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ON THE MATHEMATICAL WORK OF JOSÉ F. ESCOBAR

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Abstract

In this article we provide an overview of the work of José F. Escobar, who gave many important contributions to the fields of Differential Geometry and Partial Differential Equations throughout his mathematical career.

1 Introduction

The mathematician José Fernando Escobar made many important contributions to the fields of Differential Geometry and Partial Differential Equations throughout his productive life. This article intends to provide an overview of his mathematical work. The content of the paper essentially appeared on a lecture given by the author, in conjunction with H. Araújo and L. Rodriguez, at the XIII Brazilian School of Differential Geometry, held at IME-USP in 2004. Due to the limitations of space and to the inherent subjectivity of the task, we do not wish to describe in details the entire work of J. F. Escobar, but only a part of it considered most relevant from the author's personal perspective.

Escobar's powerful technical skills and his wide range of interests were certainly influenced by his Ph. D. studies under the supervision of Richard Schoen. During his mathematical career he touched several different and difficult problems lying in the domain of Geometric Analysis - the part of Differential Geometry concerned with the partial differential equations that naturally arise in geometrical questions. We can safely classify his results into two broad classes of problems.

The first class encompasses problems of prescribing the scalar curvature and the boundary mean curvature under conformal deformations of metrics. What is perhaps his greatest contribution falls in this class - the solution of the Yamabe Problem on manifolds with boundary in almost all cases. As a consequence of that achievement he proved a generalization of the profound Riemann mapping theorem to higher dimensions, showing that, except possibly for dimensions 4 and 5, every smooth bounded open set of a Euclidean space admits a conformal scalar-flat metric with constant mean curvature on the boundary. Another remarkable theorem was obtained in his Ph. D. thesis in collaboration with R. Schoen. He completely determines the functions which can be realized as the scalar curvature of a conformal metric on nontrivial quotients of the three-sphere, extending the well-known Moser's result for $\mathbb{RP}^2$.

In the second class we can distinguish his work concerned with the spectral properties of the Laplacian on manifolds and isoperimetric inequalities. He was interested in the Laplace-Beltrami operator on complete and noncompact manifolds of nonnegative sectional curvature, together with similar questions on the differential form spectrum of the Hodge Laplacian. Motivated by his work on the Yamabe Problem he also became interested in the Steklov eigenvalue problem on manifolds, which somehow had been forgotten in the literature. He obtained various comparison results for the first nontrivial Steklov eigenvalue and studied its relation to isoperimetric constants. Very recently he also worked on isoperimetric inequalities in three dimensional PL-manifolds of nonpositive curvature ([4]).

This paper is organized as follows. In Section 2 we describe his contributions to the Yamabe Problem on manifolds with boundary. In Section 3 we discuss his work on the problem of conformally prescribing the scalar and mean curvatures. In Section 4 we describe some of his work on the spectrum of the Laplace-Beltrami operator and the Hodge Laplacian on nonnegatively curved manifolds. And in Section 5 we discuss some of his results on the first nontrivial Steklov eigenvalue. In the last section we make some final remarks of a more personal tone.

2 Yamabe Problem on Manifolds with Boundary

The classical Yamabe Problem consisted in showing that every Riemannian compact manifold, without boundary, admits a conformally related metric with constant scalar curvature. In dimension two that follows from the much celebrated Uniformization Theorem of Riemann surfaces. As for the higher dimensional setting, Yamabe was the first one to raise that question, and although he had claimed to have solved the problem, his paper ([34]) contained a crucial mistake. After the subsequent efforts, throughout approximately 25 years, of Trudinger ([33]), Aubin ([1]) and Schoen ([30]), the Yamabe problem was finally given an affirmative solution in all cases. That achievement was of special importance because it required the complete solution, for the first time, of a nonlinear elliptic equation of critical exponent.

