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ON A CONSTRUCTION OF PARAMETRIZED MINIMAL NETWORK

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Abstract

We survey a new method first introduced in [MY], which can be used to construct a minimal network in $\mathbf{R}^n$ which spans a finite set of points by minimizing a form of energy functional. The new construction is very much relevant to the much studied Steiner minimal network problem.

1 Introduction

We start with the following classical problem;

Generalized Fermat Problem *Find a point in the plane, the sum of whose distances from given n points is minimal.*

When $n = 2$, the solution is clearly any point lying on the line segment connecting the two given points. When $n = 3$, it is called the Fermat Problem [IT], whose solution attributed to Torricelli is given by using classical plane geometry.

We now formulate this problem using calculus of variations. Recall in minimizing length of paths connecting two given points, there is no sequential compactness for a minimizing sequence of maps (paths) since one can always repararametrize the curves without changing the length, and the space of such reparameterizations is not compact. For the case of $n = 2$, this problem can be skirted around by minimizing the energy functional instead. The energy minimizing sequence of maps (paths) are automatically length minimizing, and a convergent subsequence exists. An energy/length minimizer thus obtained produces a constant speed parameterization of the path connecting the two given points.

One can try to follow the same argument for $n = 3$, by minimizing the *sum* of the energy for three segments connecting the interior vertex and the given three boundary points. It turns out that the juncture point which minimizes this functional is the center of mass of the triangle formed by the three points. This solution clearly does not minimize the length, as one can see by observing that for a triangle with a vertex whose interior angles is larger than 120 degree, the juncture point which minimizes the length is the vertex itself, while it cannot possibly be the center of mass of the triangle.

In a collaboration of the arthor with C. Mese [MY], we introduced a weighted energy functional. Simply put, it is the sum of the energy from the three segments, weighted by reciprocals of weights $c_i \geq 0$, $(i = 1, 2, 3)$ with $\sum c_i = 1$. As the space of such weights are compact, a minimizing sequence in the product space of the space of maps and the space of weights has a convergent subsequence, which then produces a point which minimizes the distances from the three given points.

To be more precise, we start with definitions, which allow us to deal with much more general settings. Let G be a connected finite graph. Denote by ∂G the set of vertices of G incident with only one edge. The vertices not in ∂G are called interior vertices. In this paper we assume that the interior vertices of the graphs are of valence greater than two. Suppose there are m edges in G and n vertices in ∂G. Label the edges of G by $e_1, ..., e_m$ so that $e_1, ..., e_n$ corresponds to the n edges incident to a vertex in ∂G. Now let the vertices incident to the edge e_i be labeled $e_{i,0}$ and $e_{i,1}$. Here, the labeling for an edge e_i with $i = 1, ..., n$ is chosen so that $e_{i,0} \in \partial G$. Let $e_{i,j} \sim e_{i',j'}$ $(i, i' = 1, ..., m,\ j, j' = 0, 1)$ if $e_{i,j}$ and $e_{i',j'}$ represent the same vertex in G.

Let $\mathcal{C}$ be defined by

$$\mathcal{C} = \{c = (c_1, ..., c_m) : c_1 + ... + c_m = 1,\ \ c_i \geq 0\ (i = 1, ..., m)\}.$$

Fix $p_1, ..., p_n \in \mathbf{R}^n$ and define

$$\begin{aligned}\mathcal{A}_G &= \{\alpha = (\alpha_1, ..., \alpha_m) : \alpha_i : I \to \mathbf{R}^n \in C^\infty \text{ so that } \alpha_i(0) = p_i \ (i = 1, ..., n) \\ &\qquad \text{and } \alpha_i(j) = \alpha_{i'}(j') \ (i = 1, ..., m, j = 0, 1) \text{ if } e_{i,j} \sim e_{i',j'}\}.\end{aligned}$$

Note that $\alpha \in \mathcal{A}_G$ can be seen as a map from G to $\mathbf{R}^n$ satisfying $\alpha(\partial G) = \{p_1, ..., p_n\}$.

