.1.*
- *if $\kappa + 4H^2 \leq 0$, then Σ^2 is a complete* open *surface. Depending on the sign of the function $4H^2 + \kappa \cos^2(\theta)$, it is either congruent to a disk-like surface D^2_H or a particular parabolic surface P^2_H or a surface C^2_H of catenoidal type.*

Remark. The disk-like cmc surfaces D^2_H and the cmc surfaces C^2_H of catenoidal type are *rotationally-invariant*. They are homeomorphic to disks or annuli, respectively. The parabolic cmc surfaces P^2_H on the other hand are orbits under suitable 2-dimensional *solvable* subgroups $\mathsf{A}\,\mathsf{N} \subset \mathsf{SO}^+(2,1) \times \mathbb{R}$.

For the purposes of this survey we refrain from giving a precise definition of the noncompact cmc surfaces with vanishing holomorphic quadratic differential Q. We rather illustrate their meridian curves in Figures 3–6.

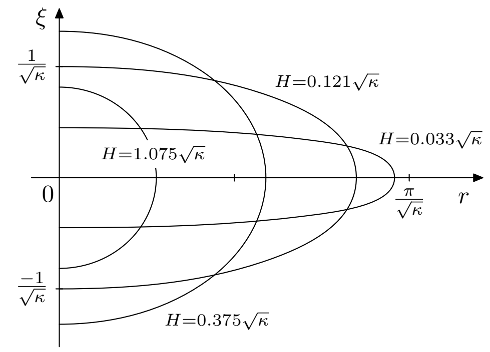

Figure 3: Meridians of the cmc spheres S^2_H in case $\kappa > 0$.

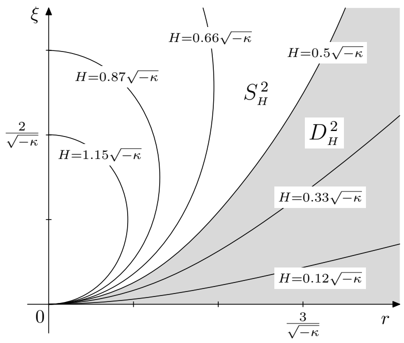

Figure 4: Meridians of several cmc spheres S^2_H and disk-like cmc surfaces D^2_H when $\kappa < 0$.

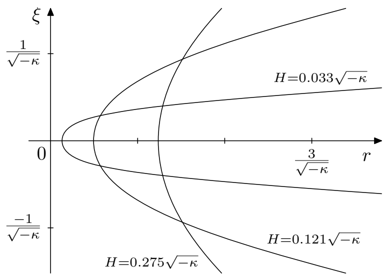

Figure 5: Meridians of the cmc surfaces C^2_H of catenoidal type.

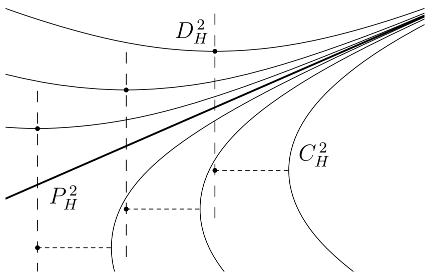

Figure 6: The parabolic cmc surfaces P^2_H come as limits.

4.3 On the Proof of Theorem 1.

We begin explaining the three basic ingredients used in various computations throughout the proof.

First of all, we need a way to compute the $\bar\partial$*–operator* on the space of quadratic differentials on an oriented Riemann surface $(\Sigma^2, \iota^\star g)$. Since the almost complex structure J on such a surface is *parallel*, the $\bar\partial$-operator can be expressed in terms of the Levi-Civita connection ∇ as follows:

$$\begin{aligned}\bar\partial\, Q(X; Y_1, Y_2) \;&=\; \tfrac{1}{2}\big(\nabla_X\, Q + i\cdot \nabla_{JX}\, Q\big)(Y_1, Y_2)\\ &=: \;\nabla_{\frac{1}{2}(1+iJ)X}\, Q\,(Y_1, Y_2)\ .\end{aligned}$$

Note that the preceding formula is not the definition of the $\bar\partial$–operator. It even fails for complex manifolds when the hermitean metric under consideration is not Kaehlerian; this does not happen in the case of Riemann surfaces though.

