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SOME GEOMETRIC FORMULAS AND CANCELLATIONS IN ALGEBRAIC AND DIFFERENTIAL TOPOLOGY

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Abstract

We survey several phenomena in algebraic and differential topology by explicit formulas and cancellations that are of geometric origin. This covers some homotopy groups of the classical groups and some exotic diffeomorphisms and exotic involutions of spheres.

1 Introduction

In the XIII Escola de Geometria in São Paulo, the authors gave talks about the beautiful interaction between geometry and topology that is obtained when one gets explicit formulas in topology via geometric constructions.

In this article we will give an overview of the recent papers [Du, PR, DMR, Pü, ADPR1]. Rather than presenting the results of these papers in detail we will focus on their viewpoint here.

2 A geometric view on some homotopy groups

2.1 Introduction

The homotopy groups of the classical groups have been of continuous interest to topologists, geometers, and mathematical physicists since the discovery/invention of the notion of homotopy groups by Hurewicz in the 1930th. At the beginning, the approach towards these homotopy groups was rather explicit (see [St]), i.e., one wrote down explicit maps and explicit homotopies and determined the group structure with their help. The main theoretical tools at that point of time were the exact homotopy sequence for fiber bundles

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and the Hopf degree theorem. Of course, the explicit approach was limited and a sophisticated machinery was developed rather quickly to compute the isomorphism classes of many homotopy groups. We refer to the book [Di] for a detailed history of algebraic topology.

We would briefly (and very roughly) like to compare this development to the development of another mathematical field, namely differential equations. At the beginning of this field, people were mainly interested in explicit solutions. Qualitative statements were based on comparison with such solutions. In modern analysis the emphasis shifted to more abstract questions like existence, uniqueness, dependence on initial conditions, and regularity.

In both fields (homotopy theory, differential equations), however, some of the results that the modern machinery provides can be considered to be a challenge instead of the final solution of the problem. For some differential equations an existence result for solutions still leaves the problem to actually find the solutions explicitly, and for a homotopy group which is, for example, isomorphic to $\mathbb{Z}_k$ the next goal would be to actually see the geometry behind a nullhomotopy of the k-th power of a map that generates the homotopy group.

Of course, it does not make any sense to try to find such nullhomotopies for all or even a large class of known homotopy groups, just like it does not make sense to ask for an explicit solution of a generic differential equation. There are, however, still some basic homotopy groups (in which the machinery is anchored) for which a geometric understanding is desirable but has not been found in the early era of homotopy theory and ever since.

The perhaps most important problem of this kind in homotopy theory (in a weakened form this problem traces back to Eilenberg – see the comment in [Sa]) is the question of how the commutator of unit quaternions becomes nullhomotopic in the 12th power. We will now explain this exemplary problem and an approach towards a solution. Finally, we will give a brief overview of analogous results for other homotopy groups of the classical groups.

2.2 The commutator of quaternions

The unit quaternions $S^3 \subset \mathbb{H}$ form a non-commutative group under quaternionic multiplication. The amount of non-commutativity is measured by the commutator

$$c : S^3 \times S^3 \mapsto S^3, \quad (q_1, q_2) \mapsto q_1 q_2 q_1^{-1} q_2^{-1}.$$

It was shown by successive efforts of many of the leading figures of homotopy theory around 1950 that this map generates $\pi_6(S^3)$ and that $\pi_6(S^3)$ is isomorphic to $\mathbb{Z}_{12}$. Since for a Lie group the product of two homotopy classes is represented by the value-by-value product of two representing maps, this amounts to say that the 12th power c^{12} of the commutator is nullhomotopic. The obvious problem is now to find such a nullhomotopy. A particularly nice nullhomotopy should considerably deepen our understanding of how non-commutative the quaternions actually are.

The reader might wonder how the map c (beeing defined on $S^3 \times S^3$) can be considered to be an element of $\pi_6(S^3)$. This is based on the property that c is constant to 1 if one of its arguments is equal to ± 1, so that c factorizes over the smash product $S^3 \wedge S^3 \approx S^6$ (the commutator c is the prototyp of what became known as Samelson product, see e.g. [Sa, Bt, Wh]).

