Matemática Contemporânea, Vol 29, 79-97

HOMOTHETIES AND ISOMETRIES OF METRIC SPACES

Mauro Patrão *

Abstract

Let (M, d) be a metric space. We prove that when the group of homotheties $H(M, d)$ is a locally compact group, with respect to the compact-open topology, it is a Lie group if, and only if, the group of isometries $I(M, d)$ is a Lie group. Then we prove that when (M, d) is a Heine-Borel metric space, its group of homotheties $H(M, d)$ is also a Heine-Borel metric space and, if (M, d) is a Heine-Borel ultrametric space, its group of isometries is an increasing union of compact subgroups. We also prove that when (M, d) is locally compact and the space $\Sigma(M)$ of the connected components of M is quasi-compact, its group of homotheties $H(M, d)$ is locally compact. As applications we give some generalizations of classical results in Riemannian geometry. Namely, if (M, d) is a Finsler manifold or an Alexandroff space, then its group of homotheties is a Lie group. With some additional hypothesis, this is also true for a Hadamard space.

1 Introduction

Let (M, d) be a metric space. The group of homotheties $H(M, d)$ is defined by

$$H(M, d) = \{f \in C(M) : f \text{ is onto and } d(f(x), f(y)) = \lambda(f)d(x, y), \ \forall x, y \in M\},$$

where $C(M)$ is the set of all continuous mappings of M into itself and $\lambda(f) > 0$ is a constant. The group of isometries $I(M, d) = \{f \in H(M, d) : \lambda(f) = 1\}$ is a closed normal subgroup of $H(M, d)$. It is a classical result of the theory

*I thank the help of my friends L. Seco and L. San Martin and the careful comments of the referee.
Supported by FAPESP: proc. $n°$ 02/13839-4
Keywords: homothety group, isometry group, proper action, Lie group, Heine-Borel metric space, ultrametric space.
Mathematics Subject Classification: Primary: 54E40, 54E45; Secondary: 22E99, 54H15.

of topological groups due to Gleason and Yamabe (cf. [10]) that a topological group G is a Lie group if, and only if, it is locally compact and has no small subgroups. Using this deep result, we prove that when the group of homotheties $H(M,d)$ is a locally compact group, it is a Lie group if, and only if, the group of isometries $I(M,d)$ is a Lie group.

Then we prove that, under some reasonable conditions over (M,d), the group of homotheties $H(M,d)$ is locally compact. Since the work of van Dantzig and van der Waerden [4], it is well known that if M is connected and locally compact then its group of isometries $I(M,d)$ is locally compact with respect to the compact-open topology. Recently, Manoussos-Strantzalos [9] replaced the connectivity of M by the weaker hypothesis that the space $\Sigma(M)$ of the connected components of M is quasi-compact (compact but not necessarily Hausdorff) with respect to quotient topology. Also, Gao-Kechris [13] proved a stronger result about the group of isometries $I(M,d)$ encompassing the above ones.

We prove some extensions and variations of these results for the group of homotheties $H(M,d)$. First we consider Heine-Borel metric spaces, i.e., metric spaces whose the compact subsets are the bounded and closed ones. We prove that for a Heine-Borel metric space (M,d) its group of homotheties $H(M,d)$ is also a Heine-Borel metric space. In particular, it follows that $I(M,d)$ is a Heine-Borel metric space. For this result we do not require any further assumption, like e.g. the quasi-compactness of [9]. The set $M = \mathbb{Z}$ of integers with the standard distance $d\,(x,y) = |x-y|$, is a simple example of a metric space where $\Sigma(M)$ is not quasi-compact but M is a Heine-Borel space, so our results apply to it. As a consequence of our methods we give a new proof of the fact [13] that action of $I(M,d)$ on M is proper if (M,d) is Heine-Borel.

The class of Heine-Borel metric spaces is interesting for two facts. First, because there is a consequence of the Hopf-Rinow theorem for length spaces, (cf. [2]), which claims that (M,d) is Heine-Borel if and only if it is complete and locally compact. Second, because of the following fact: if (M,d) is a separable and locally compact metric space then there is a metric d' equivalent to d such that (M, d') is Heine-Borel [8].