Decomposing the Weingarten map A as $A = H \cdot \mathbb{1} + A_0$, where A_0 denotes the traceless part, turns out to be extremely useful, since on surfaces traceless symmetric endomorphisms *anti-commute* with the almost complex structure J.

Finally, the *Codazzi equations* play a key role. For surfaces Σ^2 in 3–manifolds the curvature term on their r.h.s. can be factored through the curvature ellipsoid and can thus be expressed in terms of the Einstein tensor G:

$$\begin{aligned} &\langle \nabla_X A \cdot Y - \nabla_Y A \cdot X \,, Z \rangle \\ &\qquad = \langle X \times Y \,, G\,(\nu \times Z) \rangle \;=\; \langle (X \times Y) \times Z \,, G\,\nu \rangle \;. \end{aligned}$$

Since $\nu \perp X, Y, Z$, the final simplification step on the r.h.s. is a consequence of the following identity for cross products:

$$G\,(X \times Y) \;=\; \mathrm{tr}(G) \cdot X \times Y - (G\,X) \times Y - X \times (G\,Y) \;.$$

The key steps in the argument.

a) The Codazzi equations imply that

$$\bar\partial\big(\pi_{2,0}(h_\Sigma)\big)(X; Y_1, Y_2) \;=\; \langle\, \psi(X; Y_1, Y_2) \,,\, G\,\nu \,\rangle$$

where

$$\begin{aligned} &\psi(X; Y_1, Y_2) := \tfrac{1}{2} \left[\langle X^-, Y_1^+ \rangle\, Y_2^+ + \langle X^-, Y_2^+ \rangle\, Y_1^+ \right] \,, \\ &X^- := \tfrac{1}{2}\,(1 + iJ)\,X \;, \qquad\qquad Y_\mu^+ := \tfrac{1}{2}\,(1 - iJ)\,Y_\mu \;. \end{aligned}$$

Even without going into all the details of the computation, one can see that ψ is a trilinear map of type $(2,1)$ that depends just on the metric and the almost complex structure J. It seems worthwhile to point out that the space of such maps is just 1–dimensional. This readily yields the claimed formula for $\bar\partial\big(\pi_{2,0}(h_\Sigma)\big)$ — at least up to a constant factor.

b) In order to compute the $\bar\partial$–derivative of the second term which is the pullback of the field $L := \mathrm{d}\xi^2$ of vertical projectors, it suffices to express the covariant derivative ∇ on the surface Σ^2 in terms of the covariant derivative D of the

3-manifold and split the Weingarten map as $A = H \cdot \mathbb{1} + A_0$:

$$\bar\partial\bigl(\pi_{2,0}(\iota^\star L)\bigr)(X; Y_1, Y_2)$$
$$= \langle\, Y_1^+ \,, \mathrm{D}_{(X^-)}\, L \cdot Y_2^+ \,\rangle - 2H \cdot \langle\, \psi(X; Y_1, Y_2) \,, L\,\nu \,\rangle$$

The same reasoning as in (a) can be used in order to see without much computation that the terms involving the mean curvature H constitute a multiple of the second term on the r.h.s. of the preceding formula. Moreover, it is possible to argue that there does not exist any trilinear map of type $(2,1)$ that depends linearly on a traceless symmetric endomorphism, hence the absence of terms involving A_0.

c) For the product spaces $M^2_\kappa \times \mathbb{R}$ the field of vertical projectors is parallel, i.e., $\mathrm{D}\,L = 0$. Furthermore, because of the way in which the Einstein tensor G describes the curvature ellipsoid, it is clear that $G = -\kappa \cdot L$. With this additional information, it is evident that the linear combination of $\bar\partial\bigl(\pi_{2,0}(h_\Sigma)\bigr)$ and $\bar\partial\bigl(\pi_{2,0}(\iota^\star L)\bigr)$ that expresses the $\bar\partial$–derivative of Q evaluates to zero.

$\square$

A more conceptual point of view.