2.3 Enlarged target spaces

Instead of considering the original problem we first enlarge the target space of c from S^3 to the larger groups $\mathrm{SU}(3)$ and G_2. Since there is more space in these groups than in the original target space S^3, smaller powers of the commutator c become nullhomotopic. Indeed, it is easy to see from the relevant exact homotopy sequences that c generates $\pi_6(\mathrm{SU}(3)) \approx \mathbb{Z}_6$ and $\pi_6(\mathrm{G}_2) \approx \mathbb{Z}_3$. In other words, the sixth power of c is nullhomotopic in $\mathrm{SU}(3)$ and the third power of c is nullhomotopic in G_2.

One of the main results of our recent work [PR, Pü, DMR] is now the following.

Result. *We provide explicit nullhomotopies of* c^6 *in* $\mathrm{SU}(3)$ *and of* c^3 *in* G_2.

The first step for obtaining these nullhomotopies is to pass from the commutator c to a homotopic generator $b : S^6 \to S^3$ of $\pi_6(S^3)$. (Actually, we do not really deform maps here but consider a suitable homotopy equivalence between the smash product $S^3 \wedge S^3$ and the standard S^6.)

In the second step we "unfold" b in the larger target spaces, i.e., we deform b to maps $\phi : S^6 \to \mathrm{SU}(3)$ and $\chi : S^6 \to \mathrm{G}_2$ that are less collapsed and that are more natural for the enlarged target spaces.

The third step is to arrange copies of the generators ϕ of $\pi_6(\mathrm{SU}(3))$ and χ of $\pi_6(\mathrm{G}_2)$

by symmetries in such a way that the value-by-value product of the arranged maps is the constant map to the identity matrix.

We will skip the first step here. It is not difficult but a bit technical and when we later see the formula for b it will be evident that b is related to the commutor of unit quaternions.

2.4 Unfolding via lifting geodesics

For the second step and the construction of the map b the following inclusion of principal bundles is essential:

$$\begin{tikzcd}
\mathrm{Sp}(1) \arrow[r] \arrow[d] & \mathrm{SU}(3) \arrow[r] \arrow[d] & \mathrm{G}_2 \arrow[d] \\
\mathrm{Sp}(2) \arrow[r] \arrow[d] & \mathrm{SU}(4) \arrow[r] \arrow[d] & \mathrm{Spin}(7) \arrow[d] \\
S^7 \arrow[r, equal] & S^7 \arrow[r, equal] & S^7
\end{tikzcd}$$

Note that Sp(1) is just the group S^3 of unit quaternions. From the exact homotopy sequences of the bundles above it follows that the characteristic maps of the bundles generate $\pi_6(S^3)$, $\pi_6(\mathrm{SU}(3))$, and $\pi_6(\mathrm{G}_2)$. In [St] an explicit construction of these characteristic maps is described using two local trivializations of each bundle. The characteristic maps obtained this way, however, do not suggest how to arrange multiple copies of them in order to see algebraic cancellations. Such arrangements and also natural homotopies between each two characteristic maps in the bigger fiber can be seen much better with the following lifting construction that uses only one local trivialisation. This construction was not available at Steenrod's time since there was no precise notion of a Riemannian submersion and one was not used to think of non-biinvariant Riemannian metrics on Lie groups.