The length $L(\alpha)$ of the network $\alpha[G]$ is given by the sum of the lengths of the curves defined by $\alpha_1, ..., \alpha_m$;

$$L(\alpha) = \sum \int_0^1 \left| \frac{d\alpha_i}{dt} \right| dt.$$

Let $c = (c_1, ..., c_m) \in \mathcal{C}$. We say that $\alpha = (\alpha_1, ..., \alpha_m) \in \mathcal{A}_G$ is compatible with c if $\alpha_i(t)$ is a constant map whenever $c_i = 0$. Otherwise, we say α is incompatible with c. Furthermore, we define the c-energy of $\alpha \in \mathcal{A}_G$ as

$$E_c(\alpha) = \begin{cases} \displaystyle\sum^{c} \frac{1}{c_i} \int_0^1 \left|\frac{d\alpha_i}{dt}\right|^2 dt & \text{if } \alpha \text{ is compatible with } c \\ \infty & \text{if } \alpha \text{ is incompatible with } c \end{cases}$$

Here, $\displaystyle\sum^{c}$ denotes the sum over i with $c_i \neq 0$.

We consider the variational problem of achieving

$$\inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{A}_G} E_c(\alpha) \tag{1}$$

and show that its minimizing element is a length minimizer in an appropriate sense.

Note here that the Fermat Problem corresponds in the setting where $m = n = 3$ and the graph G is the standard tripod, whose boundary ∂G is prescribed to be sent to a set of given three points $\{p_1, p_2, p_3\}$ in $\mathbf{R}^2$.

2 Minimizing Length and Energy Functionals

Lemma 1 *For every $c \in \mathcal{C}$ and $\alpha \in \mathcal{A}_G$, $L(\alpha) \leq (E_c(\alpha))^{1/2}$. The equality $L(\alpha) = (E_c(\alpha))^{1/2}$ is achieved if and only if $\left|\frac{\partial \alpha_i}{\partial t}\right| = l_i$ and $c_i = \frac{l_i}{\sum_{i=1}^m l_i}$ $(i = 1, ..., m)$.*

Proof. By the Cauchy-Schwartz inequality, as well as the fact the energy of a parameterized curve bounds from above the square of the length, gives

$$L(\alpha) = \sum^{c} l_i = \sum^{c} \sqrt{c_i}\frac{l_i}{\sqrt{c_i}} \leq \left(\sum^{c} c_i\right)^{1/2} \left(\sum^{c} \frac{l_i^2}{c_i}\right)^{1/2}$$
$$\leq 1 \cdot \left(\sum^{c} \frac{1}{c_i}\int_0^1 \left|\frac{\partial \alpha_i}{\partial t}\right|^2\right)^{1/2} = (E_c(\alpha))^{1/2}.$$

We have equality in the first inequality above if and only if $c_i = \frac{l_i}{\sum_{i=1}^m l_i}$ and in the second inequality if and only if $\left|\frac{\partial \alpha_i}{\partial t}\right| = l_i$.

□

We next show that the minimizing element of our *inf inf* variational problem in fact minimizes the total length L.

Theorem 2 *If*

$$E_{c^*}(\alpha^*) = \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}_G} E_c(\alpha)$$

for $c^* \in \mathcal{C}$ *and* $\alpha^* \in \mathcal{A}_G$*, then* $L(\alpha^*) \leq L(\alpha)$ *for all* $\alpha \in \mathcal{A}_G$*.*

Proof. First, note that if $c \in \mathcal{C}$ and $\alpha^c \in \mathcal{A}_G$ satisfies $E_c(\alpha^c) = \inf_{\alpha\in\mathcal{A}_G} E_c(\alpha)$, then $\alpha_i^c : I \to \mathbf{R}^n$ $(i = 1, ..., m)$ must be energy minimizing; i.e.

$$\int_0^1 \left|\frac{d\alpha_i^c}{dt}\right|^2 \leq \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt$$

for all $\gamma : I \to \mathbf{R}^n \in C^\infty$ with $\gamma(0) = \alpha_i^c(0)$ and $\gamma(1) = \alpha_i^c(1)$. (Otherwise, we can replace α_i^c by γ to lower the c-weighted energy.)

