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THE GEOMETRY OF GRAVITATIONAL COLLAPSE

Roberto Giambò

Abstract

This review aims to give a first glance to some aspect related to singularities in General Relativity, in particular connection with the mathematical formulation of gravitational collapse theory. Some models of spacetime providing counterexamples to Penrose's Cosmic Censorship conjecture are reviewed, and related open issues are discussed.

1 Singularities in general relativity

By *singularity* it is usually meant a region of spacetime where laws of physics break down. Giving a general definition in a rigorous way, however, could be a hard task by itself. First of all, we recall that a spacetime with energy–momentum tensor T is a 4–dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$, time orientable, solving Einstein field equation

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} S\, g_{\mu\nu} = 8\pi\, T_{\mu\nu}, \tag{1.1}$$

where R_{ij} are Ricci tensor components, S is the scalar curvature, and the tensor T encodes all properties of matter – momentum, energy, stresses and strains. In order to avoid "fake" examples of singularities, the spacetime is also supposed to be *inextensible*, that is it cannot be isometrically embedded into another larger spacetime.

By a singularity we will mean a boundary of the spacetime where i) non–spacelike geodesic incompleteness occurs, and ii) at least one curvature invariant diverges along non–spacelike geodesics. We briefly comment on this definition.

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First, let us observe that i) does not cover all cases of singular behavior – see for instance the example by Geroch [5] of a geodesically complete spacetime possessing a future inextensible timelike curve with finite proper length. Here, however, the singularity will be determined by non–spacelike geodesics, representing motions of free falling particles and photons. Of course, we will also want to model physical genuine singularity, since in principle one may have examples of geodesic incomplete spacetime with bounded curvature – see for instance properties of the so–called Taub–NUT spacetime described in [14] – and this is the reason for requirement ii).

The easiest and most famous example of such a singularity is probably represented by Schwarzschild solution

$$g = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin\theta \, \mathrm{d}\phi^2\right). \tag{1.2}$$

This is the only possible spherically symmetric solution of field equation in vacuum, that is when the tensor T vanishes, and for $r > 2m$ represents the geometry of the spherical massive star exterior, with mass m. Actually, it is well known that the $r < 2m$ portion of the spacetime contains a true singularity at $r = 0$, and that the two patches can be continuously "glued" together at $r = 2m$ using appropriate coordinate transformations [2, 3, 12, 18]. Photons or particles may travel from the $r > 2m$–region to the other patch, but the contrary is forbidden, since all their trajectories will end in the boundary $r = 0$, according with requirement i) above. The surface $r = 2m$ therefore acts like an *horizon* which covers the singularity from observers in the outer region, and that is the reason why the $r < 2m$ solution is called Schwarzschild *black–hole*.

Many other examples of singularities are known, even accepting a lower degree of symmetry – for instance, Kerr spacetime that is only axisymmetric. In the first years of dvelopment of general relativity, it was conjectured that singularities could be avoided removing symmetry of the spacetime. Hawking and Penrose's singularity theorems [4], instead, predict geodesic incompleteness under assumptions not relying on any kind of symmetry. These theorems, however, do not provide any information about the causal structure of the singularity, that is whether it will be covered by an event horizon – as for Schwarzschild – or not. For instance, there is a subclass of Kerr solutions that possesses the singularity but no horizon, and therefore the singularity is globally visible by a faraway observer. But this solution is actually stationary, that is the singularity is eternally naked, whereas it would be of great interest to investigate properties of dynamically forming singularities. This category contains scalar field singularities investigated by Christodoulou [1], as well as the singularities forming in radiation collapse described by Vaidya spacetime [10]. However, solutions of (1.1) describing solid–elastic matter probably represent the most interesting field of investigation, since they arise for instance as a model for the indefinite collapse of a massive stellar body. This issue will be addressed in the following.

2 A short introduction to spherical gravitational collapse

When a massive star exhausts its nuclear fuel, it collapses under the effect of its own gravitation. If the star is unable to radiate away a sufficient amount of mass to fall below the neutron star limit (about 3 solar masses) no final stable state is available and then singularities are formed.