As explained in (a) and (b), the key terms in the expressions for the $\bar\partial$–derivatives of $\pi_{2,0}(h_\Sigma)$ and $\pi_{2,0}(\iota^\star L)$ are already determined by *representation theory* up to some universal complex-valued factors. This argument readily implies that both these terms are multiples of $\langle\, \psi(X; Y_1, Y_2) \,, L\,\nu \,\rangle$. Thus there is a *fixed* linear combination of $\pi_{2,0}(h_\Sigma)$ and $\pi_{2,0}(\iota^\star L)$ that is holomorphic on *all* cmc surfaces in $M^2_\kappa \times \mathbb{R}$ whose mean curvature equals the number used in the definition of Q.

By construction the quadratic differential Q itself vanishes identically on the rotationally-invariant cmc spheres S^2_H described in Subsection 3.1, and so $\bar\partial\, Q \equiv 0$, too. The tensor field $\langle\, \psi(X; Y_1, Y_2) \,, L\,\nu \,\rangle$ on the other hand does not vanish identically on these cmc spheres. This *identifies* Q as the linear combination that is holomorphic on all cmc surfaces.

4.4 On the Proof of Theorem 3.

Here the basic idea is to *prolong* and work with the unit normal field

$$\nu\colon \Sigma^2 \looparrowright N^5_\kappa := T_1(M^2_\kappa \times \mathbb{R}) \ .$$

The formula $\iota = \pi \circ \nu$ can then be used to recover the immersion itself.

In this setting the problem of classifying cmc surfaces with vanishing holomorphic quadratic differential boils down to studying integral surfaces of some explicitly given 2–dimensional distribution in the tangent bundle of a 5–manifold. More precisely,

Lemma. *Immersions ι with constant mean curvature H and $Q \equiv 0$ correspond to maps $\nu\colon \Sigma^2 \to N^5_\kappa$ that are integral surfaces of some 2-dimensional distribution $E_H \subset TN^5_\kappa$.*

In fact, this lemma is just a slightly unusual way of writing the fundamental equations of submanifold geometry.

Observation. The distribution E_H is *invariant* under the action of the isometry group $\mathrm{Iso}_0(M^2_\kappa \times \mathbb{R})$ of the product space. This action has 4-*dimensional orbits* that are *separated* by the invariant function

$$\begin{aligned} \Theta\colon N^5_\kappa &\to \left[\, -\tfrac{1}{2}\pi \,,\, \tfrac{1}{2}\pi \,\right] \\ v &\mapsto \arcsin(\mathrm{d}\xi \cdot v) \ . \end{aligned}$$

In particular, it is sufficient to analyze for *one* point p in each fiber whether or not there exists an integral surface of E_H through p and, if so, to determine this integral surface.

Of course, in general Θ will not be constant along such an integral surface. The *range* of this function can easily be studied with the help of the integral curves $s \mapsto c(s)$ of the component of $\operatorname{grad}\xi$ that is perpendicular to ν. The explicit formulas for E_H reveal that the function $\theta\colon s \mapsto \Theta \circ c(s)$ satisfies the following differential equation:

$$\tfrac{\partial}{\partial s}\,\theta \;=\; \tfrac{1}{4H}\left(4H^2 + \kappa\,\cos^2(\theta)\right) \ .$$

Analyzing this differential equation, it is not hard to determine the integral surfaces of E_H explicitly. As explained above, the various congruence classes are characterized by the corresponding ranges of Θ. The corresponding meridian curves have been depicted in Figures 3–6.

Since the ranges discovered in the preceding step cover the entire interval $[-\frac{1}{2}\pi\,,\frac{1}{2}\pi\,]$, it follows a posteriori that E_H *is integrable* everywhere.

5 Further Generalizations

Next we investigate the *scope of the construction* introduced in the preceding section. In particular, we ask: For which (orientable) Riemannian 3–manifold (M^3,g) does there exist a correction field L that induces a holomorphic quadratic differential on any immersed cmc surface. In this generality, it is of course no longer possible to define the correction field L by means of an explicit expression.