If the compact manifold being considered has a nonempty boundary one can think of several possible boundary conditions to ask for. Escobar figured out that, from the point of view of conformal geometry, a natural geometrical condition would have to involve the mean curvature. That important observation naturally led Escobar ([12]) to study the problem of finding smooth metrics with constant scalar curvature and minimal boundary inside a given conformal class. Throughout this paper R_g will denote the scalar curvature with respect to the Riemannian metric g, while h_g will be the mean curvature on the boundary. We can precisely formulate the problem as follows:

Yamabe Problem I (on manifolds with boundary)
Let (M^n, g) be a compact Riemannian manifold with boundary ∂M, and assume $n \geq 3$. Find a metric $\tilde{g}$, conformally related to g, such that for a certain constant $c \in \mathbb{R}$ one has

$$\begin{cases} R_{\tilde{g}} &= c \text{ in } M, \\ h_{\tilde{g}} &= 0 \text{ on } \partial M. \end{cases}$$

If we write the conformal change as $\tilde{g} = u^{\frac{4}{n-2}} g$, where u is a smooth positive function defined on the manifold, we will find that the transformation laws for the scalar and mean curvatures are given by:

$$\begin{aligned} R_{\tilde{g}} &= -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} L_g u, \\ h_{\tilde{g}} &= \frac{2}{n-2} u^{-\frac{n}{n-2}} B_g u, \end{aligned} \tag{2.1}$$

where $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ is the so-called conformal Laplacian and $B_g = \frac{\partial}{\partial \eta} + \frac{n-2}{2} h_g$ is an associated boundary operator. Here Δ_g stands for the Laplace-Beltrami operator and η is the outward unit normal to the boundary.

The Yamabe Problem I is therefore equivalent, in the language of partial differential equations, to find a smooth solution $u \in C^\infty(M)$ to the nonlinear boundary-value problem

$$\begin{cases} L_g u + \lambda u^{\frac{n+2}{n-2}} = 0 \\ B_g u = 0 \\ u > 0, \end{cases} \tag{2.2}$$

where λ is an arbitrary constant.

The difficulty in solving problem (2.2) comes almost entirely from the fact that $\frac{n+2}{n-2}$ is a critical exponent for the so-called Sobolev embeddings. This means that a direct variational approach will necessarily fail due to the noncompactness of the inclusion $H^1(M) \subset L^{\frac{2n}{n-2}}(M)$. That drawback was already present in the classical Yamabe Problem, and the basic technique to overcome it consists in first lowering the exponent of the equation. One then finds a family of subcritical solutions (u_p), $p < \frac{n+2}{n-2}$, and investigates its behavior as $p \to \frac{n+2}{n-2}$. The hope is that this family will stay uniformly bounded, in which case standard machinery from the theory of elliptic partial differential equations implies the existence of a convergent subsequence.

The main theorem in [12] succesfully covers almost all manifolds, its precise statement being:

Theorem 1 (Escobar). *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M, $n \geq 3$. Assume that one of the following conditions is satisfied:*

1. $n = 3,\ 4$ or 5*;*

2. *M has a nonumbilic point on ∂M;*

3. *M is locally conformally flat;*

4. *$n \geq 6, \partial M$ is umbilic and there exists a point $P \in \partial M$ such that the Weyl tensor $\mathrm{W}_g(P)$ is not zero.*

Then there exists a smooth metric $\tilde{g} = u^{\frac{4}{n-2}} g$ with constant scalar curvature in M and zero mean curvature on the boundary ∂M.

Let us now briefly discuss the main ideas behind the proof of Theorem 1.

First one can check that solutions to the Yamabe Problem I are critical points of the functional

$$F(\tilde{g}) = \frac{n-2}{4(n-1)} \int_M R_{\tilde{g}} d\mu_{\tilde{g}} + \frac{n-2}{2} \int_{\partial M} h_{\tilde{g}} d\sigma_{\tilde{g}}$$

restricted to the constraint set

$$C_g^I = \{\tilde{g} = u^{\frac{4}{n-2}} g : \ \mathrm{Vol}\ (M^n, \tilde{g}) = 1\},$$

where $d\mu_g$ denotes the Riemannian volume form on M with respect to the metric g, and $d\sigma_g$ is the induced volume form on the boundary ∂M.