In particular, this implies that α_i^* is a one-dimensional harmonic map, which in turn implies it is linear and thus $\left|\frac{d\alpha_i^*}{dt}\right|$ is a constant, say λ_i. Now note that if $\Lambda = \sum_{i=1}^m \lambda_i$, then $\lambda = (\frac{\lambda_1}{\Lambda}, ..., \frac{\lambda_m}{\Lambda}) \in \mathcal{C}$ and

$$E_{c^*}(\alpha^*) \leq E_\lambda(\alpha^*) = \sum^{\lambda} \frac{\lambda_i^2}{\left(\frac{\lambda_i}{\Lambda}\right)} = \Lambda\left(\sum^{\lambda} \lambda_i\right) = L(\alpha^*)^2 \qquad (2)$$

by the minimality of c^*. Furthermore, $L(\alpha^*)^2 \leq E_{c^*}(\alpha^*)$ by Lemma 1 and this shows $L(\alpha^*)^2 = E_{c^*}(\alpha^*)$.

For an arbitrary choice of $\alpha = (\alpha_1, ..., \alpha_m) \in \mathcal{A}_G$, we wish to show $L(\alpha^*) \leq L(\alpha)$. Since reparameterizing a curve does not change the length of the image, without the loss generality, we may assume that α_i $(i = 1, ..., m)$ is parameterized proportional to arclength and let l_i equals the length of the image of α_i. By the minimizing property of c^* and α^*, $E_{c^*}(\alpha^*) \leq E_c(\alpha)$ for every $c \in \mathcal{C}$ and every $\alpha \in \mathcal{A}_G$. Thus, if we set $c = (c_1, ..., c_m) \in \mathcal{C}$ where $c_i = \frac{l_i}{\sum_{j=1}^m l_j}$, we obtain

$$E_c(\alpha) = \sum^c \frac{1}{c_i} l_i^2 = \sum^c \frac{1}{\left(\frac{l_i}{\sum_{j=1}^m l_j}\right)} l_i^2 = \left(\sum_{j=1}^m l_j\right)\left(\sum^c l_i\right) = L(\alpha)^2,$$

and this implies

$$L(\alpha^*)^2 \leq E_{c^*}(\alpha^*) \leq E_c(\alpha) = L(\alpha)^2.$$

Therefore, $L(\alpha^*) \leq L(\alpha)$.

□

Lemma 3 *If*

$$E_{c^*}(\alpha^*) = \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{A}} E_c(\alpha)$$

for $c^* \in \mathcal{C}$ *and* $\alpha^* \in \mathcal{A}_G$, *we have* $c_i^* = \frac{\lambda_i}{\sum_{j=1}^m \lambda_j}$ *where* $\left|\frac{d\alpha_i^*}{dt}\right| = \lambda_i$.

Proof. Immediate from (2) and Lemma 1.

□

3 Existence of the Minimizers

Proposition 4 *For each* $c \in \mathcal{C}$, *there exists a c-energy minimizer* $\alpha^c \in \mathcal{A}_G$. *In other words, there exists* $\alpha^c \in \mathcal{A}_G$ *so that*

$$E_c(\alpha^c) = \inf_{\alpha \in \mathcal{A}_G} E_c(\alpha).$$

Proof. Fix $c = (c_1, ..., c_m) \in \mathcal{C}_G$ and let $\{\alpha^j = (\alpha_1^j, ..., \alpha_m^j)\} \subset \mathcal{A}_G$ be a minimizing sequence, i.e.

$$\lim_{j \to \infty} E_c(\alpha^j) = \inf_{\alpha \in \mathcal{A}_G} E_c(\alpha).$$

If we reparameterize α_i^j with respect to arclength and call it $\bar{\alpha}_i^j$, then

$$E_c(\bar{\alpha}^j) = L(\bar{\alpha}^j)^2 = L(\alpha^j)^2 \leq E_c(\alpha^j).$$

Therefore, we may assume that α_i^j is arclength parameterized with speed l_i^j.

Assume $E_c(\alpha^j) \leq M$. Thus, $(l_i^j)^2 \leq c_i E_c(\alpha_i^j) \leq c_i M \leq M$. This in turn implies that α^j is an equicontinuous family of maps. By the Arzela-Ascoli Theorem, there exists a subsequence of α^j (which we still denote by α^j by abuse of notation) which converges uniformly to $\alpha^c \in \mathcal{A}_G$. In particular, $\lim_{j\to\infty} l_i^j = l_i$ where l_i is the arclength of $\alpha^c([0,1])$. It then follows that

$$\begin{aligned}
\inf_{\alpha \in \mathcal{A}_G} E_c(\alpha) &\leq E_c(\alpha^c) \\
&= \sum^c \frac{1}{c_i} l_i^2 \\
&= \sum^c \frac{1}{c_i} \lim_{j\to\infty} (l_i^j)^2 \\
&= \lim_{j\to\infty} E_c(\alpha^j) \\
&= \inf_{\alpha \in \mathcal{A}_G} E_c(\alpha)
\end{aligned}$$

and this shows the existence of a c-energy minimizer α^c.

