Matemática Contemporânea, Vol 28, 91-110

WARPED PRODUCT METRICS AND LOCALLY CONFORMALLY FLAT STRUCTURES

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To the memory of Prof. José Fernando Escobar

Abstract

Our purpose is to show the usefulness of warped product metrics in constructing new examples of complete locally conformally flat manifolds of nonpositive curvature.

1 Introduction

Locally conformally flat structures are natural generalizations of isothermal coordinate systems, which are available on Riemann surfaces. However, not every higher dimensional Riemannian manifold admits such a coordinate system, and there are some well-known equivalent conditions to local conformal flatness. Necessary and sufficient conditions for the existence of a locally conformally flat structure are given by the nullity of the Weyl tensor (if $\dim M \geq 4$) and the fact that the Schouten tensor is a Codazzi tensor in dimension three. Here $W = R - C \odot g$ and $C = \frac{1}{n-2}\left(Ric - \frac{Sc}{2(n-1)}g\right) \odot g$ denote the Weyl tensor and the Schouten tensor respectively, where $\odot$ represents the Kulkarni-Nomizu product (see, for example [12]). The important fact that the Schouten tensor is Codazzi in any locally conformally flat manifold motivates the consideration of locally conformally flat structures on manifolds equipped with a warped product metric. Indeed, although the local structure of Codazzi tensors is not

Supported by project BFM 2003-02949 (Spain)
2000 M. S. C.: 53C15, 53C50
Key words and phrases: Locally conformally flat, multiply warped product, nonpositive curvature.

yet completely understood, they lead to warped product decompositions of the manifold in many cases [2], [21].

On the other hand, a classification of locally conformally flat manifolds remains, in general, incomplete. There are, however, some important results:

- A locally decomposable locally conformally flat manifold must be of constant curvature or locally isometric to a product of an interval and a space of constant curvature, or locally isometric to the product of two spaces of constant opposite sectional curvature [22]. As a consequence locally symmetric (or even the more general class of semi-symmetric) locally conformally flat manifolds are completely determined [5].

- A compact and simply connected locally conformally flat manifold must be conformal to an Euclidean sphere (cf. [13], [23]).

- The universal cover of a complete locally conformally flat manifold with nonnegative Ricci curvature is in the conformal class of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{R}\times\mathbb{S}^{n-1}$, where $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ are spheres of constant sectional curvature [24]. Moreover, such conformal equivalence can be specialized to isometric equivalence under some additional conditions on the scalar curvature and the norm of the Ricci tensor (see, for example [6], [7], [11], [20]).

In spite of the results above, there is a lack of information as concerns locally conformally flat manifolds of nonpositive curvature. Therefore, a first step in understanding such manifolds is the construction of representative examples in that class. Here again warped products appear as good candidates, since they provide of a powerful tool to construct examples of complete metrics of negative curvature [1]. In a search for new examples of locally conformally flat metrics, we will pay attention to warped product metrics and some generalizations like multiply warped products. Other generalizations like twisted products are not of interest in studying locally conformally flat manifolds, since they reduce to warped products in many cases.

The paper is organized as follows. In Section 2 we recall some basic facts on the geometry of warped and multiply warped products, and we fix the notation to be used throughout the paper. Locally conformally flat warped products are characterized in §3. We devote §4 to obtain a description of multiply warped products of constant sectional curvature with one-dimensional base, which is the key ingredient in obtaining a complete classification of locally conformally flat multiply warped metrics with one-dimensional base. Finally, as an application, we show new explicit examples of complete locally conformally flat manifolds of nonpositive curvature in §5.

2 Preliminaries

Let us start fixing some notation and criteria to be used in what follows. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. The Riemann curvature tensor R is taken with the sign convention $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, for all vector fields $X, Y, Z \in \mathfrak{L}(M)$. The Ricci tensor is the contraction of the curvature tensor given by $Ric(X, Y) = \operatorname{trace}\{U \rightsquigarrow R(X, U)Y\}$, for all $X, Y \in \mathfrak{L}(M)$, and the scalar curvature is obtained by contracting the Ricci tensor, $Sc = \operatorname{trace} Ric$.