The functional F is bounded from below on C_g^I, and in the paper Escobar introduces the Sobolev quotient

$$Q(M^n) = \inf_{\tilde{g} \in C_g^I} F(\tilde{g}),$$

which clearly is a conformal invariant of the manifold (M^n, g). By applying the transformation laws (2.1) one can also compute that number through the use of an appropriate energy over a space of functions, as indicated by

$$Q(M^n) = \inf_{\varphi \in C^\infty(M)} Q_g(\varphi) := \frac{E_g(\varphi)}{||\varphi||^2_{\frac{2n}{n-2}}},$$

where

$$E_g(\varphi) = \int_M (|\nabla \varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2) d\mu_g + \frac{n-2}{2} \int_{\partial M} h_g \varphi^2 d\sigma_g,$$

and

$$||\varphi||_{\frac{2n}{n-2}} = \left(\int_M |\varphi|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}}.$$

A general local construction proves that $Q(M^n) \leq Q(\mathbb{S}^n_+)$ for any manifold M, where $\mathbb{S}^n_+$ denotes a hemisphere with a metric of constant sectional curvature one. An argument similar to the one given by Aubin in [1] shows that if $Q(M^n) < Q(\mathbb{S}^n_+)$, the Palais-Smale condition holds and then there exists a minimizer in $H^1(M)$. That minimizing function is obtained as a limit of a sequence of subcritical solutions, and it will be a positive solution to the Problem (2.2). Its smoothness follows from the work of Cherrier ([6]).

Hence, to solve Yamabe Problem I one has to exhibit a test function $\varphi \in C^\infty(M)$ satisfying $Q_g(\varphi) < Q(\mathbb{S}^n_+)$. There are essentially two different ways of doing that. One is to construct a local test function based on a rotationally symmetric family of standard solutions to the corresponding equation in the Euclidean case. The function vanishes outside a small neighborhood of a point on the boundary and the idea is to exploit the local conformal geometry of the manifold by computing an expansion for $Q_g(\varphi)$. When there is not enough local information one needs a more sophisticated construction involving the Green function of the conformal Laplacian L_g with boundary condition given by the operator B_g. The inspiration originates in the work of Schoen ([30]) and the final argument in this case requires the Positive Mass Theorem for manifolds with boundary, to which Escobar gives a proof (in the Appendix of [12]) for the cases he needs.

In [11], Escobar considers the - in some aspects different - problem of existence of conformal metrics which are scalar-flat with constant mean curvature on the boundary :

Yamabe Problem II (on manifolds with boundary)
Find a metric $\tilde{g}$, conformally related to g, such that for a certain constant $c \in \mathbb{R}$ one has

$$\begin{cases} R_{\tilde{g}} &= 0 \text{ in } M, \\ h_{\tilde{g}} &= c \text{ on } \partial M. \end{cases}$$

The solutions are again critical points of the same functional F above defined, restricted this time to a different set:

$$C_g^{II} = \{\tilde{g} = u^{\frac{4}{n-2}} g : \text{ Vol } (\partial M^n, \tilde{g}) = 1\}.$$

The associated analytical problem is given by

$$\begin{cases} L_g u &= 0 \\ B_g u &= \lambda u^{\frac{n}{n-2}} \\ u &> 0, \end{cases}$$

the nonlinearity now being on the boundary condition. This already introduces some new phenomena, for instance the functional F is not necessarily bounded from below on the constraint set C_g^{II}, as it can be verified for some manifolds. A direct variational approach again will not work because of the lack of compactness of the Sobolev trace embedding $H^1(M) \subset L^{\frac{2(n-1)}{n-2}}(\partial M)$. We remark that Escobar ([9]) determined the best constant in a related Sobolev trace inequality on a halfspace in $\mathbb{R}^n$.