We also consider the group of isometries $I(M, d)$ of Heine-Borel ultrametric spaces (M, d), i.e., metric spaces where

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

for all $x, y, z \in M$. It is proved that if (M, d) is a Heine-Borel ultrametric space then its group of isometries is an increasing union of compact subgroups. This result is an improvement of a theorem due to Gao-Kechris [13], which states that the group of isometries is the closure of an increasing union of compact subgroups. Also, we prove that if (M, d) is a Heine-Borel ultrametric space and G is a finitely generated subgroup of $I(M, d)$ then $\mathrm{cl}(G)$ is compact . Heine-Borel ultrametric spaces are extensively used in Number Theory because of the Ostrowski theorem [3], [12] which states that every nontrivial norm on $\mathbb{Q}$ is equivalent to the standard absolute value or to the Heine-Borel ultrametric p-adic norm for some prime p.

Next we consider metric spaces (M, d) such that the space $\Sigma(M)$ of the connected components of M is quasi-compact with respect to quotient topology. First we generalize the result of [9] for the group of homotheties: if M is a locally compact metric space and $\Sigma(M)$ is quasi-compact then its group of homotheties $H(M, d)$ is locally compact with respect to the compact-open topology.

In this setting we also look at the quasi-metric spaces, i.e., a space (M, d) where d satisfies all the distance axioms, except perhaps that $d(x, y)$ is not necessarily equal to $d(y, x)$ for all $x, y \in M$, and such that the topology generated by forwards metric balls is equal to the topology generated by backwards metric balls. We prove that if M is a locally compact quasi-metric space and $\Sigma(M)$ is quasi-compact then its group of homotheties $H(M, d)$ is locally compact with respect to the compact-open topology. An interesting example of quasi-metric spaces are the Finsler manifolds, where its Finsler function F is positively homogeneous but not necessary absolutely homogeneous and, for all $x, y \in M$, we define $d(x, y)$ as the greatest lower bound of the length of all smooth curves joining x to y.

As applications we give some generalizations of classical results in Riemannian geometry. If (M,d) is a Finsler manifold or an Alexandroff space, then its group of homotheties is a Lie group. With some additional hypothesis, this is also true for a Hadamard space.

2 Homotheties and isometries

Let (M,d) be a metric space. We denote the closed and open metric balls of (M,d) centered in $x \in M$ with radius $r > 0$, respectively, by $B[x,r] = \{y \in M : d(x,y) \le r\}$ and by $B(x,r) = \{y \in M : d(x,y) < r\}$. We also denote the closure of $N \subset M$ by $\mathrm{cl}(N)$. The group of homotheties of M is defined by

$$H(M,d) = \{f \in C(M) : f \text{ is onto and } d(f(x),f(y)) = \lambda(f)d(x,y),\ \forall x,y \in M\},$$

where $C(M)$ is the set of all continuous mappings of M into itself and $\lambda(f) > 0$ is a constant. The group of isometries of M is defined by $I(M,d) = \{f \in H(M) : \lambda(f) = 1\}$. We show next that there is a continuous homomorphism from $H(M,d)$ to the multiplicative group $(0,\infty)$:

Lemma 2.1. *The mapping $f \mapsto \lambda(f)$ is a continuous homomorphism, with respect to the compact-open topology, from $H(M,d)$ to the multiplicative group $(0,\infty)$.*

Proof. Clearly, for all $f,g \in H(M,d)$, we have $\lambda(f \circ g) = \lambda(f)\lambda(g)$ and $\lambda(f^{-1}) = \lambda(f)^{-1}$. To see that λ is continuous and $H(M,d)$ is closed, with respect to the compact-open topology, let $(f_i)_{i\in I}$ be a net in $H(M,d)$ such that $f_i \to f$ in the compact-open topology, which implies that $f_i(x) \to f(x)$, for all $x \in M$. Let $w,z \in M$ be such that $w \neq z$. Then

$$\lambda(f_i) = \frac{d(f_i(w),f_i(z))}{d(w,z)} \to \frac{d(f(w),f(z))}{d(w,z)} = \lambda(f). \tag{1}$$