For a vector field $X \in \mathfrak{L}(M)$ the *divergence* of X is defined by $div\, X = \operatorname{trace} \nabla X$. Now, for a map $f : (M, g) \longrightarrow \mathbb{R}$, the *gradient* of f is determined by $g(\nabla f, X) = df(X) = X(f)$, for all $X \in \mathfrak{L}(M)$. Also, the linear map $h_f(X) = \nabla_X \nabla f$ is called the *Hessian tensor* of f on (M, g), and $H_f(X, Y) = g(h_f(X), Y)$ is called the *Hessian form* of f on (M, g). Finally, the *Laplacian* of f on (M, g) is defined by $\Delta f = div\, \nabla f$, and it satisfies $\Delta f = \operatorname{trace} h_f$.

2.1 Multiply warped metrics

Now we recall the definition of a multiply warped product. Let (B, g_B), (F_1, g_1), …, (F_k, g_k) be Riemannian manifolds. The product manifold $M = B \times F_1 \times \cdots \times F_k$ equipped with the metric

$$g = g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k,$$

where $f_1, \ldots, f_k : B \longrightarrow \mathbb{R}$ are positive functions, is called a *multiply warped product*. B is the *base*, $F_1, \ldots, F_k$ are called the *fibers* and $f_1, \ldots, f_k$ are referred to as the *warping functions*. In what follows we will denote a multiply warped product manifold as above by $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$. (See [16], [17], [21] and the references therein for more information on multiply warped products and some physical applications in the construction of multidimensional models).

Remark 1 Note that warped products are just multiply warped products with a unique fiber. Also, it is possible to refine the above definition in order to identify multiply warped products which are essentially the same manifold although with different form. In this sense, we adopt the following criteria:

C1. Warping functions are supposed to be nonconstant and any two warping functions which are multiples one to each other are written as the same function and the metric of the fiber is multiplied by the corresponding constant in order to do not modify the metric of the multiply warped product.

C2. Fibers with the same warping function are joined in one fiber.

Next we collect some basic facts about multiply warped spaces which will be used throughout the paper. In particular, we describe the Levi-Civita connection and the Riemann curvature tensor of a multiply warped product (for a more detailed information see [4]). In what follows let X, Y, $Z \in \mathcal{L}(B)$ and U_r, V_r, $W_r \in \mathcal{L}(F_r)$ $(r = 1, \ldots, n)$. The nonvanishing components of the Levi-Civita connection are determined by:

$$\begin{aligned}
&\nabla_X Y = \nabla^B_X Y \\
&\nabla_X V_i = \nabla_{V_i} X = \tfrac{X(f_i)}{f_i} V_i \\
&nor \nabla_{U_i} V_i = \mathbb{I}_i(U_i, V_i) = -\tfrac{\langle U_i, V_i \rangle}{f_i} \nabla f_i \\
&tan \nabla_{U_i} V_i = \nabla^{F_i}_{U_i} V_i
\end{aligned} \tag{1}$$

where ∇^B and ∇^{F_i} denote the Levi Civita connection on B and F_i, respectively, and $\mathbb{I}_i$ is the second fundamental form of F_i. Moreover, the nonvanishing components of the Riemann curvature tensor are given by:

$$\begin{aligned}
&R_{XY}Z = R^B_{XY}Z\\
&R_{V_iX}Y = \tfrac{H_{f_i}(X,Y)}{f_i}V_i\\
&R_{XU_i}V_i = \tfrac{\langle U_i,V_i\rangle}{f_i}\nabla_X(\nabla f_i)\\
&R_{U_iV_i}W_i = R^{F_i}_{U_iV_i}W_i - \tfrac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2}\{\langle U_i,W_i\rangle V_i - \langle V_i,W_i\rangle U_i\}\\
&R_{U_jU_i}V_i = \tfrac{\langle U_i,V_i\rangle\langle\nabla f_i,\nabla f_j\rangle}{f_if_j}U_j
\end{aligned} \tag{2}$$

where R^B and R^{F_i} denote the Riemann curvature tensor of (B, g_B) and (F_i, g_i), respectively.

Recall that the sectional curvature of a plane $\pi = \langle\{X, Y\}\rangle$ is given by $K(\pi) = \frac{R(X,Y,X,Y)}{\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2}$. Then the sectional curvature of $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ satisfies

$$\begin{aligned}
&K_{XY} = K^B_{XY}\\
&K_{XU_i} = -\tfrac{H_{f_i}(X,X)}{f_i\langle X,X\rangle}\\
&K_{U_iV_i} = \tfrac{1}{f_i^2}K^{F_i}_{U_iV_i} - \tfrac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2}\\
&K_{U_iU_j} = -\tfrac{\langle\nabla f_i,\nabla f_j\rangle}{f_if_j}
\end{aligned} \tag{3}$$

where K^B and K^{F_i} denote the sectional curvature on the base B and the fiber F_i, respectively.