Endow the Lie groups Sp(2), SU(4), and Spin(7) with left invariant metrics that are right invariant under $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, U(3), and G_2, respectively, and induce the standard metric of constant curvature 1 on the sphere S^7 by Riemannian submersion. Now lift unit speed geodesics that start at the north pole of S^7 to horizontal geodesics of the Lie groups that pass through the identity elements. At time π these geodesics are evidently contained in the fibers over the south pole of S^7. Hence, we obtain maps from the S^6 in the tangent space of the north pole of S^7 to the fibers over the south pole. (Note that these fibers are diffeomorphic to the groups Sp(1), SU(3), and G_2, respectively, but not in an a priori canonical way. We emphazise this by the upper index "−".) A picture of this, in the special case of Sp(2), is given in Figure 1 of section 3. It can be seen e.g. from [Br], page 452, that the resulting maps

$$b : S^6 \to \mathrm{Sp}(1)^-, \quad \phi : S^6 \to \mathrm{SU}(3)^-, \quad \chi : S^6 \to \mathrm{G}_2^-$$

represent the images of id_{S^7} under the boundary maps

$$\pi_7(S^7) \to \pi_6(\mathrm{Sp}(1)^-), \quad \pi_7(S^7) \to \pi_6(\mathrm{SU}(3)^-), \quad \pi_7(S^7) \to \pi_6(\mathrm{G}_2^-)$$

in the exact homotopy sequences of the bundles. It follows immediately from Bott periodicity and the exact homotopy sequences that the maps b, ϕ, and χ generate the relevant homotopy groups.

Before we actually write down formulas for b, ϕ, and χ we would like to show how b is "unfolded" to ϕ and χ, i.e., how each two of these maps are homotopic to each other in the bigger fiber. Let γ_v be a geodesic of S^7 starting at the north pole with initial tangent vector v and let $\tilde{\gamma}_v^{\mathrm{Sp}}$ and $\tilde{\gamma}_v^{\mathrm{SU}}$ denote the unique horizontal lifts of γ to Sp(2) and SU(4), respectively. Since both lifts project to the same curve the bundle inclusion above implies

$$\tilde{\gamma}_v^{\mathrm{Sp}}(t) \in \tilde{\gamma}_v^{\mathrm{SU}}(t) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{SU}(3) \end{smallmatrix}\right)$$

for all $t \in \mathbb{R}$. Therefore, the homotopy

$$H(t, v) = \tilde{\gamma}_v^{\mathrm{SU}}(\pi) \cdot \tilde{\gamma}_v^{\mathrm{SU}}(t)^{-1} \cdot \tilde{\gamma}_v^{\mathrm{Sp}}(t)$$

attains values in $\mathrm{SU}(3)^-$. We clearly have

$$\begin{aligned} H(0, v) &= \tilde{\gamma}_v^{\mathrm{SU}}(\pi) = \phi(v), \\ H(\pi, v) &= \tilde{\gamma}_v^{\mathrm{Sp}}(\pi) = b(v). \end{aligned}$$

Hence, H is a homotopy between b and ϕ with values in $\mathrm{SU}(3)^-$. In a completely analogous way we obtain homotopies between b, ϕ, and χ with values in G_2^-. Thus we have completely described the "unfolding" of the map b in larger target spaces.

2.5 Geometric/algebraic cancellations

We will now discuss the structure of the maps b, ϕ, and χ and the related explicit cancellations in $\pi_6(\mathrm{SU}(3)) \approx \mathbb{Z}_6$ and $\pi_6(\mathrm{G}_2) \approx \mathbb{Z}_3$.

The easiest to understand is the generator $\chi : S^6 \to \mathrm{G}_2$ of $\pi_6(\mathrm{G}_2) \approx \mathbb{Z}_3$ (see [CR2, Pü]). This map parametrizes an isolated orbit of the adjoint action of G_2 on itself, namely, an orbit that passes through one of the two nontrivial elements in the center

$$\{1, e^{2\pi i/3}, e^{-2\pi i/3}\} \cdot \text{identity matrix}$$

of the subgroup $\mathrm{SU}(3) \subset \mathrm{G}_2$. Let us denote this element by C. Since the third power of ACA^{-1} is constant to the identity element of G_2 for any matrix $A \in \mathrm{G}_2$, the value-by-value third power of χ is the constant map to the identity element. This shows the cancellation in $\pi_6(\mathrm{G}_2)$ in the clearest and easiest way possible.

Combining this cancellation with the homotopy between b (i.e. the commutator, essentially) and χ we obtain the desired nullhomotopy of the third power of the commutator c in G_2.