Theorem 4. *Fix some constant $H\in\mathbb{R}$. Let (M^3,g) be an oriented Riemannian manifold, and let L_0 be a $\mathbb{C}$–valued, traceless, symmetric bilinear form on M^3. Then the expression*

$$Q \;:=\; \pi_{2,0}(h_\Sigma + \iota^\star L_0)$$

defines a holomorphic *quadratic differential on any surface $\iota\colon\Sigma^2 \looparrowright (M^3,g)$ with constant mean curvature H, if and only if L_0 solves the differential equation*

$$\mathrm{D}_X\, L_0 \;=\; \tfrac{1}{2}\, i\cdot\bigl[\star X, G-2H\,L_0\bigr]\;. \tag{$*$}$$

Here the square brackets denote the commutator, and $\star X$ stands for the skew-symmetric endomorphism $Y\mapsto X\times Y$ induced by the cross-product.

Remark. Focusing on traceless fields L_0 *does not restrict* the class of quadratic differentials Q. It is a mere normalization, as the projector $\pi_{2,0}$ clearly annihilates all multiples of the induced metric $\iota^\star g$ on the surface Σ^2.

The preceding theorem should *not make one expect to find holomorphic quadratic differentials* for cmc surfaces in a generic Riemannian 3–manifold. First, thinking about 3–manifolds with bumpy metrics, it is absurd to expect getting any kind of uniqueness result for minimal surfaces quite in contrast to the rigidity that conceivably follows analyzing the holomorphic quadratic differential in more detail.

On a more technical basis, the ODE-system $(*)$ is strongly *overdetermined*. So, one should expect that the corresponding integrability conditions impose serious *restrictions on the geometry* of the underlying 3–manifold (M^3, g).

Theorem 5. *Let* $(\tilde{M}^3, g)$ *be a simply-connected, oriented Riemannian manifold, and let* $H \in \mathbb{R}$ *be some real constant. Then equation* $(*)$ *is solvable if and only if* $(\tilde{M}^3, g)$ *is a* homogeneous space *with an at least* 4*-dimensional isometry group.*

Recall that homogeneous Riemannian 3–manifolds $(\tilde{M}^3, g)$ come with 6–, 4–, or 3–dimensional isometry groups. The ones with 6*–dimensional isometry groups* are the space forms.

Observe that all simply-connected, *homogeneous* 3*–manifolds with* 4*–dimensional isometry groups* admit natural equivariant Riemannian submersions with 1–dimensional, totally-geodesic fibers. They are classified up to isometry by the curvature κ of the quotient surface and the bundle curvature τ of these submersions. The range of this invariant is the entire plane except for the curve $\kappa = 4\tau^2$ which corresponds to spaces of constant curvature. In this family one distinguishes six different *homogeneous structures*:

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_{\text{Berger}}$	$\mathsf{Nil}(3)$	$\widetilde{\mathsf{Sl}}(2, \mathbb{R})$

The first row consists of the *product spaces* $M^2_\kappa \times \mathbb{R}$. When discussing these spaces in Section 4, we have ignored the case $\kappa = \tau = 0$, as it boils down to euclidean 3–space with the standard flat metric; only the automorphisms are restricted to isometries that preserve the splitting as $\mathbb{R}^2 \oplus \mathbb{R}$. In the second row one encounters 3 *new classes of simply-connected Riemannian* 3*–manifolds*: the Berger spheres $\mathbb{S}^3_{\text{Berger}}$, the Heisenberg group $\mathsf{Nil}(3)$, and the universal covering of $\mathsf{Sl}(2,\mathbb{R})$. These are the target spaces where the existence of a holomorphic quadratic differential Q on immersed cmc surfaces has not been known beforehand.

The isometry group of the Berger spheres is an index-2 extension of the unitary group $\mathsf{U}(2)$ contained in $\mathsf{O}(4)$. In the other two cases, however, we are dealing with maximal homogeneous structures. Altogether, with Theorem 5 we have constructed holomorphic quadratic differentials for cmc surfaces in homogeneous 3–manifolds corresponding to 7 of the eight maximal homogeneous structures that appear in Thurston theory [16, 17]. Only $\mathsf{Solv}(3)$ is missing; the reason is that it only admits a 3–dimensional isometry group.