As said before, singularity theorems cannot predict the final state of such singularities. Although it is commonly believed that a *cosmic censor* exists who safely hides the singularity inside a black hole, actually this is just a conjecture, first formulated by Penrose [17], and cannot be proved using field equations. Indeed, counterexample may be given – and some of them will be sketched below – where the horizon forms but ”not at time” to prevent the singularity from communicating with light signals to a faraway observer. In the following, examples of spacetimes exhibiting naked singularities will be reviewed, but first, it is necessary to restate Einstein field equation in spherical symmetry. Of course, one can raise objections on physical reasonability of the solutions found – see last paragraph – but what turned out to be the truth in the last twenty years of research is that a general theory – and, in fact, even the underlying hypotheses – are extremely difficult to be stated (see e.g. [10]).

The general spherical line element in a *comoving* reference frame is given by the metric

$$g = -e^{2\nu(t,r)}\,\mathrm{d}t^2 + e^{2\lambda(t,r)}\,\mathrm{d}r^2 + R(t,r)^2\left(d\theta^2 + \sin\theta\,\mathrm{d}\phi^2\right). \tag{2.1}$$

We now introduce the *Misner–Sharp mass*, defined in terms of the gradient of R induced by g:

$$1 - \frac{2m}{R} = g(\nabla R, \nabla R),$$

which gives

$$m(r,t) = \frac{R}{2}\left[1 - R'^2\,e^{-2\lambda} + \dot{R}^2 e^{-2\nu}\right]. \tag{2.2}$$

With this auxiliary function, Einstein field equations (1.1) in the unknown functions λ, ν and R of r and t read (prime and dot denote derivation with respect to r and t)

$$m' = 4\pi\varepsilon\,R^2\,R', \tag{2.3}$$

$$\dot{m} = -4\pi\,p_r\,R^2\,\dot{R}, \tag{2.4}$$

$$\dot{R}' = \dot{\lambda}R' + \nu'\dot{R}, \tag{2.5}$$

$$p_r' = -(\varepsilon + p_r)\,\nu' - \frac{2R'}{R}(p_r - p_t), \tag{2.6}$$

where ϵ, p_r, p_t are the nonzero components of the energy momentum tensor:

$$T^\mu_\rho = \mathrm{diag}\left(-\varepsilon(r,t), p_r(r,t), p_t(r,t), p_t(r,t)\right). \tag{2.7}$$

2.1 Spherical dust collapse

The collapse of an incoherent spherical dust cloud is the described by the metric satisfying (1.1) with $T_{\mu\nu} = \epsilon u_\mu\,u_\nu$, where u_μ is a velocity (i.e. unit timelike) vector field and ϵ is the energy density of the system. With reference to the energy momentum tensor (2.7), it amounts to require the equation of state $p_r = p_t = 0$ to hold. This spacetime is described by the Tolman–Bondi–Lemaitre solution[1], that, in a reference frame (t, r, θ, ϕ) comoving with the collapsing

[1] equation (2.8) actually represent only a particular case of TBL class of solutions, called *marginally bound*, which anyway is enough for the purpose of this review.

matter ($u^\nu = -\delta^\nu_t$), reads

$$g = -\mathrm{d}t^2 + \left(\frac{\partial R}{\partial r}\right)^2 \mathrm{d}r^2 + R^2 \left(\mathrm{d}\theta^2 + \sin\theta\,\mathrm{d}\phi^2\right), \quad R(r,t) = r\left(1 - \frac{3}{2}t\sqrt{\frac{2m(r)}{r^3}}\right)^{2/3}, \tag{2.8}$$

where the mass function (2.2) m is a function of the coordinate r only, and then it is completely determined by the initial state at $t = 0$. The initial condition

$$R(r, t = 0) = r, \tag{2.9}$$

amounts to label the 2–dimensional shells of matter $r =$ const. in such a way that the map $t \mapsto R(r_0, t)$ denotes the evolution of the shell labelled r_0.

Equation (1.1) implies

$$\epsilon = \epsilon(r,t) = \frac{m'(r)}{4\pi\, R^2\, R'},$$

and then it is clear that, if m' is regular and bounded away from zero, the energy density ϵ diverges as $R = 0$ or $R' = 0$. The latter case corresponds to the so–called *shell crossing* singularities, which are of some interest by themselves but are not a topic of this study, that deals instead with *shell–focussing* singularities, i.e. those occurring as $R = 0$ [11].