The next map to discuss is the generator $\phi : S^6 \to \mathrm{SU}(3)$ of $\pi_6(\mathrm{SU}(3)) \approx \mathbb{Z}_6$. Here we will encounter a much subtler cancellation. The map ϕ can be described as follows: Consider the shortest path c from the identity matrix in $\mathrm{SU}(3)$ to a nontrivial element in the center. Concretely, let $c(t)$ be the diagonal matrix with entries e^{-2it}, e^{it}, and e^{it} where $t \in [0, \frac{2\pi}{3}]$. Regard the sphere S^5 as the unit sphere in $\mathbb{C}^3$ and consider the map

$$[0, \tfrac{2\pi}{3}] \times S^5 \to \mathrm{SU}(3), \quad (t, z) \mapsto Ac(t)A^{-1}$$

where $A \in \mathrm{SU}(3)$ is any matrix whose first column is z. For $t = 0$ or $t = \frac{2\pi}{3}$ the values of the map are constant to the identity matrix. Hence, this map induces a map $S^6 \to \mathrm{SU}(3)$, which is precisely the generator ϕ of $\pi_6(\mathrm{SU}(3))$ obtained from the lifting construction explained above.

The question is now how the order 6 of $\pi_6(\mathrm{SU}(3))$ can be recognized from this generator. It is obvious that the sixth power ϕ^6 does not differ in its structural appearance from ϕ itself, since the sixth power goes straight down to the diagonal entries of the path c. Nevertheless, ϕ^6 is nullhomotopic while ϕ is not. The trick that uncovers the essential cancellation is the following: Instead of ϕ consider the map $\phi_1 : S^6 \to \mathrm{SU}(3)$ that is induced by

$$[0, \tfrac{2\pi}{3}] \times S^5 \to \mathrm{SU}(3), \quad (t, z) \mapsto \eta(z)c(t)\eta(z)^{-1}$$

where $\eta : S^5 \to \mathrm{SU}(3)$ denotes any map that generates the stable homotopy group $\pi_5(\mathrm{SU}(3)) \approx \mathbb{Z}$ (particularly nice examples of such maps have been given and investigated in [CR1, PR, Pü]). The important property of such maps $S^5 \to \mathrm{SU}(3)$ is that the composition with the projection $\mathrm{SU}(3) \to S^5$ leads to a map $S^5 \to S^5$ of degree 2. From this it is easily seen that the homotopy class of ϕ_1 is twice the class of ϕ in $\pi_6(\mathrm{SU}(3))$. But now there are two homotopic copies ϕ_2 and ϕ_3 of ϕ_1 obtained by permuting the diagonal entries of the path c, and the value-by-value product of ϕ_1, ϕ_2, and ϕ_3 is the constant map to the identity matrix in $\mathrm{SU}(3)$. Thus we clearly see how ϕ_1^3 is nullhomotopic.

It remains to provide an explicit homotopy between ϕ_1 and ϕ^2. This homotopy can be obtained by the standard techniques that are used to prove the Hopf degree theorem (see [Pü] for details).

Analogous to the case of $\pi_6(\mathrm{G}_2)$ the cancellation revealed here gives in combination with the homotopy between the commutator of quaternions and ϕ the desired nullhomotopy of the sixth power of the commutator in $\mathrm{SU}(3)$.

Finally, we come to the most important map $b : S^6 \to S^3$. This map was constructed by the lifting construction mentioned above in [Du]. In order to give its explicit formula, let p be an imaginary quaternion and w be a quaternion such that $|p|^2 + |w|^2 = 1$, i.e., the vector (p, w) is contained in the Euclidean sphere S^6. Moreover, let $e^p = \cos \pi|p| + \frac{p}{|p|} \sin \pi|p|$ denote the exponential map of the unit sphere S^3 in the quaternions from 1. Then

$$b : S^6 \to S^3, \quad (p, w) \mapsto \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0 \\ -1, & w = 0. \end{cases}$$