3 Locally conformally flat warped product spaces

Recall that a Riemannian manifold (M, g) is locally conformally flat if and only if every point in M admits a coordinate neighborhood $\mathfrak{U}$ which is conformal to the Euclidean space $\mathbb{R}^n$. Note that any Riemann surface is locally conformally flat, but not every higher dimensional Riemannian manifold admits a locally conformally flat structure. Necessary and sufficient conditions for local conformal flatness are expressed by means of the Schouten tensor $C = \frac{1}{n-2}\left(Ric - \frac{Sc}{2(n-1)}g\right) \odot g$, which has to be Codazzi in dimension three, and the Weyl tensor $W = R - C \odot g$, which vanishes on any locally conformally flat $(4 \leq n)$-dimensional manifold, where $\odot$ denotes the Kulkarni-Nomizu product

$$(A \odot B)(X,Y,Z,V) = A(X,Z)B(Y,V) + A(Y,V)B(X,Z) - A(X,V)B(Y,Z) - A(Y,Z)B(X,V).$$

Locally conformally flat warped product metrics have been investigated by several authors [9], [10], [19], who obtained necessary and sufficient conditions in terms of the curvatures of the base and the fiber of the warped product, together with some PDE's involving the warping function. The approach followed on those works focused on the investigation of the Weyl tensor through the curvature expressions (2). An alternative approach was considered in [3], based on the fact that any warped product metric is in the conformal class of a suitable direct product metric. Then, the following was obtained as an application of the results in [22] (see also [14]).

Theorem 2 [3, Theorem 1] *Let $M = B \times_f F$ be a Riemannian warped product. Then the following hold:*

(i) *If dim $B = 1$, then $M = B \times_f F$ is locally conformally flat if and only if (F, g_F) is a space of constant curvature.*

(ii) *If dim $B > 1$ and dim $F > 1$, then $M = B \times_f F$ is locally conformally flat if and only if*

  (ii.a) *(F, g_F) is a space of constant curvature c_F.*

  (ii.b) *The function $f : B \to \mathbb{R}^+$ defines a global conformal deformation on B such that $(B, \frac{1}{f^2} g_B)$ is a space of constant curvature $\tilde{c}_B = -c_F$.*

(iii) *If dim $F = 1$, then $M = B \times_f F$ is locally conformally flat if and only if the function $f : B \to \mathbb{R}^+$ defines a conformal deformation on B such that $(B, \frac{1}{f^2} g_B)$ is a space of constant curvature.*

Remark 3 Warped products with one-dimensional base have a very rich behavior as concerns local conformal flatness, since any positive function makes $I \times_f F$ locally conformally flat if F has constant sectional curvature. However, there is a kind of uniqueness for the warping function if the base is assumed to be of higher dimension. For instance, let (B, g_B) be a compact manifold and (F, g_F) a manifold of constant curvature such that there exist functions f and $\hat{f}$ that make the warped products $B \times_f F$ and $B \times_{\hat{f}} F$ locally conformally flat; then $(B, \frac{1}{f^2} g_B)$ is isometric to a sphere $\mathbb{S}^s$ and $\hat{f}/f = -\frac{s-1}{Sc}\psi + C$, where ψ is an eigenfunction for the first eigenvalue of the Laplacian in the sphere, Sc denotes the scalar curvature of $\mathbb{S}^s$ and C a suitable constant (cf. [3]).

Remark 4 Complete locally conformally flat manifolds with base a model space of constant curvature are obtained from the table below, which shows the possible warping functions corresponding to each different case [3].

		Fibers				**Curvature of**
Base	**Warping function**	$\mathbb{R}$	$\mathbb{R}^d$	$\mathbb{S}^d$	$\mathbb{H}^d$	$(B, \frac{1}{f^2} g_B)$
$\mathbb{R}^s$	$f(\overrightarrow{\mathbf{x}}) = a\Vert\overrightarrow{\mathbf{x}}\Vert^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}} \rangle + c$	✓	×	×	✓	$4\,a\,c - \Vert\overrightarrow{\mathbf{b}}\Vert^2$
$\mathbb{S}^s$	$f(\overrightarrow{\mathbf{x}}) = -\frac{s-1}{Sc}\langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}} \rangle + C$	✓	×	×	✓	$C^2 - \frac{(s-1)^2}{Sc^2}\Vert\overrightarrow{\mathbf{b}}\Vert^2$
$\mathbb{H}^s$	$f(\overrightarrow{\mathbf{x}}) = \frac{a\Vert\overrightarrow{\mathbf{x}}\Vert^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}} \rangle + c}{x_s}$	✓	✓	✓	✓	$4\,a\,c - \Vert\overrightarrow{\mathbf{b}}\Vert^2$

