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GEOMETRY AND THE COMMUTABILITY OF SYMMETRIC OPERATORS

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Abstract

By studying the Ricatti equation, we show that the Ricci operator commutes with its derivative along geodesics. We also prove that, for any v tangent to a closed manifold of nonpositive curvature, the Ricci operator $K_{\tilde{v}}$ commutes with the second fundamental form of the horosphere at $\tilde{v}$ associated with the ray $\gamma_{\tilde{v}}$, where $\tilde{v}$ is the lifting of v to the universal covering with the induced metric.

Let M be a Riemannian manifold. Several works in Riemannian geometry have used the fact that the Ricatti equation

$$U' + U^2 + K = 0 \tag{0.1}$$

is satisfied along small geodesics γ on M or along complete geodesics or rays in some special cases. See for example [Gr], [Gu], [A], [E], [K], [FM], [OS], [MZ1], [MZ2]. The Ricci operator $K = K_{\gamma'(t)} := R(\gamma'(t), \,.\,)\gamma'(t)$ and $U = U_{\gamma'(t)}$ are symmetric operators on $\gamma'(t)^{\perp} \subset T_{\gamma(t)}M$, where R is the curvature tensor. The derivative $U' = U'_{\gamma'(t)} : \gamma'(t)^{\perp} \to \gamma'(t)^{\perp}$ is defined by $\langle U'(v), w\rangle = \langle U(v), w\rangle'$, whenever $v, w \perp \gamma'$ are parallel vector fields along γ.

Let R be a tensor in a Riemannian manifold M satisfying the usual algebraic conditions of the curvature tensor. It would be interesting to ask what conditions R should satisfy so that it is the curvature tensor of some Riemannian metric in an open subset of M. Our first result below shows that there

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is some condition on the first derivatives of R, which is a surprising fact, since the curvature is a tensor. Another surprise is that we used Ricatti equations on some closed manifold to prove this local result.

Theorem 1. *Let M be any Riemannian manifold. Then the Ricci operator K_v, defined by $K_v(u) = R(v,u)v,\ u \in \{v\}^{\perp}$, commutes with K'_v.*

To prove Theorem 1 we need to show the following general result about Ricatti equations on closed manifolds.

Theorem 2. *Let M be a complete manifold with finite volume and SM the unit tangent bundle of M with standard Liouville measure which is invariant under the geodesic flow. We consider symmetric operators U_v and K_v satisfying the Ricatti equation along geodesics as above, except that now K_v is not necessarily related to the Ricci operator. Assume that U and K are uniformly bounded. Then, for almost all $v \in SM$, U_v, K_v and U'_v commute.*

As another consequence of Theorem 2 we obtain the following result:

Corollary. *Let M be a closed manifold with nonpositive sectional curvature. Take $v \in T_pM$ and $\tilde{v} \in T_{\tilde{p}}\tilde{M}$ with $\pi(\tilde{p}) = p,\ d\pi_{\tilde{p}}\tilde{v} = v$, where $\pi\colon \tilde{M} \to M$ is the universal covering of M with the induced metric. Then the second fundamental form $U_{\tilde{v}}$ of the horosphere associated with the ray $\gamma_{\tilde{v}}$ starting at $\tilde{v}$ commutes with $K_{\tilde{v}}$, where $K_{\tilde{v}} = R(\tilde{v},\,.\,)\tilde{v}$.*

1. Proof of Theorem 2 and its consequences

Proof of Theorem 2. Let U and K be as in Theorem 2. Let $U_c = U + cI$, where c is a positive constant and I is the identity map restricted to the orthogonal complement $\{\gamma'\}^{\perp}$. Since there exists $A > 0$ such that $|U| < A$, for sufficiently large c we have that $\langle (U_c)_v\, x, x\rangle \geq B|x|^2$, for a positive constant B, any v in the unit tangent bundle SM and any $x \perp v$. We can write

$$U_c' + U_c^2 + K_c = 0,$$

along each geodesic in M, where $K_c := K - 2cU - c^2 I$. For sufficiently large c we have that K_c is negative definite, hence $-K_c$ has a positive square root, that is, an operator $\sqrt{-K_c}$ with the same eigen spaces as $-K_c$ and whose corresponding eigenvalues are the positive square roots of the eigenvalues of $-K_c$. Applying the proof in [OS], page 459, to U_c and K_c, we get

$$\int_{SM} \operatorname{tr} \sqrt{-K_c}\, d\mu \leq \int_{SM} \operatorname{tr} U_c\, d\mu = \int_{SM} \operatorname{tr} U\, d\mu + \int_{SM} \operatorname{tr} cI\, d\mu. \tag{1.1}$$

We will use the following trivial result.