The obvious problem is now to find a cancellation analogous to the ones described above. To uncover this cancellation is much harder than to uncover the previous cancellations and we do not have a complete solution so far. The central problem is to come from b to some abelian structure. In $\mathrm{SU}(3)$ above the path c was contained in just one maximal torus, which made the three homotopic maps ϕ_1, ϕ_2, and ϕ_3 commute. For $\pi_6(S^3)$ one might only speculate that one of the finite subgroups of $\mathrm{SO}(3)$ or S^3 gives an arrangement of several "copies" of b or maps derived from b such that the product of all these maps turns out to be constant to 1. We summarize this in the following problem:

Problem. Explain the order 12 of $\pi_6(S^3)$ in a geometric way similar to what we have done for $\pi_6(\mathrm{SU}(3))$ and $\pi_6(\mathrm{G}_2)$.

We view the fact that there exist many different proofs for the fact that $\pi_6(S^3)$ is isomorphic to $\mathbb{Z}_{12}$ as a strong indication that such a geometric explanation exists.

2.6 Presentations of other homotopy groups

The cancellations in $\pi_6(\mathrm{SU}(3))$ and $\pi_6(\mathrm{G}_2)$ above are prototypes of what can be found for some other homotopy groups: In [PR, Pü] we describe similar cancellations for the entire series $\pi_{2n}(\mathrm{SU}(n)) \approx \mathbb{Z}_{n!}$ (which was essential in the first proofs of the fact that S^1, S^3, and S^7 are the only parallelizable spheres, see [BMi, Ke]), for the series $\pi_{2n}(\mathrm{SU}(n-1))$ with even n, for $\pi_4(\mathrm{Sp}(2))$ and for $\pi_5(\mathrm{Sp}(2))$. Moreover, we obtain structurally nice generators of all homotopy groups $\pi_{4n-2}(\mathrm{Sp}(n-1))$. Also, the last stable homotopy groups $\pi_{2n-1}(\mathrm{SU}(n))$ and $\pi_{4n-1}(\mathrm{Sp}(n))$ are represented in several ways and the representing maps are used to determine some previously unknown nonstable homotopy groups of the symmetric space $\mathrm{SU}(n)/\mathrm{SO}(n)$. A particular important case is the stable homotopy group $\pi_7(\mathrm{Sp}(2)) \approx \mathbb{Z}$. We provide the first explicit formula for a generator of this group. With this structurally nice map we come closer to a nullhomotopy of the twelfth power of the commutator of unit quaternions. In fact, our formula contains a nullhomotopy of a map $S^6 \to S^3$ that represents the 12-th element of $\pi_6(S^3)$. However, the relation between this map and c^{12} still has to be given in a way such that the algebraic cancellations turn out as clearly and comprehensibly as possible.

3 Exotic diffeomorphisms and involutions of spheres

In 1956, J. Milnor made the surprising discovery of differentiable manifolds Σ^7_k that are homeomorphic but not diffeomorphic to the sphere. From our point of view, we want to remark that his discovery came through the analysis of explicit presentation of S^3-bundles over S^4 with SO(4)-structure group; this allowed him to 1) Construct a Morse function with just two critical points (therefore these are topological spheres) and 2) Compute the intersection form and Pontryagin classes of a bounding manifold M^8, thus proving that they are not diffeomorphic to spheres.

After a lot of topological work in the classification of differentiable structures on spheres (see [KM, Ko, Mi2, Le]), the next big step, for geometers, was the construction of a metric of non-negative curvature by Gromoll and Meyer ([GM]), via the beautiful geometric model of the Milnor exotic sphere $\Sigma^7_{2,-1}$ as a quotient of an isometric action of S^3 on $\mathrm{Sp}(2)$ with the biinvariant metric.