Despite these differences Escobar ([11]) gives, by using similar methods, an affirmative solution to Yamabe Problem II for almost all manifolds. As a corollary (dimension 6 follows from Theorem 3.1 in [14]), he is able to prove:

Theorem 2 (Escobar). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose $n \neq 4, 5$. Then there exists a conformal metric $g = u^{\frac{4}{n-2}}\delta$ with zero scalar curvature $R_g = 0$ in M and constant mean curvature $h_g = c$ on the boundary ∂M.*

The above result can be thought of as a generalization of the famous Riemann mapping theorem, which says that every simply connected, proper domain in the plane is conformally equivalent to the unit disk. One certainly cannot expect such a statement to hold in higher dimensions because there are very few conformal diffeomorphisms between Euclidean open sets when $n \geq 3$. The Theorem 2 asserts that at least one can achieve, by a conformal deformation, zero scalar curvature and constant mean curvature on the boundary. Those are geometric properties of a Euclidean ball. The result is still open in dimensions 4 and 5, to the author's knowledge.

Escobar also obtained similar results for the Yamabe problem ([13]) under mixed constraints:

$$a \text{ Vol } (M^n, g) + b \text{ Vol } (\partial M^n, g) = 1,$$

where $a, b > 0$.

The solution (in almost all cases) of the correct versions of the Yamabe Problem for manifolds with boundary is perhaps the greatest contribution of J. F. Escobar, helping to understand the general geometrical question of what is the nicest metric on a given manifold.

3 Prescribed Scalar and Mean Curvatures

In this section we shall describe some of the contributions made by Escobar to the problem of prescribing scalar and mean curvatures under conformal deformations.

Let (M^n, g) be a compact Riemannian manifold, $n \geq 3$, and suppose a smooth function $K : M \to \mathbb{R}$ is given. The so-called **Prescribed Scalar Curvature Problem** consists in finding a metric $\tilde{g}$, conformal to g, such that

$$R_{\tilde{g}} = K \text{ in } M.$$

This problem has been extensively studied when the underlying manifold is the sphere $\mathbb{S}^n$ with a metric g_0 of constant sectional curvature one. In this case it becomes especially interesting due to the presence of a noncompact group of conformal diffeomorphisms, and surprisingly there are nontrivial obstructions first discovered by Kazdan and Warner ([25]).

When there is no boundary the Sobolev quotient, also called Yamabe invariant, takes the form $Q(M^n, g) = \inf_{\tilde{g} \in C_g} Y(\tilde{g})$, where

$$Y(\tilde{g}) = \frac{n-2}{4(n-1)} \int_M R_{\tilde{g}} d\mu_{\tilde{g}}$$

and

$$C_g = \{\tilde{g} = u^{\frac{4}{n-2}} g : \text{ Vol } (M^n, \tilde{g}) = 1\}.$$

The conformal invariant $Q(M^n, g)$ plays a prominent role in the classical Yamabe Problem, where the manifolds fall into three different categories according to the sign of it. The most difficult case to settle being when $Q(M^n, g) > 0$ or, equivalently, when the first eigenvalue of the conformal Laplacian $\lambda_1(L_g)$ is positive. The Yamabe invariant of an Einstein manifold was studied by Escobar in [2], with P. Aviles as a collaborator.

The Sobolev quotient is also important when dealing with the Prescribed Scalar Curvature Problem. For example one cannot prescribe a nonpositive function as the scalar curvature of a conformal metric on M when $Q(M^n, g) > 0$.

In a joint paper with his scientific advisor R. Schoen, Escobar ([23]) showed that on a three dimensional compact Riemannian manifold of positive Yamabe invariant, not conformally equivalent to the standard sphere, being positive somewhere is a necessary and sufficient condition for a function to be the scalar curvature of a conformally related metric. That result, which also appeared in his Ph. D. thesis, completely solves the Prescribed Scalar Curvature Problem when $M^3 = \mathbb{S}^3/\Gamma$, where Γ is a nontrivial finite group of isometries acting without fixed points. It should be noted that this extends a well-known result for $\mathbb{RP}^2$ due to Moser (see [26]).