□

Theorem 5 *There exists an absolute minimizer for the inf inf variational problem. In other words, there exists $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{A}_G$ so that*

$$E_{c^*}(\alpha^*) = \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{A}_G} E_c(\alpha).$$

Proof. Let $c^j = (c_1^j, ..., c_m^j) \in \mathcal{C}$ be a minimizing sequence. In other words, if we let $\alpha^j \in \mathcal{A}_G$ be a c^j-energy minimizer whose existence is guaranteed by Proposition 4, then

$$\lim_{j\to\infty} E_{c^j}(\alpha^j) = \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{A}_G} E_c(\alpha).$$

Since α_i^j is energy minimizing (see proof of Theorem 2), it is parameterized by arclength.

Since $\mathcal{C} \subset \mathbf{R}^n$ is compact, there exists a subsequence of c^j (still denoted by c^j) which converges to $c^* \in \mathcal{C}$. Without the loss of generality, we may assume $E_{c^j}(\alpha^j) \leq M$ and $c_i^j \neq 0$ for all $j = 1, 2, ...$ and $i = 1, 2, ..., m$. If l_i^j is the arclength of $\alpha_i^j([0,1])$, then $(l_i^j)^2 \leq c_i^j E_{c^j}(\alpha^j) \leq c_i^j M \leq M$. Thus, α_i^j $(j = 1, 2, ...)$ is an equicontinuous family of map and there exists a subsequence of α_i^j (still denoted by α_i^j) which converges uniformly to α_i^*. Let l_i^* is the arclength of $\alpha^*([0,1])$.

If $c_i^* \neq 0$, then

$$\begin{aligned} \frac{1}{c_i^*}(l_i^*)^2 &= \lim_{j\to\infty} \frac{1}{c_i^*}(l_i^j)^2 \\ &= \lim_{j\to\infty} \frac{1}{c_i^j}(l_i^j)^2 + \lim_{j\to\infty} \left(\frac{1}{c_i^*} - \frac{1}{c_i^j} \right) (l_i^j)^2. \end{aligned}$$

Since $(l_i^j)^2 \leq M$, the last term on the right hand side equals 0. Therefore,

$$\begin{aligned} \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}_G} E_c(\alpha) &\leq E_{c^*}(\alpha^*) \\ &= \sum^{c^*} \frac{1}{c_i^*}(l_i^*)^2 \\ &= \lim_{j\to\infty} \sum^{c^*} \frac{1}{c_i^j}(l_i^j)^2 \\ &\leq \lim_{j\to\infty} E_{c^j}(\alpha^j) \\ &= \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}_G} E_c(\alpha) \end{aligned}$$

and thus $E_{c^*}(\alpha^*) = \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}_G} E_c(\alpha)$.

□

Remark. The absolute minimizer as above can have various degenerations of edges. In particular, when the domain graph G has a nontrivial topology, there are various ways the topology of the image network $\alpha(G)$ result from degenerations of edges. Consequently we do not expect uniqueness in this length minimizing solution. For some examples where the uniqueness fails, see, for example [IT].

4 Geometry of the Minimizers

For the energy minimizers thus obtained, we demonstrate a particular balancing phenomena, due to the stationarity of the energy functional. Note that the stationarity condition is used here as a local property.