The set of differentiable structures on the topological sphere of dimension $n+1$ is the same as the number of path-connected components of the group $\mathrm{Diff}^+(S^n)$ of orientation-preserving diffeomorphisms of the sphere in one dimension less, the identification being given by , if $\sigma \in \mathrm{Diff}^+(S^n)$,

$$\sigma \mapsto \mathbb{R}^n \cup_{\tilde{\sigma}} \mathbb{R}^n\,,$$

the union of two disks by the map $\tilde{\sigma} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ defined by $\tilde{\sigma}(x) = \frac{1}{|x|}\sigma(x/|x|)$; thus σ provides charts in S^n that define a differentible structure. It is then easy to prove that the spheres corresponding to different diffeomorphisms σ, τ are diffeomorphic if and only if the σ and τ are isotopic, i.e., they belong to the same connected component of the group $\mathrm{Diff}^+(S^n)$. This has been known since Milnor's discovery in 1956; however, there was no explicit example at all of an *exotic* diffeomorphism, that is, an orientation-preserving diffeomorphism of a sphere that is not isotopic to the identity.

This embarrasing situation was solved in 2001 in [Du], where such an example is given. Part of the beauty of this example is that it was constructed using geometry: to be precise, the geometry of geodesics of the Gromoll-Meyer fibration $S^3 \cdots \mathrm{Sp}(2) \to \Sigma^7_{2,-1}$ in relation to the canonical fibration $S^3 \cdots \mathrm{Sp}(2) \to S^7$. Let us describe this construction briefly:

Let $\mathrm{Sp}(2)$ be the group of 2×2 matrices with quaternionic entries satisfying

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Gromoll-Meyer fibration is given as follows: the group $\mathrm{Sp}(1) = S^3$ acts freely on $\mathrm{Sp}(2)$ as follows: if $q \in \mathrm{Sp}(1)$ and $A \in \mathrm{Sp}(2)$,

$$q\bigstar \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix},$$

therefore producing a principal fibration $S^3 \to \mathrm{Sp}(2) \to \Sigma^7$.

The group S^3 also acts freely in Sp(2) as follows:

$$q \bullet \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{q} \end{pmatrix},$$

producing a principal fibration $S^3 \to \mathrm{Sp}(2) \to S^7$, where S^7 is the standard 7-sphere. In fact, the projection of Sp(2) onto S^7 is just $A \to 1^{st}$ column of A.

Note that the action of Sp(2) on itself by left translation commutes with the $\bullet$-action. Therefore, the Sp(2) acts by bundle maps of $S^3 \to \mathrm{Sp}(2) \to S^7$; the induced action on S^7 is given by matrix multiplication: if $A \in \mathrm{Sp}(2)$,

$$A \begin{pmatrix} a & c \\ b & d \end{pmatrix} \to A \begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore, we have two different fibrations with Sp(2) as total space:

$$\begin{tikzcd}
 & S^3 \arrow[d, "\star"'] & \\
S^3 \arrow[r, "\bullet"] & \mathrm{Sp}(2) \arrow[r] \arrow[d] & S^7 \\
 & \Sigma^7 &
\end{tikzcd}$$

In general, the $\bullet$-fibers and the $\bigstar$-fibers have no relation, but the fibers through any point in $O(2) \subset \mathrm{Sp}(2)$ of the $\bigstar$ and $\bullet$-actions coincide (in particular, the fibers through $\pm Id \in \mathrm{Sp}(2)$). This observation is crucial in the construction: We lift the round metric of the sphere to a left-invariant metric on Sp(2) (let us remind the reader that this is *not* the bi-invariant metric; the quotient of $S^3 \cdots \mathrm{Sp}(2) \to S^7$ under the bi-invariant metric on the sphere is not the round sphere).