When the dimension is greater than three, Escobar and Schoen were able to show in [23] that the result is still true if the manifold is locally conformally flat and the derivatives of order less than or equal to $n-2$ of the prescribed function vanish at a maximum point. Notice that those functions constitute a dense subset, in the C^1 topology, of the set of functions which are positive somewhere. At present it is not known whether the general result holds without those additional assumptions.

In that same paper, Escobar and Schoen also consider manifolds with Sobolev quotient equal to zero. Given the solution to the Yamabe Problem those are the manifolds which admit conformal metrics that are scalar flat. They provide a complete answer to the problem in dimensions 3 and 4, by showing that the only obstructions are:

1. K changes sign and
2. $\int_M K dv_g < 0$.

That theorem is a full extension, to those dimensions, of the results obtained by Kazdan and Warner ([24]) on the two dimensional torus.

Another related problem that attracted Escobar's attention is the **Prescribed Mean Curvature Problem** on the unit ball $B^n \subset \mathbb{R}^n$ - the mean curvature version of the Prescribed Scalar Curvature Problem on the standard sphere. Given a smooth function h on the boundary ∂B^n of the ball, the problem consists in finding a metric g conformal to the Euclidean metric δ and satisfying

$$\left\{ \begin{array}{rcll} R_g & = & 0 & \text{in } B^n, \\ h_g & = & h & \text{on } \partial B^n. \end{array} \right. \tag{3.1}$$

If one writes $g = u^{\frac{4}{n-2}}\delta$, the transformation laws in section 2 yield the equivalent analytical problem:

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in } B^n, \\ \dfrac{\partial u}{\partial \eta} + \frac{n-2}{2} u & = & \frac{n-2}{2} h u^{\frac{n}{n-2}} & \text{on } \partial B^n, \\ u & > & 0. & \end{array} \right. \tag{3.2}$$

A first obstruction to the solution of problem (3.2) comes from integration by parts. Given a solution u, one readily obtains

$$\frac{n-2}{2} \int_{\partial B^n} h u^{\frac{2(n-1)}{n-2}} d\sigma = \frac{n-2}{2} \int_{\partial B^n} u^2 d\sigma + \int_{B^n} |\nabla u|^2 dx,$$

which shows that h has to be positive somewhere.

This is not the only known necessary condition. In [14] Escobar found a nontrivial Kazdan-Warner type obstruction given by:

$$\int_{\partial B^n} \langle X, \nabla h_g \rangle d\sigma_g = 0,$$

where X is a conformal Killing vector field on ∂B^n. For instance this condition can be used to show that the problem (3.2) has no solutions if $h = ax_i + b$, where $a \neq 0$ and $i = 1, \ldots, n$.

In [22], in collaboration with G. Garcia, Escobar studied the Prescribed Mean Curvature Problem when h is a Morse function, positive somewhere, with $\Delta h \neq 0$ at its critical points. After a detailed blowup analysis of subcritical solutions they obtain strong results in low dimensions and prove a general existence theorem of nonminimizers for the three dimensional ball. Namely, if D_μ denotes the number of critical points $\overline{x}$ of h such that $\Delta h(\overline{x}) < 0$ and the Morse index of $-h$ at $\overline{x}$ is μ, they show the existence of solutions to problem (3.2) when $D_0 - D_1 \neq 1$. If $D_0 - D_1 > 1$ there exists a solution of index 1. Fundamental to this work was the description given by Escobar ([10]) of the moduli space of solutions when $h = 1$, a result in the spirit of Obata's theorem on the sphere ([27]). He proved that scalar flat metrics of constant mean curvature one on the boundary, which are pointwise conformal to the Euclidean metric δ on the unit ball, can only occur as pull-backs of the form $F^*\delta$, where $F : B^n \to B^n$ is a conformal diffeomorphism. In [18] Escobar obtains other related results on uniqueness and non-uniqueness of solutions.