$\square$

Let G be a subgroup of $H(M,d)$ and denote $[G,G]$ the commutator group, that is, the smallest closed subgroup of G containing all elements of the form

$[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$. From what was shown above, we obtain the following corollary for the groups $H(M, d)$ and $I(M, d)$ of a metric space (M, d):

Proposition 2.1. *Let (M, d) be a metric space and G a subgroup of $H(M, d)$. Then the following statements holds:*

1. *$I(M, d)$ is a closed normal subgroup of $H(M, d)$.*

2. *If $G/[G, G]$ is compact, then G is a subgroup of $I(M, d)$.*

3. *If G is compact then G is a compact subgroup of $I(M, d)$.*

Proof.

1. Clearly $I(M, d)$ is the kernel of the continuous homomorphism λ.

2. Since λ is a homomorphism and $(0, \infty)$ is an abelian group, it follows that $[G, G] \subset \ker(\lambda)$. Hence, since λ is a continuous homomorphism, $\lambda(G)$ is isomorphic to $G/\ker(\lambda)$ and therefore $\lambda(G)$ is a compact subgroup of $(0, \infty)$. Hence $\lambda(G) = \{1\}$ implying that $G \subset I(M, d)$.

3. If G is compact, $G/[G, G]$ is compact.

□

Since the group of isometries $I(M, d)$ is a closed subgroup of $H(M, d)$ with respect to the compact-open topology most of the results for $H(M, d)$ apply immediately to the group of isometries. A topological group G has no small subgroups if there is a neighborhood V of the identity element $e \in G$ with the following property: if $H \subset V$ is a subgroup of G, then $H = \{e\}$. The next result is a consequence of a theorem due Gleason and Yamabe (cf. [15], [10]) which states that G is a Lie Group if, and only if, G is a locally compact group and has no small subgroups.

Theorem 2.1. *Suppose that $H(M, d)$ is locally compact. Then $I(M, d)$ is a Lie group if, and only if, $H(M, d)$ is a Lie group.*

Proof. If $H(M,d)$ is a Lie group, then $I(M,d)$ is a Lie group, since $I(M,d)$ is a closed subgroup. Assume that $I(M,d)$ is a Lie group. Then $I(M,d)$ has no small subgroups, i.e., there is a neighborhood $U \subset I(M,d)$ of the identity id in I(M,d) such that if $G \subset U$ is a subgroup of $I(M,d)$, then $G = \{\mathrm{id}\}$. Let $V \subset H(M,d)$ be a neighborhood of the identity in $H(M,d)$ such that $U = V \cap I(M,d)$ and $W \subset V$ another neighborhood of the identity in $H(M,d)$ such that $\mathrm{cl}(W)$ is compact. Let $G \subset W$ be a subgroup of $H(M,d)$. Thus $\mathrm{cl}(G)$ is compact and hence G is a subgroup of $I(M,d)$ (cf. Proposition 2.1). Since $G \subset W \cap I(M,d) \subset U$, $G = \{\mathrm{id}\}$. Therefore $H(M,d)$ is locally compact and has no small subgroups and hence is a Lie group.

□

In the sections 3 and 5, we prove that, under some reasonable conditions over (M,d), the group of homotheties $H(M,d)$ is locally compact.

3 Heine-Borel metric spaces

Let (M,d) be a Heine-Borel metric space, i.e., a metric space such that a subset K is compact if and only if it is bounded and closed. If we fix a point $x_0 \in M$, then $M = \cup_{n\in\mathbb{N}} B[x_0,n]$. As (M,d) is a Heine-Borel metric space, all closed metric balls are compact. Therefore M is separable and $C(M)$ is a metric space with respect to the compact-open topology (cf. [8]) with the distance Δ defined by

$$\Delta(f,g) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{d_n(f,g)}{1+d_n(f,g)} \tag{2}$$

where $d_n(f,g) = \sup_{x\in B[x_0,n]} d(f(x),g(x))$. For $f \in C(M)$, we denote

$$\alpha(f) = \inf_{x\neq y} \frac{d(f(x),f(y))}{d(x,y)} \qquad \text{and} \qquad \beta(f) = \sup_{x\neq y} \frac{d(f(x),f(y))}{d(x,y)}. \tag{3}$$