Since the metric of the standard sphere is *wiedersehen*, i.e. every geodesic from a point meets after length π at the antipodal point (in particular the geodesics emanating from $(1,0) \in S^7 \subset \mathbb{H} \times \mathbb{H}$ meet again after length π at $(-1,0)$), the horizontal lifts from the identity of Sp(2) of these geodesics, which can be computed, lie after length π on the fiber over $(-1,0)$, that is, we have the following picture:

However, since the fibers through $-Id \in \mathrm{Sp}(2)$ of the $\bigstar$ and $\bullet$-actions coincide, it follows that, in the subduced metric from Sp(2), every geodesic in the exotic sphere $\Sigma^7_{2,-1}$

Figure 1:

emanating from $N = [Id]_\bigstar$ meets again, at length π, at the point $S = [-Id]_\bigstar$. Thus this metric in $\Sigma^7_{2,-1}$ has the wiedersehen property at the points N and S. In particular, this implies (see Theorem I of [Du]) that the exponential map is a diffeomorphism from the open ball of radius π on each tangent space $T_N\Sigma^7_{2,-1}$ and $T_S\Sigma^7_{2,-1}$ onto its image, which just misses the opposite wiedersehen point. In particular, the diffeomorphism $\sigma := \exp_N^{-1} \circ \exp_S |_{B(0,\pi/2)}$, composed with adequate identifications of the respective tangent spaces with $\mathbb{R}^7$, provides a diffeomorphism σ of S^6 that by reason of $\Sigma^7_{2,-1}$ being exotic, is not isotopic to the identity. See the picture:

Figure 2: This figure illustrates the construction of the exotic diffeomorphism σ of S^6. Regard an arbitrary $v \in S^6$ as the initial velocity vector of a geodesic in Σ^7 starting at the north pole. Assign to v the unique initial velocity vector $\sigma(v)$ of a geodesic in Σ^7 that starts at the south pole and yields at time $\frac{\pi}{2}$ the same point of the equator than the geodesic from the north pole.

All this would be idle talk if the geodesics could not be computed. However, they *can* be computed, and this produces a formula for the diffeomorphism σ:

Consider the Blakers-Massey map $b : S^6 \to S^3$ given at the end of section 2.4. The map b is a real analytic map whose homotopy class generates $\pi_6(S^3)$; note that Figure 1 express exactly what was done in section 2 with respect to finding homotopy groups by lifting geodesics. Then let

$$\sigma(p, w) = (b(p, w)p\,\overline{b(p, w)}, b(p, w)w\,\overline{b(p, w)})\,.$$

Then σ is a diffeomorphism, $\Sigma^7 = D^7 \cup_\sigma D^7$, and σ is not isotopic to the identity. In fact, σ is a generator of the group $\pi_0 \operatorname{Diff}^+(S^6)$ since it is known that $\Sigma^7_{2,-1}$ is a generator of the group of differentiable structures on the spheres ([EK, KM]).

There are deep algebraic relationships between the Blakers-Massey map b and σ, note that

$$\sigma^n(p,w) = (b^n(p,w)p\,\overline{b^n(p,w)}, b^n(p,w)w\,\overline{b^n(p,w)})$$

(this is not immediate since b is not constant, but is nontheless true). These kind of relationships were essential in [DMR] where this results were extended to the Cayley numbers and exotic diffeomorphisms of S^{14}. Topologically, the relation is more complicated: since $\pi_6(S^3) \cong \mathbb{Z}_{12}$, b^{12} is homotopic to a a constant, but σ^{12} is not isotopic to the identity. From the other side, σ^{28} is isotopic to the identity, but b^{28} is not homotopic to the identity. This is a good place to state the following:

Problem. Find an explicit isotopy between σ^{28} and the identity of S^6.

Having an explicit generator σ of $\pi_0 \operatorname{Diff}^+(S^6)$ is just the beginning, now we can play with it and study in depth the structure of diffeomorphism groups explicitly. In [DMR], it was shown the following: substitute quaternions by Cayley numbers in the definitions of b and σ, we get that $b : S^{14} \to S^7$ generates $\pi_{14}(S^7)$, and σ is an exotic diffeomorphism of S^{13}, which generates $bP^{16} \subset \pi_0 \operatorname{Diff}^+(S^{14})$. This last inclusion has index two, which prompts the following:

Problem. Find an explicit diffeomorphism $\tau : S^{14} \to S^{14}$ such that τ and σ generate $\pi_0 \operatorname{Diff}^+(S^{14})$.