Remark. Inspecting the proof of Theorem 5, one finds that equation $(*)$ always admits a *homogeneous solution* L_0, i.e., a solution that is invariant under the action of the full isometry group $\mathrm{Iso}_0(\tilde{M}^3, g)$ of the homogeneous space. This solution is necessarily a multiple of the traceless Einstein tensor G_0.

Following the argument from the proof of Theorem 3, it is possible to classify the cmc surfaces on which the holomorphic quadratic differential Q corresponding to these homogeneous solutions L_0 vanishes identically. As a result, we can generalize Hopf's result even further:

Theorem 6. *Any immersed cmc sphere S^2 in a simply-connected homogeneous space $(\tilde{M}^3, g)$ with an at least* 4*–dimensional isometry group is in fact an embedded, rotationally-invariant cmc sphere.*

For the proofs of all 3 theorems presented in this section we refer the reader to the forthcoming paper [3].

Remark. There are special situations where the ODE-system $(*)$ has other solutions L_0 than the homogeneous ones. This occurs for instance in hyperbolic 3–space $\mathbb{H}^3$ when studying cmc surfaces whose mean curvature H equals the mean curvature of the horospheres. In the literature these surfaces are referred to as *Bryant surfaces.* They are known to have more than one nontrivial holomorphic quadratic differential [7].

Discussion of the results.

In contrast to the 5 symmetric spaces, all three bundle geometries $\mathbb{S}^3_{\text{Berger}}$, $\mathsf{Nil}(3)$, and $\widetilde{\mathsf{Sl}}(2,\mathbb{R})$ are *not* locally-conformally flat. Thus one cannot say that the additional symmetries of the Willmore functional are responsible for obtaining holomorphic quadratic differentials on immersed cmc surfaces. On the other hand, it is also not correct to believe that the existence of an at least 2–dimensional Ricci eigenspace is the distinctive geometric property. This time, the problem is that the standard metric on $\mathsf{Solv}(3)$ has a double Ricci eigenvalue, too.

However, it seems natural to think of the holomorphic quadratic differential Q constructed in Theorems 4 and 5 as a *family of first integrals for the cmc equation* that is due to the 1–dimensional isotropy groups of the bundle geometries and the 3–dimensional isotropy groups of the space forms, respectively.

Observation. The *isotropy group* G_p of any point p in a simply-connected, homogeneous 3–manifold $(\tilde{M}^3, g)$ with a 4–dimensional isometry group contains the 180°–rotations around all horizontal geodesics through p.

In fact, G_p is isomorphic to the orthogonal group $\mathsf{O}(2) \subset \mathsf{SO}(T_p\tilde{M}^3)$ generated by these rotations, provided that the bundle curvature τ is nonzero. Otherwise, $(\tilde{M}^3, g)$ is a product space, and G_p is isomorphic to the slightly larger group $\mathsf{O}(2) \times \mathsf{O}(1) \subset \mathsf{O}(T_p\tilde{M}^3)$.

This simple observation has a lot of impact for the global theory of minimal surfaces in this class of homogeneous spaces.

Corollary (Schwarz symmetrisation). *In homogeneous bundles with* 4*–dimensional isometry groups, it is possible to extend any minimal surface* Σ^2*, whose boundary consists only of horizontal and vertical edges, to a possibly immersed global minimal surface* $\hat{\Sigma}^2$ *consisting of patches congruent to* Σ^2*.*

As usual, the basic patch Σ^2 can be obtained by first constructing an appropriate boundary polygon and then solving Plateau's problem for this contour.

In our opinion, the principal results presented in this section, i.e., Theorems 5 and 6 and the Corollary, strongly *suggest* that the homogeneous 3–manifolds with at least 4–dimensional isometry groups are the proper setting for studying global properties of minimal surfaces and cmc surfaces.

6 Minimal Surfaces in the Heisenberg Group $\mathsf{Nil}(3)$

In order to test the thesis at the end of the preceding section, we have started investigating global properties of minimal surfaces in the Heisenberg group $\mathsf{Nil}(3)$. As in the previous section, we only consider left-invariant Riemannian metrics. These metrics come in a 1–parameter family g_τ that is naturally indexed by bundle curvature.