Escobar also studied the Prescribed Mean Curvature Problem on other manifolds with boundary. As a consequence he proves in ([14]) the following remarkable theorem:

Theorem 3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain different from a ball, and let $h : \partial\Omega \to \mathbb{R}$ be a smooth function. There exists a scalar flat metric on Ω, conformal to the Euclidean metric, and with boundary mean curvature given by h if and only if h is positive somewhere.*

Escobar also considered a Dirichlet boundary condition in [8], where he proves the existence of positive solutions to a perturbation of the conformal scalar curvature equation in dimension 4, extending a result due to Brézis and Nirenberg ([3]).

4 Spectrum of the Laplacian on Complete Manifolds

In this section we will describe some of Escobar's work on the spectral properties of both the Laplace-Beltrami and the Hodge Laplacian operators on open manifolds of nonnegative curvature.

The Laplace-Beltrami operator is a formally self-adjoint operator $\Delta_g : C_0^\infty(M) \to C^\infty(M)$, which has a unique unbounded self-adjoint extension acting on $L^2(M)$. Its spectrum $\mathrm{Spec}(\Delta_g)$ is an important geometric invariant and therefore it becomes natural to study its relation to other geometric quantities, like the curvature of the manifold.

A particularly special manifold is the Euclidean space, where some spectral properties of the Laplacian are very well known. For example one knows that its spectrum is the nonnegative halfline and that there are no eigenfunctions in L^2. It is generally expected that these properties should hold for a much bigger class of manifolds, for example it has been conjectured by S. T. Yau (see Chapter VIII in [31]) that they should be true for any complete noncompact manifold of nonnegative sectional curvature. In his Ph. D. thesis Escobar succesfully approached (see [7]) that problem on $\mathbb{R}^n$ endowed with a rotationally symmetric metric of nonnegative sectional curvature. He proved that the spectrum of such a metric is $[0, \infty)$ and its Laplace-Beltrami operator does not have any eigenvalues.

The idea for proving that is to use the rotational symmetry for reducing the problem to one concerning a Schrödinger operator on the half line. The proof is then based on an asymptotic formula developed by Escobar for non-integrable potentials, such as the one that appears when the curvature is nonnegative.

In a subsequent paper in collaboration with A. Freire, Escobar ([19]) proved that the Laplacian on nonnegatively curved manifolds has a pure continuous spectrum, this time assuming the existence of a pole and some quadratic decay condition on the curvatures when $n \geq 3$. Recall that a pole is a point $p \in M$ such that the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. The proof is based on an integral identity for eigenfunctions discovered by the authors, very similar to formulas obtained classically by F. Rellich ([29]) and others. In dimension 2 no decay hypothesis is needed.

The topological structure of a complete, noncompact and nonnegatively curved manifold is revealed by a classical theorem ([5]) due to Cheeger and Gromoll. It states that such a manifold contains an embedded compact submanifold S, referred to as a soul, which is also totally convex (hence connected and totally geodesic), and such that its normal bundle NS is diffeomorphic to M. When this diffeomorphism can be given by the exponential map $\exp_S : NS \to M$, which is not always the case, Escobar and Freire ([19]) were able to show that $\mathrm{Spec}(\Delta_g) = \mathrm{Spec}_{\mathrm{ess}}(\Delta_g) = [0, \infty)$ under the extra assumption when $n \geq 3$:

$$\int_1^\infty \frac{1}{v(r)} \int_{S_r} \mathrm{Ric}(\nabla r) dr < \infty.$$

Here $r(x) = d(x, S)$, $S_r = \{x \in M : d(x, S) = r\}$ and $v(r) = \mathrm{vol}(S_r)$. They also verified this result without the integrability condition assuming only the manifold has a soul of codimension one.