Theorem 6 *Choose an interior vertex p in G, and let $e_1, ..., e_k$ be the edges incident to p. For $c^0 \in \mathcal{C}$ denote the weights assigned to the edges $\{e_j\}$ by $\{c_j^0\}$. Provided that each c_j^0 is strictly positive, a c^0-energy minimizing map α^0 satisfies*

$$\sum_{j=1}^{k} \frac{1}{c_j^0} \frac{\partial \alpha_j^0}{\partial t}(p) = 0.$$

Proof. Let V be an arbitrary vector field on $\cup_{i=1}^{k} \alpha_i^0(I)$, vanishing at all the vertices but $\alpha(p)$. In this sense, V defines a local interior deformation of the image graph $\alpha^0[G]$ around $\alpha(p)$. Let $\alpha^s \in \mathcal{A}_G$ be defined by setting $\alpha_i^s(t) = \alpha_i^0 + sV(\alpha_i^0(t))$ $(i = 1, ..., k)$. Without loss of generality, we may assume that each $\alpha_j^0 : I \to \mathbf{R}^n$ is oriented so that $\alpha_j^0(1) = \alpha_j^0(p)$ By the minimizing property of α^0, we have

$$\begin{aligned}
0 &= \frac{d}{ds} E_{c^0}(\alpha^s)|_{s=0} \\
&= \frac{d}{ds} \left(\sum_j \frac{1}{c_j^0} \int_0^1 \left\langle \frac{\partial \alpha^s}{\partial t}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt \right) |_{s=0} \\
&= \sum_j \frac{1}{c_j^0} \left(\frac{\partial}{\partial s} \int_0^1 \left\langle \frac{\partial \alpha^s}{\partial t}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt \right) |_{s=0} \\
&= \sum_j \frac{1}{c_j^0} \left(\int_0^1 \frac{\partial}{\partial s} \left\langle \frac{\partial \alpha^s}{\partial t}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt \right) |_{s=0} \\
&= \sum_j \frac{1}{c_j^0} \left(\int_0^1 2 \left\langle \frac{\partial^2 \alpha^s}{\partial t \partial s}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt \right) |_{s=0} \\
&= 2 \sum_j \frac{1}{c_j^0} \left(\int_0^1 \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha^s}{\partial s}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt \right) |_{s=0} \\
&= 2 \sum_j \frac{1}{c_j^0} \cdot \left\langle V(\alpha_i^0(p)), \frac{\partial \alpha^0}{\partial t}(p) \right\rangle
\end{aligned}$$

where $\frac{\partial\alpha^0}{\partial t}(p)$ is defined by continuity. Since $V(\alpha_i^0(p))$ is arbitrary, we have shown $\sum_j \frac{1}{c_j^0}\frac{\partial\alpha_i^0}{\partial t}(p) = 0$.

□

minimizers, this balancing around an interior vertex with nondegenerate edges has a more direct geometric meaning.

Corollary 7 *Let α^* and c^* be as in Theorem 2. For an interior vertex p in G with all the edges $(e_1, ..., e_k)$ incident to p nondegenerate with respect to α^*, the k unit vectors outward and tangent to the edges sums up to a zero vector. In particular for $k = 3$, the three edges meet at 120 degree angle.*

Proof. Let $\lambda_i = \left|\frac{d\alpha_i^*}{dt}\right|$ and $\Lambda = \sum \lambda_i$. From Lemma 3, we obtain $c_i^* = \frac{\lambda_i}{\Lambda}$ and thus Theorem 6 implies

$$\sum_{j=1}^{k} \frac{1}{\lambda_j}\frac{\partial\alpha_j^0}{\partial t}(1) = \Lambda \sum_{j=1}^{k} \frac{1}{c_j^*}\frac{\partial\alpha_j^*}{\partial t}(1) = 0.$$

Since $\frac{1}{\lambda_j}\frac{\partial\alpha_j^*}{\partial t}(1)$ is a unit vector which indicates the outward direction of the image of α_i^*, the statement follows.

□

Recall the well-known fact [IT] that the solution to the Steiner minimal network problem has vertices at which three edges meet at 120 degree angles; a special case of the general phenomena for c-energy minimizer described in Theorem 6 above. It should be noted here that the so-called Steiner problem is about finding a network of the least length among *all* possible graphs. As it has been known that only graphs with vertices of valence three appear as solutions to the Steiner problem, the inf inf approach introduced in this article would produce the Steiner network only when we start with an appropriate graph G. On the hand the generalized Fermat problem $(n > 3)$ can be solved using our method, while the it has little to do with the Steiner problem, for the same reason that a vertex of valence greater than three is inefficient in minimizing length. Hence our method is potentially useful in situations where introducing a juncture is costly compared to the lines, so that it only make sense to have junctures with the number of valence much higher than three.

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