Lemma 1.1. *Consider the function* $f(x) = \sqrt{-ax^2 + 2bx + 1} - 1$*, where* $|a|$ *and* $|b|$ *are small compared with* 1*. Then we have* $f(0) = 0,\ f'(0) = b,\ f''(0) = -(b^2 + a),\ f'''(0) = 3(b^2 + a)b$*.*

Now consider, at each point $v \in SM$, an orthonormal basis $\{e_i\} \subset \{v\}^\perp$ which diagonalizes the positive symmetric matrix $-K_c$. We choose $\{e_i\}$ in such a way that $g(v) = \sum_{i=0}^{n-1} \langle U(e_i), e_i\rangle^2$ is a minimum between all orthonormal bases which diagonalize K_c. This will imply that g is measurable (and integrable, by the uniform boundedness of $\langle U(e_i), e_i\rangle$). To see this, consider the fiber bundle $F \to SM$ where the fiber over $v \in SM$ is the set of orthonormal bases of $\{v\}^\perp \subset T_vM$. The set $S = \{(v, \{e_i\}) \in F | \sum_i (\langle K_c^2 e_i, e_i\rangle - \langle K_c e_i, e_i\rangle^2) = 0\}$ is a measurable subset of F and consists of the set of orthonormal bases that diagonalize K_c. The function g can be defined as $g(v) = \min\{\sum_{i=0}^{n-1} \langle U(e_i), e_i\rangle^2 |(v, \{e_i\}) \in S\}$. So g is clearly measurable. We have

$$\begin{aligned} &\int_{SM} \operatorname{tr} \sqrt{-K_c}\, d\mu - \int_{SM} \operatorname{tr} cI\, d\mu \qquad (1.2)\\ &= \int_{SM} \sum_{i=0}^{n-1} \left[\sqrt{-\langle K(e_i), e_i\rangle + 2c\langle U(e_i), e_i\rangle + c^2} - c\right] d\mu \end{aligned}$$

$$= \int_{SM} \sum_{i=0}^{n-1} c \left[\sqrt{\frac{-\langle K(e_i), e_i \rangle}{c^2} + \frac{2\,\langle U(e_i), e_i \rangle}{c} + 1} - 1 \right] d\mu.$$

From (1.2), Lemma 1.1 and the fact that $\langle U(e_i), e_i \rangle$ and $\langle K(e_i), e_i \rangle$ are uniformly bounded, we obtain

$$\int_{SM} \operatorname{tr} \sqrt{-K_c}\, d\mu - \int_{SM} \operatorname{tr} cI\, d\mu \tag{1.3}$$
$$= \int_{SM} \sum_{i=0}^{n-1} \left[\langle U(e_i), e_i \rangle + \frac{-\langle K(e_i), e_i \rangle - \langle U(e_i), e_i \rangle^2}{2c} + O\left(\frac{1}{c^2}\right) \right].$$

Since $\int_{SM} \sum_{i=0}^{n-1} \langle U(e_i), e_i \rangle \; d\mu = \int_{SM} \operatorname{tr} U\, d\mu$, we obtain

$$\int_{SM} \sum_{i=0}^{n-1} \left[-\langle K(e_i), e_i \rangle - \langle U(e_i), e_i \rangle^2 \right] \, d\mu \leq 0. \tag{1.4}$$

In fact, if the integral in (1.4) would be positive, this would contradict (1.1) and (1.3) for sufficiently large c.