The sectional curvature of the conformal metrics $\frac{1}{f^2} g_B$ in the table shows the nonexistence of complete locally conformally flat warped products with base the sphere or the Euclidean space and fibers of nonnegative curvature because of the positivity of the warping function. Moreover the Poincaré half-space model has been used to describe the hyperbolic space $\mathbb{H}^s$ (cf. [15]).

Remark 5 Sufficient conditions for a warped product to have negative or nonpositive sectional curvature are given in [1] in terms of the curvature of both factors (more exactly, that the base and the fiber have negative or nonpositive sectional curvature, respectively) and the convexity of the warping function.

There are several generalizations of warped products metrics that could provide *a priori* new examples of locally conformally flat manifolds with negative curvature. One of the most important is the class of twisted products: the *twisted product* $B \times_f F$ of (B, g_B) and (F, g_F) with twisting function f is the product manifold $B \times F$ with metric tensor $g = g_B \oplus f^2 g_F$, where f is defined on the whole manifold, i.e., $f : B \times F \to \mathbb{R}^+$. However, this sort of products can not provide new examples, unless the base or the fiber are one-dimensional, as pointed out in the next result:

Theorem 6 [3] *Let* $M = B \times_f F$ *be a Riemannian twisted product with* $\dim B \geq 2$ *and* $\dim F \geq 2$. *If* M *is locally conformally flat then it can be expressed as a warped product.*

This result is obtained as a consequence of the so-called mixed Ricci flatness (it means that the Ricci tensor vanishes when applied to a vector field on the base and another one on the fiber of the twisted product), which is a nice key to discern when a twisted product can be reduced to a warped one [8]. So, roughly speaking, the search of new locally conformally flat examples on twisted products is limited to the particular case where the warping function is constant on the fiber, that is, to warped products.

4 Locally conformally flat multiply warped product spaces

Multiply warped products are a richer structure than that of warped products, since it obviously contains a higher number of fibers and warping functions. However when one tries to investigate curvature-properties of those spaces, some compatibility conditions among the different warping functions are required (see for example (3), which strongly restricts both the warping functions and the number of fibers (cf. Theorem 9). Since our aim is to obtain new examples of locally conformally flat manifolds, motivated by the results in Theorem 2, in what follows we restrict ourselves to multiply warped spaces with one-dimensional base. Therefore, we consider a Riemannian manifold (M, g) with the underlying structure of a multiply warped product space of the form

$$M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k \qquad (I \subset \mathbb{R}), \tag{4}$$

with metric tensor

$$g = (dt)^2 \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k,$$

where $g_1, \ldots, g_k$ are Riemannian metrics on $F_1, \ldots, F_k$, respectively. We will refer to such a manifold as a multiply warped space of type (4).

Our purpose in this section is to obtain a complete characterization of locally conformally flat metrics of type (4). This will be achieved in Theorem 11, which reduces the problem to the determination of all metrics of type (4) which are of constant sectional curvature.

4.1 Multiply warped spaces of constant sectional curvature

Next we will obtain a complete description of multiply warped spaces of constant sectional curvature. We recall that Theorem 7 has been previously obtained by Mignemi and Schmidt [17], but we include a different proof which is much shorter that the one followed at there. (Further note that the result in Theorem 7 can also be obtained from the classification of warped product representations of space forms in [18]). First of all note that, as a consequence of (3), if $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a space of constant sectional curvature, then each fiber F_i must be of constant sectional curvature c_{F_i}.