Since the collapse is spherical, there is no dependence of angular parameters, and then the whole framework can be expressed by a 2–dimensional picture, where the singularity curve

$$t_s(r) = \frac{2}{3}\sqrt{\frac{r^3}{2m(r)}}$$

represents the time of complete collapse of the shell labelled r, and, by analogy with Schwarzschild case, the curve $t_h(r)$ implicitly defined by

$$R(r, t_h(r)) = 2m(r) \tag{2.10}$$

represents the apparent horizon, i.e. the time at which the shell gets trapped.

The situation is sketched in Figure 1(a). In order for the initial time to represent a nonsingular state, the mass $m(r)$ must be a regular function with

$m(0) = m'(0) = m''(0) = 0$. A choice of $m(r) = m_0 r^3$ models a homogeneous situation – i.e. all shells get singular at the same time – and it is not significant for our discussion. The interesting case is when

$$m(r) = m_0 r^3 + m_n r^{3+n} + o(r^{3+n}), \qquad n \in I\!N,\, n > 0. \tag{2.11}$$

For the sake of simplicity one chooses m_0 such that $t_s(r) \to 1$ as $r \to 0^+$. The constant m_n must be negative so that the initial energy ϵ at $t = 0$ is a not–increasing function of r.

In order to study light rays emission from the singularity, let us recall the following facts. First, if we restrict ourselves to look for *radial* null geodesics, i.e. curves with constant angular part θ and ϕ, then it suffices to study solution of the first order ODE

$$\frac{\mathrm{d}t_g(r)}{\mathrm{d}r} = R'(r, t_g(r)). \tag{2.12}$$

It is a remarkable property that the apparent horizon $t_h(r)$ is a subsolution of equation (2.12) for $r > 0$. Although it can be straightforwardly proved using (2.8) and (2.10), this is a more general fact that actually depends almost only on the spherical symmetry of the system, as proved in [7]. In view of this fact, comparison theorems in ODE straightly show that *noncentral* singularity – i.e. points $(r, t_s(r))$ with r strictly positive – cannot be naked, since the comoving time at which the shell labelled r (with $r > 0$) enters the trapped region is strictly smaller that the time at which the same shell gets singular.

Things may be different for the *central* shell ($r = 0$), where singularity occurs at the same time the shell gets trapped. The central singularity can give rise to a violation of cosmic censorship, since it is naked in case it emanates a null (and future–pointing) geodesic below the horizon. Unfortunately, studying equation (2.12) for the initial data $t_g(0) = 1$ cannot be performed using classical existence theorems in ODE, since the right hand side of (2.12) is not defined at the point $(0, 1)$. In order to give a satisfactory answer to the existence of solutions of the ODE, a technique first employed in [15] and therefore developed in [8, 9] can be used. It actually relies on looking for curves which coincide with the horizon at $r = 0$, stay below it for $r > 0$ and are supersolution of (2.12).

The existence of such curves – denoted as $t_*(r)$ in Figure 1(a) – implies, using comparison theorems in ODE, the existence of a null radial geodesic below the horizon that can be traced back until the singularity, which is therefore naked. Using Taylor developments, it can be seen [6] that, with reference to (2.11), the central singularity is at least locally naked if $n = 1, 2$ (if $n = 3$ the endstate is related to the sign of a quantity involving m_0 and m_n).

It must be also remarked that, although it is just a point of the curve $t_s(r)$ to violate cosmic censor, a faraway observer does not see it like a pointwise object, since there may be nonradial null geodesic emanating from the singularity as well, as pointed out in [16]. The study of nonradial null geodesic, anyway, does not affect the spectrum of endstates sketched before with reference to the radial geodesic equation (2.12), since nonradial geodesics are supersolutions and their existence would imply the existence of a radial light ray. In other words, a singularity radially censored is censored, and it must be remarked that this is again a general fact only relying on the symmetry of the problem, as shown in [8].