In the class of inner metric spaces this family has a limit with very special properties, which shows up naturally in many contexts in analysis and geometry. However, it is a Carnot-Caratheodory metric rather than a Riemannian metric. In this section we restrict the discussion to the Riemannian case.

6.1 Equivariant Examples.

Equivariant minimal surfaces in the Heisenberg group, i.e., complete minimal surfaces $\Sigma^2 \subset (\mathsf{Nil}(3), g_\tau)$ that are invariant w.r.t. some 1–parameter subgroups of isometries, have been *classified* in a paper by Ch. Figueroa, F. Mercuri, and R. Pedrosa [14]. There are 4 distinct classes of 1–parameter subgroups in $\mathrm{Iso}_0(\mathsf{Nil}(3))$, a group that comes with a canonical homomorphism onto the group of motions $\mathrm{Iso}_0(\mathbb{R}^2)$ in the quotient plane $\mathbb{R}^2 = \mathsf{Nil}(3)/\mathrm{Center}$. The list of the corresponding *equivariant minimal surfaces* is as follows:

1. Vertical Planes:

   These are the total preimages of straight lines, and thus they are invariant under *vertical translations*. In fact, each vertical plane is the orbit of a 2–dimensional abelian subgroup of $\mathsf{Nil}(3)$.

2. Catenoids and Horizontal Umbrellas:
   These are the minimal surfaces in $\mathsf{Nil}(3)$ that are invariant under the group ϕ_t of *rotations* around some vertical axis.

3. Helicoids and Helicoidal Catenoids:
   These are the minimal surfaces in $\mathsf{Nil}(3)$ that are invariant under some group ϕ_t of *screw motions* with a vertical axis.

4. Saddle-Type Surfaces:
   These are the minimal surfaces in $\mathsf{Nil}(3)$ other than the vertical planes that are invariant under a group ϕ_t of isometries that projects to a 1–parameter group of *translations*. They come as a 1–parameter family of noncongruent surfaces indexed by slope.

Observation. Among the equivariant minimal surfaces in $(\mathsf{Nil}(3), g_\tau)$, the umbrellas and the saddle-type surfaces are the ones that are *graphs* w.r.t. the Riemannian submersion $(\mathsf{Nil}(3), g_\tau) \to \mathbb{R}^2$. The holomorphic quadratic differentials Q and the conformal types of these global minimal graphs are as listed below:

	umbrellas	vertical planes	min. surfaces of saddle-type
conformal type	hyperbolic	parabolic	parabolic
hol. quad. diff.	$Q \equiv 0$	$Q \equiv 0$	$Q = \mathrm{c} \cdot \mathrm{d}z^2$

The properties of the vertical planes have been listed here too, as these surfaces occur both as limits of families of umbrellas and as limits of families of minimal surfaces of saddle-type. In the first case the idea is to let the vertical axis move to infinity, whilst in the second case one lets the slope parameter approach infinity.

6.2 Scherk Surfaces in $\mathsf{Nil}(3)$.

The doubly-periodic Scherk surface is a global minimal surface in euclidean 3–space that is a graph over each of the black squares in a suitable checkerboard tiling of the plane and that also contains the vertical lines over the vertices of this tiling. Up to scaling the piece over a single square is congruent to the graph of the function

$$f(x,y) := \ln\circ\cos(y) - \ln\circ\cos(x)$$

with $(x,y) \in \bigl(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\bigr)^2$.

In the Heisenberg group there exist minimal surfaces with similar properties. They have been constructed in cooperation with Harold Rosenberg; the details will be given in a forthcoming joint paper.