When $f \in H(M,d)$ then f is onto and $\alpha(f) = \beta(f) = \lambda(f) > 0$. The following lemma is essential for the results in the next section:

Lemma 3.1. *For all $w, z \in M$ and all $a, b, c > 0$, the set*

$$V[w,z,a,b,c] = \{f \in C(M) : f \textit{ is onto}, a \leq \alpha(f) \leq \beta(f) \leq b \textit{ and } d(f(w),z) \leq c\} \tag{4}$$

is compact with respect to the compact-open topology.

Proof. For all $f \in V = V[w,z,a,b,c]$ and all $x \in M$, we have $d(f(x),z) \leq d(f(x),f(w)) + d(f(w),z) \leq bd(x,w) + c$, and, since M is Heine-Borel, $V(x) = \{f(x) : f \in V\}$ is relatively compact in M. Since for all $f \in V$, we have $d(f(x),f(y)) \leq bd(x,y)$, then V is a uniformly equicontinuous family and so, by Arzelá-Ascoli's theorem for spaces that are an increasing countable union of compact subsets (cf. [8]), V is relatively compact in $C(M)$ with respect to the compact-open topology. It remains to prove that V is closed with respect to the compact-open topology. Let $\bar{f} \in \mathrm{cl}(V)$, where $\mathrm{cl}(V)$ is the closure of V in $C(M)$, with respect to the compact-open topology, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in V such that $f_n \to \bar{f}$. By taking limits in the inequalities of (4), we have $a \leq \alpha(\bar{f}) \leq \beta(\bar{f}) \leq b$ and $d(\bar{f}(w),z) \leq c$. To show that f is onto, let $y \in M$. Since each f_n is onto, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $f_n(x_n) = y$. Hence $d(x_n,y) \leq a^{-1}d(f_n(x_n),f_n(y)) = a^{-1}d(y,f_n(y))$. Since $f_n(y) \to \bar{f}(y)$, we obtain that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in the Heine-Borel space M. Hence there are $x \in M$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \to x$. Thus $d(\bar{f}(x),y) \leq d(\bar{f}(x),f_{n_k}(x)) + d(f_{n_k}(x),f_{n_k}(x_{n_k})) \leq d(\bar{f}(x),f_{n_k}(x)) + bd(x,x_{n_k})$, and, since $f_{n_k}(x) \to \bar{f}(x)$ and $x_{n_k} \to x$, it follows that $d(\bar{f}(x),y) = 0$. Therefore $\bar{f}$ is onto showing that V is compact with respect to the compact-open topology. □

We have that $\log : (0,\infty) \to \mathbb{R}$ is a continuous homomorphism from the multiplicative group $(0,\infty)$ to the additive group $\mathbb{R}$. Thus the mapping $f \mapsto \log(\lambda(f))$ is a continuous homomorphism from $H(M,d)$ to the additive group $\mathbb{R}$ and we can define a distance δ equivalent to Δ (cf. [8]) by

$$\delta(f,g) = \Delta(f,g) + d(f(x_0),g(x_0)) + |\log(\lambda(f)) - \log(\lambda(g))|. \tag{5}$$

We denote the closed metric ball of $(H(M,d),\delta)$ centered in $f \in H(M,d)$ with radius $r > 0$ by $\mathcal{B}[f,r] = \{g \in H(M,d) : \delta(f,g) \leq r\}$.