Theorem 7 *Let $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space of type (4). Then M is of constant sectional curvature K if and only if $k \leq 2$ and, moreover, one of the following holds:*

(i) If $K = 0$, then $M = I \times_{\alpha_1} F_1$ or $M = I \times_{\alpha_1} F_1 \times_{\alpha_2} F_2$, with warping functions given by

$$\alpha_i(t) = a_i\, t + b_i, \qquad i = 1, 2.$$

Moreover, the fibers (F_i, g_i) *are necessarily of constant sectional curvature* $K^{F_i} = a_i^2$*, provided that* $\dim F_i \geq 2$ $(i = 1, 2)$*, and the warping functions satisfy the compatibility condition* $a_1 a_2 = 0$ *in the case of two fibers.*

(ii) If $K = c^2$*, then* $M = I \times_{\beta_1} F_1$ *or* $M = I \times_{\beta_1} F_1 \times_{\beta_2} F_2$*, with warping functions given by*

$$\beta_i(t) = a_i \sin ct + b_i \cos ct, \qquad i = 1, 2.$$

Moreover, the fibers (F_i, g_i) *are necessarily of constant sectional curvature* $K^{F_i} = c^2(a_i^2 + b_i^2)$*, provided that* $\dim F_i \geq 2$ $(i = 1, 2)$*, and the warping functions satisfy the compatibility condition* $a_1 a_2 + b_1 b_2 = 0$ *in the case of two fibers.*

(iii) If $K = -c^2$*, then* $M = I \times_{\gamma_1} F_1$ *or* $M = I \times_{\gamma_1} F_1 \times_{\gamma_2} F_2$*, with warping functions given by*

$$\gamma_i(t) = a_i \sinh ct + b_i \cosh ct, \qquad i = 1, 2.$$

Moreover, the fibers (F_i, g_i) *are necessarily of constant sectional curvature* $K^{F_i} = c^2(a_i^2 - b_i^2)$*, provided that* $\dim F_i \geq 2$ $(i = 1, 2)$*, and the warping functions satisfy the compatibility condition* $a_1 a_2 - b_1 b_2 = 0$ *in the case of two fibers.*

Remark 8 Note that two consequences follow from previous theorem on the multiply warped structure of a space of constant curvature.

1. No more than two fibers are admissible for a space $I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ to be of constant sectional curvature.

2. The (constant) sectional curvatures of the fibers, are subject to some restrictions.

Proof of Theorem 7. Note that for the particular case of one-dimensional base, the first equation in (3) disappears and the other three can be rewritten in a much simpler way as follows:

(ii) $f_i''(t) + K\, f_i(t) = 0$,

(iii) $f_i'(t)^2 + K\, f_i(t)^2 = K^{F_i}$,

(iv) $f_i'(t) f_j'(t) + K\, f_i(t) f_j(t) = 0$.

Equations (ii) only depend on the warping functions and the value of the sectional curvature. Therefore, such equations give us the general form of the warping functions depending on K being zero, positive or negative, respectively, as follows:

$$\begin{aligned} K = 0: &\quad \alpha_i(t) = a_i\, t + b_i \\ K = c^2: &\quad \beta_i(t) = a_i \sin ct + b_i \cos ct \\ K = -c^2: &\quad \gamma_i(t) = a_i \sinh ct + b_i \cosh ct \end{aligned}$$

Now, equations (iii) express the sectional curvature of a plane generated by two vectors in the same fiber, so we can interpret them saying that, for $\dim F_i \geq 2$, such equations show the compatibility between the fiber and the corresponding warping function. Thus, in each one of the above three cases and for fibers of dimension ≥ 2 we get

$$\begin{aligned} K = 0: &\quad K^{F_i} = a_i^2 \\ K = c^2: &\quad K^{F_i} = c^2(a_i^2 + b_i^2) \\ K = -c^2: &\quad K^{F_i} = c^2(a_i^2 - b_i^2) \end{aligned}$$

Equations (ii) and (iii) are sufficient when M is a warped product, but if M has more than one fiber then we must also consider equations (iv) which correspond to the sectional curvature of a plane generated by vectors in different fibers. Thus, equations (iv) can be viewed as a compatibility condition between the warping functions. Next we show that such compatibility condition implies a strong limitation in the number of fibers. Supposing there are three warping functions f_i, f_j and f_k, then from (iv) we get

$$\frac{f_i'}{f_i} = -K\,\frac{f_j}{f_j'} = \frac{f_k'}{f_k}$$

and hence one easily obtains that $f_i = \lambda f_k$ for some constant λ, which is a contradiction since warping functions are assumed to be different (cf. Remark

1). So the maximum number of fibers allowed is exactly two and, if this is the case, it follows that

$$\begin{aligned} K = 0 : &\quad a_1 a_2 = 0 \\ K = c^2 : &\quad a_1 a_2 + b_1 b_2 = 0 \\ K = -c^2 : &\quad a_1 a_2 - b_1 b_2 = 0 \end{aligned}$$

which shows the result.