Proposition 7 (Local Scherk surfaces). *Let Ω be a square in $\mathbb{R}^2$, and let $\gamma_1 \cup \gamma_2$ denote the horizontal lift of its diagonals through a given common center. Then, for any $\tau \neq 0$, there is a unique minimal graph $\Sigma^2 \subset (\mathsf{Nil}(3), g_\tau)$ w.r.t. the Riemannian submersion $(\mathsf{Nil}(3), g_\tau) \to \mathbb{R}^2$ that*

- *(i) is defined over the interior of the square Ω and contains $\gamma_1 \cup \gamma_2$, that*
- *(ii) is asymptotic to $+\infty$ over one pair of edges and to $-\infty$ over the other pair of edges, and that*
- *(iii) has the vertical lines over the 4 vertices of the square Ω as its boundary.*

Idea of the Proof. Consider the geodesic pentagon c consisting of a horizontal lift of one of the edges of the square, an adjacent segment on each of the vertical lines, and the appropriate segments of γ_1 and γ_2. This contour c is a *Nitsche graph* over the triangle Δ consisting of the given edge of the square and the two adjacent segments on the diagonals.

As in the euclidean case there is a unique stable minimal surface spanning c, and, moreover, this minimal surface is a graph over the interior of Δ. Now the Schwarz reflection asserts that we can extend this minimal surface using congruent copies obtained through 180°–rotations around γ_1 and γ_2. In this way we obtain a stable minimal graph of finite height defined over the entire square Ω. (See Figure 7.)

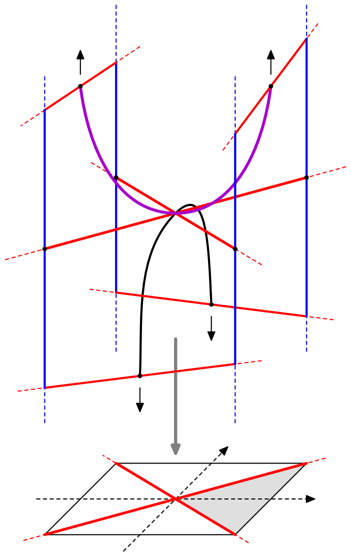

Figure 7: The local Scherk surface.

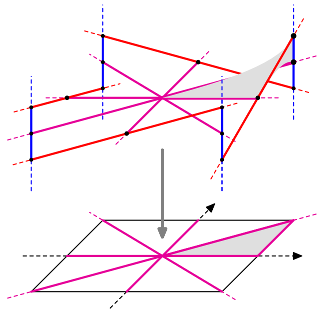

Figure 8: A triply-periodic global Scherk surface.

The *final step* in constructing the local Scherk surfaces is to show that the family of these finite height Scherk surfaces converges towards some limit surface Σ^2 when their height goes to $+\infty$. This goal is accomplished with the help of suitable barriers. By its very construction the limit surface is a minimal graph, but some further arguments are necessary to establish that this graph is actually defined on the entire open square.

$\square$

As in the euclidean case, the local Scherk surfaces in the Heisenberg group $(\mathsf{Nil}(3), g_\tau)$ naturally come as a 1–*parameter family* Σ_r^2 indexed by the size of the underlying square Ω. However, it is *not* possible anymore to recover the entire family from one of its member surfaces by scaling.

Remark. Upon enlarging the square, the local Scherk surfaces Σ_r^2 *converge* to *saddle-type surfaces* of slope zero and not to umbrellas.

Yet, it is possible to use them as comparison objects for proving curvature bounds for global minimal graphs. These bounds can be viewed as a first step towards a Bernstein theorem.

On the Schwarz extension of the local Scherk surfaces Σ^2.

As indicated above, the *local Scherk surface* Σ^2 from the Proposition can be extended using the Schwarz reflection principle; one obtains an *immersed global minimal surface* $\hat{\Sigma}^2$. Clearly, $\hat{\Sigma}^2$ is the orbit of Σ^2 under the group Γ generated by the 180°–rotations around the diagonals and the 180°–rotations around the lines containing the vertical segments of the geodesic pentagon c.

It is not hard to see that Γ is a a *discrete* subgroup of $\mathrm{Iso}_0(\mathsf{Nil}(3), g_\tau)$, and thus the global surface $\hat{\Sigma}^2$ is even *properly immersed*.