It is a well known consequence of (0.1) and the fact that μ is invariant under the geodesic flow that $\int_{SM} \operatorname{tr}(-K) = \int_{SM} \operatorname{tr} U^2$ (see [Gr]). So we have

$$\begin{aligned} \int_{SM} \sum_{i=0}^{n-1} \left[-\langle K(e_i), e_i \rangle - \langle U(e_i), e_i \rangle^2 \right] d\mu &= \int_{SM} \left[\operatorname{tr}(-K) - \sum_{i=0}^{n-1} \langle U(e_i), e_i \rangle^2 \right] \, d\mu \\ &= \int_{SM} \left[\sum_{i=0}^{n-1} \left\langle U^2(e_i), e_i \right\rangle - \langle U(e_i), e_i \rangle^2 \right] d\mu. \end{aligned} \tag{1.5}$$

From the symmetry of U and the Theorem of Cauchy-Schwarz we have

$$\sum_{i=0}^{n-1} \left[\left\langle U^2(e_i), e_i \right\rangle - \langle U(e_i), e_i \rangle^2 \right] = \sum_{i=0}^{n-1} \left[\langle U(e_i), U(e_i) \rangle - \langle U(e_i), e_i \rangle^2 \right] \geq 0, \tag{1.6}$$

By (1.4) and (1.6) we conclude that equality holds in both inequalities. By the rigidity part in the Theorem of Cauchy-Schwarz we obtain $U(e_i) = \lambda_i e_i$. So the basis $\{e_i\}$ diagonalizes U and K_c almost everywhere on SM. It diagonalizes K from the definition of K_c, and it also diagonalizes U', because of (0.1). Theorem 2 is proved.

$\square$

Proof of Theorem 1. Fix $v \in T_pM$. We want to prove that K_v and K'_v commute, where $K_v = R(v, .)v$. We consider a closed Riemannian manifold (N, g) containing a neighborhood V of p in M such that the Riemannian metric g is an extension of the Riemannian metric of $V \subset M$. We consider on N the trivial Ricatti equation

$$K' + K^2 + (-K' - K^2) = 0.$$

From Theorem 2 we conclude that K and K' commutes almost everywhere on SM, hence they commute everywhere because of the continuity of the map $u \mapsto K_u$. In particular K_v and K'_v commute. The proof is complete.

$\square$

Proof of the Corollary. In the case that M is closed with nonpositive curvature it is well known that there exists a global solution U for the Ricatti equation (with $K_v(u) = R(v, u)v$). Let $\tilde{U}$ and $\tilde{K}$ be the liftings of U and K to $\tilde{M}$. It is also known that the horosphere $S_{\tilde{v}}$, associated to the ray $\gamma_{\tilde{v}}$ in $\tilde{M}$, is C^2 and that U can be taken so that $\tilde{U}$ becomes the second fundamental form of $S_{\tilde{v}}$ at $\tilde{v}$. In particular $\tilde{U}$ and U are continuous. Thus Theorem 2 implies that U, U' and K commute everywhere on SM, hence $\tilde{U}$ and $\tilde{K}$ also commute. This concludes the proof of the corollary.

$\square$

References

[**A**] Ambrose, W., *A Theorem of Myers*, Duke Math J 24 (1057), 345–348.

[**E**] Eberlein, P., *When is a geodesic flow of Anosov type? I*, J. Differential Geometry 8 (1973), 437–463.

[FM] Freire, A., Mañé, R., *On the entropy of the geodesic flow in manifolds without conjugate points*, Invent. Math. 69 (1982), 375–392.

[Gr] Green, L. A., *A theorem of E. Hopf*, Michigan Math. J. 5 (1958), 31–34.

[Gu] Guimarães, F. F., *The integral of the scalar curvature of complete manifolds without conjugate points*, J. Differential Geom. 36 (1992) 651–662.

[K] Klingenberg, W., *Riemannian manifolds with geodesic flows of Anosov type*, Ann. Math. 99 (1974), 1–13.

[MZ1] Mendonça, S., Zhou, D., *Curvature conditions for immersions of submanifolds and applications*, Compositio Math. 137 (2003) 211–226.

[MZ2] Mendonça, S., Zhou, D., *Maximal ergodic theorems and applications to Riemannian geometry*, Israel J. of Math. 139 (2004), 319–335.

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