3 New examples of Cosmic Censorship violation

Tolman–Bondi–Lemaitre solutions were the first example of a elastic–solid medium to provide a violation of cosmic censorship, and in spite of its simplifications it represents a cornerstone in the study of gravitational collapse. Nonetheless, it is clear that it could have been of great interest to test Penrose's conjecture on models of collapsing objects with nonvanishing pressures. The first attempt was probably due to [13], that discovers solutions with $p_r = 0$, $p_t \neq 0$, but what is clear from the analysis carried out in [8] is that the class of nonvanishing tangential pressures can be further extended to a wider class of spacetimes. Indeed, for an elastic media, an equation of state is needed to close the system of Einstein field equation, i.e. a relation involving the energy–momentum tensor. We refer the reader to [13] and references therein for further details on this topic.

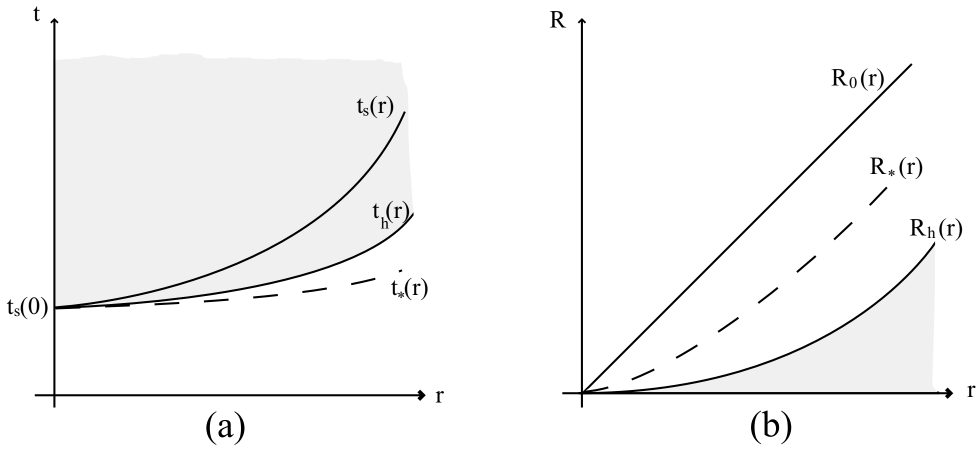

Figure 1: A 2–dimensional picture of the gravitational collapse. The grey region is the trapped region where light signals cannot escape from. In (a) the situation using comoving coordinates is sketched. In (b) the same situation using the area–radius system (r, R, θ, ϕ). In view of (2.9), the curve $R_0(r) = r$ represents the initial data of the problem.

We just recall here that this relation can be given in different ways, and a very useful one is to express $T_{\mu\nu}$ as functions of the "spatial" coordinates r, θ, ϕ and the "spatial" part of the metric e^λ, R, $R\sin\theta$. Dependance on θ, ϕ is forbidden by spherical symmetry and so we are left with r, R, e^λ as possible arguments of $T_{\mu\nu}$ to formulate an equation of state.

The class of solutions found in [8] is described by the equation of state

$$\frac{\partial p_r(r, R, e^\lambda)}{\partial e^\lambda} = 0$$

and the line element is expressed in a *not comoving* coordinate system, which promotes R as a new coordinate together with r and θ, ϕ:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{R}\right) G^2 \mathrm{d}r^2 + 2G\,\frac{Y}{u}\mathrm{d}R\mathrm{d}r - \frac{1}{u^2}\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\ . \quad (3.1)$$

In the above formula, the two functions $m(r, R)$ and $Y(r, R)$ are arbitrary (positive) functions, while

$$u^2 = Y^2 + \frac{2m}{R} - 1, \quad (3.2)$$

and the function G is given in terms of a quadrature:

$$G(r,R)=\int_R^r \frac{1}{Y(r,\sigma)}\frac{\partial(1/u)(r,\sigma)}{\partial r}\,\mathrm{d}\sigma+\frac{1}{Y(r,r)u(r,r)}. \tag{3.3}$$

For instance, spherical dust solutions correspond to a couple (m, Y) such that mass profile m only depends on r and the Y is equal to 1 (at least in the marginally bound case).