□

4.2 Locally conformally flat multiply warped product metrics

In order to obtain a precise description of all locally conformally flat manifolds $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$, first of all we get an upper bound on the number of different fibers of the product (compare with Remark 8).

Theorem 9 *Let $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space of type (4). If M is locally conformally flat then $k \leq 3$ and the fibers (F_i, g_{F_i}) are spaces of constant sectional curvature (provided that $\dim F_i \geq 2$), for all $i = 1, \ldots, k$.*

Proof. We use the fact that any multiply warped metric is in the conformal class of a suitable direct product metric as follows

$$\begin{aligned} g &= (dt)^2 \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k \\ &= f_k^2 \left(\tfrac{1}{f_k^2}(dt)^2 \oplus (\tfrac{f_1}{f_k})^2 g_1 \oplus \cdots \oplus (\tfrac{f_{k-1}}{f_k})^2 g_{k-1} \oplus g_k \right). \end{aligned}$$

Hence the multiply warped product

$$\frac{1}{f_k^2} I \times_{\frac{f_1}{f_k}} F_1 \times \cdots \times_{\frac{f_{k-1}}{f_k}} F_{k-1} \times F_k$$

is also locally conformally flat, and thus Theorem 2 implies that

$$\frac{1}{f_k^2} I \times_{\frac{f_1}{f_k}} F_1 \times \cdots \times_{\frac{f_{k-1}}{f_k}} F_{k-1} \tag{5}$$

has constant sectional curvature. Note that, since we can repeat this argument for any other warping function, then from Remark 8 it easily follows the constancy of the sectional curvature of the fibers (F_i, g_{F_i}) (provided that

$\dim F_i \geq 2$). Now, rescaling the metric on I by means of $\widetilde{t} = \int \frac{1}{f_k(t)}$, then (5) can be written as

$$I \times_{\widetilde{f}_1} F_1 \times \cdots \times_{\widetilde{f}_{k-1}} F_{k-1}$$

with $\widetilde{f}_i(\widetilde{t}) = \frac{f_i(\widetilde{t})}{f_k(\widetilde{t})}$, and since its sectional curvature is constant, (3) leads to

$$-\frac{\widetilde{f}_1'\widetilde{f}_2'}{\widetilde{f}_1\widetilde{f}_2} = K_{V_1W_2} = K_{V_1W_{k-1}} = -\frac{\widetilde{f}_1'\widetilde{f}_{k-1}'}{\widetilde{f}_1\widetilde{f}_{k-1}}$$

where $V_i, W_i \in \mathfrak{L}(F_i)$. So, it follows that either $\widetilde{f}_1$ is constant and hence $f_1 = c_{1k}f_k$ for some constant c_{1k}, or $\frac{\widetilde{f}_2'}{\widetilde{f}_2} = \cdots = \frac{\widetilde{f}_{k-1}'}{\widetilde{f}_{k-1}}$ and therefore $\widetilde{f}_2 = c_{23}\widetilde{f}_3 = \cdots = c_{2,k-1}\widetilde{f}_{k-1}$ for some constants c_{ij}. Now considering the criteria adopted in Remark 1, we have that the maximum number of different functions between $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_{k-1}$ is two. Thus, no more than three different warping functions between $f_1, \ldots, f_k$ are allowed.

$\square$

Remark 10 Note that the restriction on the number of fibers given by the previous theorem can be also obtained as a direct consequence of Theorem 7. Indeed, if $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat then $\frac{1}{f_k^2} I \times_{\frac{f_1}{f_k}} F_1 \times \cdots \times_{\frac{f_{k-1}}{f_k}} F_{k-1}$ has constant sectional curvature and hence the result follows from Theorem 7.

Now we have the desired characterization of locally conformally flat metrics of type (4).

Theorem 11 *Let $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space of type (4). Then M is locally conformally flat if and only if one of the following holds*

(i) $M = I \times_f F$ is a warped product with fiber F of constant sectional curvature (provided that $\dim F \geq 2$) and any (positive) warping function f.

(ii) $M = I \times_{f_1} F_1 \times_{f_2} F_2$ is a multiply warped product with two fibers of constant sectional curvature (provided that $\dim F_i \geq 2$) and warping functions

$$f_1 = (\xi \circ f)\frac{1}{f'}, \qquad f_2 = \frac{1}{f'}$$

where f is a strictly increasing function and ξ is a warping function making $I \times_\xi F_1$ of constant sectional curvature (cf. Theorem 7) and $(\xi \circ f) > 0$.