But, $\hat{\Sigma}^2$ is *not* embedded. In fact, the next lemma implies that $\hat{\Sigma}^2$ is invariant under a *nontrivial vertical translation*, and thus the vertical lines bounding the fundamental piece Σ^2 must be lines of self-intersection.

Lemma. *Let Γ' be the discrete subgroup in $\mathrm{Iso}_0(\mathsf{Nil}(3), g_\tau)$ generated by the 180°–rotations around the vertical lines over the vertices of the triangle Δ. Then Γ' contains a lattice. In particular, Γ' contains the vertical translation by $8h$ where h denotes the vertical displacement by which the horizontal lift of the triangle Δ fails to close up.*

One can visualize these holonomy effects in the following way:

Observation. The set $\Gamma' \cdot (\gamma_1 \cup \gamma_2)$, i.e., the union of the various images of the horizontal lift of diagonals of the basic square Ω w.r.t. the action of the discrete group Γ', is a 1–*dimensional complex* that projects to the union of the diagonals of all the black squares of the tiling and that is *invariant* under the *vertical translation* by $8h$.

Embedded global Scherk surfaces.

Because of the holonomy effects described above, it is inevitable to refrain from passing to the limit in the proof of Proposition 7. We shall rather use the stable minimal graphs of finite height that have been constructed in the first step of the proof.

The group Γ_c that is relevant for describing the Schwarz extension of a minimal surface spanning a contour c of finite height is the extension of Γ

generated by the 180°–rotation around the segment of c that is the horizontal lift of the edge of the square Ω. As a result, the Schwarz extensions of the stable minimal graphs of finite height are in general *not* properly immersed anymore. Persuing this idea a little further, one can exclude all but 3 choices for the lift of the edge of the square.

It can be shown that all these cases do actually occur.

Proposition 8 (Triply-periodic Scherk surfaces). *For each left-invariant metric g_τ on the Heisenberg group $\mathsf{Nil}(3)$, there exists an embedded, triply-periodic, global minimal surface $\hat{\Sigma}^2$ of Scherk type.*

One way to construct these triply-periodic minimal surfaces is to start out with a contour c where the end points of the horizontal lift of the edge of Ω lie at equal distances above and below the corresponding end points of γ_1 and γ_2. In fact, this contour has an additional symmetry; as indicated in Figure 8, it is invariant under the 180°–rotation around the horizontal geodesic through the center and the mid point of the lifted edge. In fact, this axis divides the geodesic pentagon c into two geodesic quadrilaterals c_1 and c_2 such that the stable minimal surfaces bounded by c_1 and c_2 are the two halves of the stable minimal surface bounded by c.

The other possibility for constructing a triply-periodic minimal surface is to work with a horizontal lift of the edge of Ω that begins at the end point of either γ_1 or γ_2. In this case the contour c degenerates to a geodesic quadrilateral, which is congruent to either one of the pieces c_1 and c_2 obtained when constructing the pentagon in the proceding paragraph from a square $\Omega' \subset \mathbb{R}^2$ that has $\sqrt{2}$ times the size of Ω.

6.3 Half-Space Theorems.

The material presented in this subsection is joint work with Harold Rosenberg, too. From the conformal point of view, the umbrellas in $\mathsf{Nil}(3)$ are *hyperbolic* surfaces and not parabolic ones. Yet, the following holds:

Theorem 9. *Let Σ^2 be a proper, possibly branched minimal surface in the Heisenberg group $\mathsf{Nil}(3)$. Suppose that Σ^2 is contained in the complement of some horizontal umbrella. Then Σ^2 is congruent to this umbrella by a vertical translation.*

The behavior of complete minimal surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$ is very different [13]; there exist plenty of complete minimal surfaces in any half-space bounded by some level set $\mathbb{H}^2 \times \{t_0\}$. In other words, there cannot be any half-space theorem at all.

Method of Proof. The same argument as in the euclidean case works, as the catenoids in $(\mathsf{Nil}(3), g_\tau)$ *collapse* to a doubly-covered punctured umbrella when their necksize shrinks to 0.

□

It seems to be an interesting question whether in the Heisenberg group there are also half-space theorems with respect to the vertical planes or the minimal surfaces of saddle-type.

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