In general, it is shown that a collapsing situation is related to Taylor expansion of the 2–variables function

$$H(r,R) := 2m(r,R) + R(Y^2(r,R)-1) \tag{3.4}$$

that must be such that the lowest order term is given by $\alpha r^3+\beta r^2 R+\gamma r R^2+\delta R^3$ with $\alpha > 0$ (compare with (2.2)). In this case, the endstates of central singularity can be studied in an analogous way as in dust case. Indeed, in [8] is shown how the endstate of central singularity is related to Taylor development of the (regular) function

$$G(r,R=0)=\xi r^{n-1}+o(r^{n-1}).$$

If $n = 1, 2$ the singularity is naked, whereas if $n = 3$ a transition situation is given depending on the sign of a quantity related to the data of the problem.

The picture is sketched in Figure 1(b). One must now pay attention to the fact that, for instance, the apparent horizon curve is now a supersolution of radial null geodesic equation. This is actually due to the change of orientation provided by the new coordinate system: the element of a 2–dimensional surface $\{\theta = \text{const.}, \varphi = \text{const.}\}$ transforms as follows:

$$\mathrm{d}r\wedge\mathrm{d}t=\frac{1}{\dot R}\,\mathrm{d}r\wedge\mathrm{d}R$$

since we are considering a collapse situation, $\dot R < 0$ and therefore this change of orientation results in an opportune restatement of the main results valid in comoving coordinates. For instance, one must now look for curves – denoted by $R_*(r)$ in Figure 1(b) – that are subsolution (instead of supersolutions) lying above (and not below) the horizon, to grant the existence of radial null geodesics.

As a matter of fact, another complication is given by the use of this coordinate system. Indeed, in comoving coordinates, it is $R(r = 0, t) = 0$ for all times prior to singularity formation: the vanishing of R at the central shell is not due to a singular behavior, but to the geometry of the spacetime in the "pole" $r = 0$. The use of (r, R, θ, ϕ) coordinate system therefore implies that the line $(r = 0, t < t_s(0))$ – which in principle contains both regular points and a singular point – is mapped into the origin of coordinates, and so one cannot make distinction anymore between regular and singular centre, unless one does not study behavior of physical quantities along curves escaping from $(r = 0, R = 0)$.

For instance, one must ensure that candidate curves for being subsolutions of null radial geodesic equation, restated in comoving coordinates, tend to the singular centre as r goes to zero. One way to do that is to study a physically relevant quantity as the energy density along those curves: if it diverges, this means that the curve tends to the singularity. Another way is to integrate the relation

$$u = -\dot{R}\, e^{-\nu}$$

to find straightly the relation between the comoving time and the "new" variables in the coordinate system:

$$t(r, R) = \int_R^r \frac{e^{-\nu}}{u}\, \mathrm{d}\sigma,$$

and study directly the asymptotic behavior of this quantity as r goes to zero.

All in all, this coordinate system provides a powerful method to escape, but not completely, the difficulties connected to the use of comoving coordinates, the latter remaining the best way to understand the underlying physics of the problem.

4 Conclusions

The questions connected with the study of gravitational collapse in General Relativity are far from being solved in a satisfactory manner. Many models providing counterexamples to Penrose's conjecture have been found, and some of them have been reviewed above, but of course it would be of great interest to investigate what happens if symmetry assumptions are removed. Unfortunately, even in case of axial symmetry – like Kerr spacetime – things become more complicated, and even the geometrical formulation of the problem presents obstacles that have not been removed so far.

In addition, whenever one will be able to give satisfactory answers in more general cases, one must keep in mind that not only classical General Relativity should be taken into account, since quantum effects may not be neglected in the extreme states of collapse, but unfortunately again, quantum gravity is a still unknown theory.

These are the main reasons why Penrose's opinion about the question of naked singularities existence in Nature, termed almost forty years ago as "the most fundamental unanswered question of general-relativistic collapse theory", is still of great interest at present time.

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Dipartimento di Matematica e Informatica
Università di Camerino, Italy
E-mail: roberto.giambo@unicam.it
URL: http://dmi.unicam.it/˜giambo