(iii) *$M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ is a multiply warped product with three fibers of constant sectional curvature (provided that dim $F_i \geq 2$) and warping functions*

$$f_1 = (\xi_1 \circ f)\frac{1}{f'}, \qquad f_2 = (\xi_2 \circ f)\frac{1}{f'}, \qquad f_3 = \frac{1}{f'}$$

where f is a strictly increasing function and ξ_i are warping functions making $I \times_{\xi_1} F_1 \times_{\xi_2} F_2$ of constant sectional curvature (as in Theorem 7) such that $(\xi_i \circ f) > 0$, $i = 1, 2$.

Proof. By Theorem 9, we know that $k \leq 3$ and that, in any case, the fibers have constant sectional curvature (provided that the dimension of the fiber is ≥ 2). Moreover, if M is a warped product ($k = 1$) then such condition is, indeed, sufficient (see Theorem 2). This proves (i). Now, if there are two fibers ($M = I \times_{f_1} F_1 \times_{f_2} F_2$) then

$$\frac{1}{f_2^2}\, dt^2 \oplus \frac{f_1^2}{f_2^2}\, g_{F_1} \tag{6}$$

has constant sectional curvature. But since f_2 is strictly positive, we can introduce a reparametrization on I by $\widetilde{t} = \int \frac{1}{f_2(t)}$, and thus (6) leads to

$$d\widetilde{t}^2 \oplus \xi(\widetilde{t})^2 g_{F_1},$$

which is a warped product with constant sectional curvature, where $\xi(\widetilde{t}) = \frac{f_1(t)}{f_2(t)}$ is given by Theorem 7. Then (ii) holds. Finally, if $k = 3$ ($M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$) then

$$\frac{1}{f_3^2}\, dt^2 \oplus \frac{f_1^2}{f_3^2}\, g_{F_1} \oplus \frac{f_2^2}{f_3^2}\, g_{F_2}$$

has constant sectional curvature, and proceeding as above we get that

$$d\widetilde{t}^2 \oplus \xi_1(\widetilde{t})^2 g_{F_1} \oplus \xi_2(\widetilde{t})^2 g_{F_2}$$

has constant sectional curvature, with $\xi_1(\widetilde{t}) = \frac{f_1(t)}{f_3(t)}$ and $\xi_2(\widetilde{t}) = \frac{f_2(t)}{f_3(t)}$ ($\widetilde{t} = \int \frac{1}{f_3(t)}$) are given by Theorem 7. Thus, (iii) is obtained.

$\square$

Remark 12 Note that the characterization above is essentially independent of the last warping function f_k in each one of the three cases. For a warped product there is no restrictions on the warping function in order to be locally conformally flat (cf. Theorem 2-(i)) and essentially the same is true in the multiply warped case as concerns the last warping function, since the only restriction at Theorem 11 is that the auxiliary function f must be increasing (but not necessarily positive).

Remark 13 Recall that a nonflat locally decomposable Riemannian manifold is locally conformally flat if and only if it is locally equivalent to the product of an interval and a space of constant sectional curvature $N(c)\times\mathbb{R}$ or to the product of two spaces of constant opposite sectional curvature $N_1(c) \times N_2(-c)$. This should be contrasted with the results in previous theorem, where the existence of locally conformally flat metrics of the form $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ is obtained.

As in Theorem 7, there are some restrictions on the possible values of the (constant) sectional curvatures of the fibers of a locally conformally flat multiply warped space $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$. Indeed, it follows from Theorem 7 (cf. Remark 8) and Theorem 11 that (see also the work in [4])

- *no more than one* $(2 \leq)$*-dimensional fiber may be of nonpositive sectional curvature.*
- *If there is a* $(2 \leq)$*-dimensional flat fiber, then only another fiber may occur, and it must be of positive curvature.*

5 Some examples of complete locally conformally flat metrics of nonpositive curvature

First of all recall from [1] that a Riemannian warped product metric is complete if and only if so are the base and the fiber. This result can be easily extended to the more general class of multiply warped products to ensure that a multiply warped product is complete if and only if so are the base and all the fibers, independently of the warping functions (which are restricted by their positivity on the whole base).