The study of exotic spheres as disks glued by σ^n led us to the following surprise: Let $\rho = \alpha\sigma$, where α is the antipodal map of the sphere, and we can think of the variables being quaternions or Cayley numbers. Then we have the following result ([ADPR1]):

Theorem. *The map ρ is a free involution of S^6 (resp. S^{14}) that is* not *differentiably conjugate to the antipodal map.*

These formulas are the first formulas for exotic involutions on *Euclidean* spheres; of course exotic involutions on Brieskorn spheres and invariant spheres inside of exotic spheres had been defined since the sixties; see [AB, HM, LdM]. In fact these two concepts – Brieskorn and invariant subspheres of exotic spheres – can be explicitly identified in at least one case in dimension 5; see [DP].

The method of proof of the previous theorem analyzes how conjugation by the antipodal map acts on the different connected components of the group of diffeomorphism of the sphere in the relevant dimension. In particular, this leads us to

Theorem. *Every orientation-reversing map of* S^6 *is isotopic to a free involution.*

These exotic involutions of S^6 and S^{14} restrict to the equators given by $\operatorname{Re} w = 0$; the restrictions are also exotic (i.e. not differentiably conjugate to the antipodal map). In dimension 5, these formulas provide a satisfyingly simple pictorial description; see [ADPR1] for the pictures or watch the movie [ADPR2].

4 Full circle: a non-trivial element of $\pi_1 \operatorname{Diff}(S^5)$

The first section of this paper describes geometrically generators of homotopy groups; then we passed to the geometric description of exotic diffeomorphisms and exotic spheres. The exoticity of these diffeomorphisms can be translated as the fact that they do not belong to the connected component of the identity, and, in fact, we gave an element in each connected component of the group $\operatorname{Diff}(S^6)$ (and half of the connected components of $\operatorname{Diff}(S^{14})$), or, in homotopy terms, we represented the group $\pi_0(\operatorname{Diff}(S^6)$. Here, we come full circle: these methods, and the special form of the maps described in the previous section, do provide explicit non-trivial elements of $\pi_1(\operatorname{Diff}(S^5))$, the fundamental group of the space of orientation-preserving homeomorphisms of S^5.

The construction goes as follows: let us look at the structure of σ:

$$\sigma(p, w) = (b(p, w)p\,\overline{b(p, w)},\ b(p, w)w\,\overline{b(p, w)})\,.$$

Note that, since σ is given by a conjugation by b, it preserves the real part of w (recall that p has no real part by definition). Let us separate the formula in the real and imaginary parts of w, we write $w = t + \omega$, $t = \operatorname{Re}(w)$, $\omega = \operatorname{Im}(w)$. Then, also putting t last, the previous formula looks like

$$\sigma(p, \omega, t) = (b(p, t + \omega)p\,\overline{b(p, t + \omega)},\ b(p, t + \omega)\omega\,\overline{b(p, t + \omega)},\ t)\,.$$

which reads like a suspension of a diffeomorphism of S^5; it is not since the diffeomorphisms in (p,ω) depend on t. Forgetting about this last variable, we have the one-parameter family of diffeomorphisms of S^5 given by

$$t \mapsto \phi_t(p,\omega) = (b(p,t+\omega)p\,\overline{b(p,t+\omega)},\ b(p,t+\omega)\omega\,\overline{b(p,t+\omega)})\,.$$

Note that $\phi_1(p,\omega) = \phi_{-1}(p,\omega) = (p,\omega)$. Thus the map ϕ can be considered as a loop based at the identity of $\mathrm{Diff}^+(S^5)$. A moments reflection will convince the reader that a homotopy between this loop and the constant identity loop would translate to an isotopy between $\sigma : S^6 \to S^6$ and the identity, which is of course not possible. Therefore, we have

Theorem. *The loop ϕ represents a non-trivial element of $\pi_1 \mathrm{Diff}(S^5)$.*

As is the case in many of these construcions, the translation to Cayley numbers works through and the respective loop ϕ represents a non-trivial element of $\pi_1 \mathrm{Diff}(S^{13})$.

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