Escobar was also interested in properties of the Hodge Laplacian $\Delta_g = d\delta + \delta d$ acting on L^2 differential forms on a nonnegatively curved manifold. One of the questions he was concerned with was to determine, under curvature assumptions, whether or not there are any nontrivial harmonic forms in L^2 of certain degrees. For instance, open surfaces of nonnegative Gauss curvature do not admit such forms. Perhaps his most important result in this subject, in a joint work with Freire ([20]), is the following vanishing theorem:

Theorem 4 (Escobar, Freire). *Let (M^n, g) be a complete noncompact Riemannian manifold with nonnegative sectional curvatures, $n \geq 3$. Suppose M has a soul S of dimension $s \leq n-2$ such that $\exp_S : NS \to M$ is a diffeomorphism. (If* dim *$S = 0$, assume S is a pole.) Given $0 < p < \frac{n-s}{2}$ or $\frac{n+s}{2} < p < n$, assume the radial sectional curvatures satisfy*

$$0 \leq K_r \leq \frac{c(1-c)}{r^2}$$

on $M \setminus S$, where $\frac{2p-1}{n-s-1} < c < 1$ and $r(x) = d(x, S)$. Then the only harmonic p-form in L^2 is the zero form.

The proof is based on Rellich-type identities for differential forms. Assuming the existence of a pole and through the use of similar techniques he also proved the nonexistence of eigenforms of the Hodge Laplacian in L^2, much in the spirit of what he had done for the Laplace-Beltrami operator acting on functions. Finally in [21], a joint work with A. Freire and M. Min-Oo, he obtains L^2 vanishing results for vector-valued differential forms.

5 Steklov Eigenvalue Problem

In this section we will discuss some of the contributions, given by Escobar in a series of papers [15], [16], [17], to the Steklov eigenvalue problem:

$$\begin{cases} \Delta\varphi = 0 & \text{in } M, \\ \dfrac{\partial\varphi}{\partial\eta} = \nu\varphi & \text{on } \partial M, \end{cases} \tag{5.1}$$

where (M^n, g) is a compact Riemannian manifold, $\nu \in \mathbb{R}$, and $\varphi \in C^\infty(M)$.

Historically the problem was introduced by Steklov ([32]) himself in 1902 for planar domains, in which case it has physical meaning. The function φ can be interpreted as the steady state temperature and the boundary condition is saying that the flux on the boundary is proportional to the temperature. One should notice that the Steklov eigenvalues are also the eigenvalues of the Dirichlet-to-Neumann map, which sends a given boundary data into the normal derivative of its harmonic extension to the interior.

The motivation Escobar had for studying the Steklov eigenvalue problem came from the Yamabe problem on manifolds with boundary, more precisely, from what we have called Yamabe Problem II in section 2 of this paper. The reason is that the behavior of the nonlinear partial differential equation (2.2) is intimately related to the sign of the first eigenvalue of the linear part

$$\begin{cases} L_g\varphi = 0 & \text{in } M, \\ B_g\varphi = \nu\varphi & \text{on } \partial M, \end{cases}$$

which definitely resembles problem (5.1).

The set of Steklov eigenvalues consists of an increasing sequence of nonnegative real numbers starting by zero and converging to infinity, as can be checked by standard variational arguments. One of the problems addressed by Escobar in his papers was to give good geometric estimates for the first non-zero Steklov eigenvalue ν_1, which can be variationally characterized as:

$$\nu_1 = \min_{\int_{\partial M} f = 0} \frac{\int_M |\nabla f|^2 dv_g}{\int_{\partial M} f^2 d\sigma_g}.$$

Many of his results focus on the two dimensional case. The first example ([15]) is a sharp two dimensional comparison result generalizing Payne's theorem ([28]) for planar domains. Escobar shows that if (M^2, g) is a compact Riemannian surface with boundary such that its Gaussian curvature is nonnegative and its geodesic curvature κ_g of the boundary curve satisfies $\kappa_g \geq \kappa_0 > 0$, then the first nontrivial Steklov eigenvalue of M is greater than or equal to κ_0. The equality holds if and only if the surface is isometric to a Euclidean disk of radius $\frac{1}{\kappa_0}$.