- It follows from Theorem 11 that any locally conformally flat multiply warped product of type (4) derives from a suitable space of constant sectional curvature as in Theorem 7. It is important to emphasize here that complete locally conformally flat metrics can be constructed from non-necessary complete metrics of constant sectional curvature. For instance, the multiply warped space $I \times_{\rho_1} \mathbb{S}^{d_1} \times_{\rho_2} \mathbb{S}^{d_2}$ with warping functions

$$\rho_1(t) = \frac{1}{\sqrt{2}}\, sin(t) + \frac{1}{\sqrt{2}}\, cos(t) \qquad \rho_2(t) = \frac{1}{\sqrt{2}}\, sin(t) - \frac{1}{\sqrt{2}}\, cos(t)$$

  is an incomplete manifold of constant sectional curvature $K = 1$. However, by making an appropriate choice $f(t) = \frac{3\pi}{8} + \frac{1}{4} arctg(t)$ and using Theorem 11, we get that

$$\mathbb{R} \times_{f_1} \mathbb{S}^{d_1} \times_{f_2} \mathbb{S}^{d_2} \times_{f_3} \mathbb{H}^{d_3}$$

  is a complete locally conformally flat space with warping functions

$$\begin{aligned} f_1(t) &= \{\tfrac{1}{\sqrt{2}}\, sin(\tfrac{3\pi}{8} + \tfrac{1}{4}\, arctan(t)) + \tfrac{1}{\sqrt{2}}\, cos(\tfrac{3\pi}{8} + \tfrac{1}{4}\, arctan(t))\} 4(1+t^2) \\ f_2(t) &= \{\tfrac{1}{\sqrt{2}}\, sin(\tfrac{3\pi}{8} + \tfrac{1}{4}\, arctan(t)) - \tfrac{1}{\sqrt{2}}\, cos(\tfrac{3\pi}{8} + \tfrac{1}{4}\, arctan(t))\} 4(1+t^2) \\ f_3(t) &= 4(1+t^2). \end{aligned}$$

Moreover, in order to obtain new examples of complete locally conformally flat manifolds of nonpositive curvature by using warped and multiply warped product metrics, there are some facts to be considered:

- Complete locally conformally flat warped product manifolds $B \times_f F$ of nonpositive curvature can be constructed on the base of Remark 5 and the results in Remark 4.

- Clearly any space of constant sectional curvature is locally conformally flat, and thus it follows from Theorem 7-(iii) the existence of complete warped products $\mathbb{R} \times_f F$ of constant negative sectional curvature. However it follows from the compatibility conditions at Theorem 7 the nonexistence of complete multiply warped manifolds $I \times_{f_1} F_1 \times_{f_2} F_2$ of constant curvature.

Remark 14 Multiply warped products of nonpositive sectional curvature can be constructed on the base of the following conditions, which are obtained in a similar way as in [1], just proceeding from the expressions in (3):

(a) Any $(2 \leq)$-dimensional fiber is of nonpositive sectional curvature.

(b) Warping functions are convex, (i.e., f_i'' is non negative),

(c) All the warping functions are increasing or decreasing functions, $f_i' \geq 0$ or $f_i' \leq 0\ \forall i$.

Moreover, conditions (a)–(c) are necessary if the base is complete (cf. [1]). Thus (a)–(c) are equivalent conditions to nonpositive sectional curvature in a complete multiply warped product of type (4).

- New examples of complete locally conformally flat manifolds of nonpositive curvature can now be constructed by using a multiply warped structure. For example, consider the multiply warped product

$$\mathbb{R} \times_{f_1} \mathbb{R} \times_{f_2} \mathbb{H}^2$$

with warping functions:

$$f_1(t) = \left(sin(\tfrac{\pi}{4} + \tfrac{1}{2}arctan(t)) + cos(\tfrac{\pi}{4} + \tfrac{1}{2}arctan(t))\right) 2(1+t^2)$$
$$f_2(t) = 2(1+t^2).$$

It follows from Theorem 11 and Remark 14 that this is a complete locally conformally flat manifold with nonpositive sectional curvature. Finally, recall that another way of checking the nonpositiveness of the sectional curvature in the above example (and, in general, for any multiply warped product with base of constant sectional curvature) consists in testing the sectional curvature with respect to each pair of vector fields in an orthogonal frame adapted to the product structure. This follows from (2), which guarantees the sign of the sectional curvature once it has been tested for pairs of vectors in such an orthogonal frame.

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