The proof of this fact is based on the well-known Weitzenböck formula

$$\frac{1}{2}\Delta(|\nabla f|^2) = |\text{Hess } f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \text{Ric}(\nabla f, \nabla f),$$

which is used to show that the function $F = |\nabla\varphi|^2$ is subharmonic if φ is a given Steklov eigenfunction. The result then follows from an application of the Maximum Principle (Hopf's Lemma) to the function F, in which calculations are done using Fermi coordinates around a point on the boundary curve.

He also investigated the first nonzero Steklov eigenvalue of subdomains of a complete simply connected surface M of constant curvature. He shows ([16]) that the geodesic balls in M maximize ν_1 among all bounded simply connected domains of same area, proving also that this topological assumption is a necessary condition. When the curvature of the ambient space is allowed to vary, but is always nonpositive, he proves in the same article the sharp inequality $\nu_1(\Omega) \leq \sqrt{\dfrac{\pi}{\mathrm{Area}(\Omega)}}$, where Ω is again a bounded simply connected subdomain and the right hand side coincides with the first nonzero Steklov eigenvalue of a Euclidean ball of area $\mathrm{Area}(\Omega)$.

As for higher dimensions Escobar ([15]) considered compact manifolds of nonnegative Ricci curvature. Again in the spirit of the Payne's theorem, he proved that if the second fundamental form π of the boundary satisfies $\pi \geq k_0 g$, then $\nu_1 > \frac{k_0}{2}$. The proof involves an application of an integral identity, obtained from the Weitzenböck formula through integration by parts, to a Steklov eigenfunction corresponding to ν_1. In [16] he conjectures that the stronger inequality $\nu_1 \geq k_0$ should always hold under the same hypotheses.

It is much more difficult to produce geometric estimates for ν_1 without any positivity assumption on the curvature. For that reason Escobar introduces ([15]) the isoperimetric constant

$$I = \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}}, \tag{5.2}$$

where $\Omega \subset M$ is an open set, $\Omega_1 = \Omega \cap \partial M$ is a nonempty subdomain of ∂M, $\Omega_2 = \partial M \setminus \Omega_1$ and $\Sigma = \partial\Omega \cap \mathrm{int}(M)$ is a hypersurface, being $\mathrm{int}(M)$ the interior of M.

He is able then to provide a Cheeger type lower bound for ν_1 in terms of the constant I and a more standard eigenvalue problem on M :

$$\nu_1 \geq \frac{(I\lambda_1(k) - ak)a}{a^2 + \lambda_1(k)},$$

where $a, k > 0$, and $\lambda_1(k)$ is the first eigenvalue of the problem

$$\begin{cases} \Delta u + \lambda_1(k)u = 0 & \text{in } M, \\ \dfrac{\partial u}{\partial \eta} + ku = 0 & \text{on } \partial M. \end{cases}$$

In [16] Escobar computes the constant I of n-dimensional Euclidean balls, and shows that two dimensional balls maximize I among all planar domains. He also gives estimates from above and below for I in terms of the geometry of the boundary, among other results.

6 Concluding Remarks

I hope I have succeeded in giving the reader some sense of the impact the work of J. F. Escobar, often referred to simply as Chepe, had in Mathematics. His achievements have been recognized by the many awards he received, notably the distinguished Presidential Faculty Fellowship in 1992.

It should also be noted that, being from Colombia, he always held strong ties to the Latin American mathematical community. Although he worked in the United States - he was a professor at Cornell University in recent years - he considered this interaction of extreme relevance, as his frequent visits to IMPA and to the Differential Geometry Schools in Brazil can attest.

I shall finish by stating that I have the honor of having been his student and, above all, the great privilege of having had him as a friend. His succesful career, combined with his captivating personality, should serve as an example for this and the generations to come.

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