The next result is an extension of the van Dantzig and van der Waerden's result for the group of homotheties $H(M,d)$ when (M,d) is Heine-Borel:

Theorem 3.1. *If (M,d) is a Heine-Borel metric space, then $H(M,d)$ is a Heine-Borel metric space.*

Proof. We have to show that $\mathcal{B}[\mathrm{id},r]$ is compact for all $r > 0$, where id is the identity mapping. This is done by showing that $\mathcal{B}[\mathrm{id},r] \subset V[w,z,a,b,c]$, for some $V[w,z,a,b,c]$ like in Lemma 3.1. For all $f \in \mathcal{B}[\mathrm{id},r]$, we have $d(f(x_0),x_0) \leq r$ and

$$|\log(\lambda(f))| = |\log(\lambda(f)) - \log(\lambda(\mathrm{id}))| \leq r. \tag{6}$$

Thus $\exp(-r) \leq \lambda(f) \leq \exp(r)$ and $\mathcal{B}[\mathrm{id},r] \subset V = V[x_0,x_0,\exp(-r),\exp(r),r]$. It remains to prove that $\mathcal{B}[\mathrm{id},r]$ is closed in $C(M)$, with respect to the compact-open topology, and hence compact. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{B}[\mathrm{id},r]$ be a sequence such that $f_n \to f$. The compactness of V implies that f is onto. By taking limits, $\delta(\mathrm{id},f) \leq r$ and, for all $x,y \in M$, with $x \neq y$ we have

$$\frac{d(f(x),f(y))}{d(x,y)} = \lim \frac{d(f_n(x),f_n(y))}{d(x,y)} = \lim \lambda(f_n) = \lambda(f) \tag{7}$$

and therefore $f \in \mathcal{B}[\mathrm{id},r]$.

□

The following corollary is a immediately consequence of Proposition 2.1 and Theorem 2.1.

Corollary 3.1. *Let (M,d) be a Heine-Borel space. Then we have:*

1. *$I(M,d)$ is a Heine-Borel metric space.*
2. *$I(M,d)$ is a Lie group if, and only if, $H(M,d)$ is a Lie group.*

The action of a group of transformations G on a metric space M is proper if, and only if, for all $w,z \in M$ there are neighborhoods U_w and U_z of w and z, respectively, such that the subset $\{g \in G : gU_w \cap U_z \neq \emptyset\}$ is relatively compact with respect to the compact-open topology. The theorem below is a special case (for Heine-Borel spaces (M, d)) of a result on the action of $I(M, d)$ on M that can be found in [13]:

Theorem 3.2. *If (M, d) is a Heine-Borel metric space, then the action of $I(M, d)$ on M is proper.*

Proof. Let $w, z \in M$, $U_w = B[w, 1]$, $U_z = B[z, 1]$. Take $f \in \{g \in I(M, d) : gU_w \cap U_z \neq \emptyset\}$. Then there is $y \in U_w$ such that $f(y) \in U_z$. Therefore $d(f(w), z) \leq d(f(w), f(y)) + d(f(y), z) \leq d(w, y) + d(f(y), z) \leq 2$. Hence $\{g \in I(M, d) : gU_w \cap U_z \neq \emptyset\} \subset V[w, z, 1, 1, 2]$ and, by Lemma 3.1, is relatively compact with respect to the compact-open topology.

□

Remark 3.1. *It is well known that Theorem 3.2 is not true if the group of isometries is replaced by the group of homotheties $H(M, d)$. For instance, let the space of real numbers $\mathbb{R}$ be endowed with the usual distance and take the sequence $(g_n)_{n \geq k}$ with $g_n(x) = nx$. We see that it does not have any convergent subsequence in the compact-open topology. On the other hand, since 0 is a fixed point for all g_n, there is $k \in \mathbb{N}$ such that*

$$(g_n)_{n \geq k} \subset \{g \in H(M, d) : gU_0 \cap U_1 \neq \emptyset\},$$

where U_0 and U_1 are arbitrary neighborhoods, respectively, of 0 and 1.

4 Heine-Borel ultrametric spaces

In this section, we consider (M, d) be a Heine-Borel ultrametric metric space, i.e., a metric space where

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

for all $x, y, z \in M$. Given a subset $F \subset I(M, d)$ of isometries and $x \in M$, we let $\langle F \rangle$ be the subgroup of isometries generated by F and $F(x) = \{f(x) : f \in F\}$, the orbit of x by F.

Lemma 4.1. *Take $x \in M$ and let $G = \langle F \rangle$, where $F \subset I(M,d)$. If $F(x) \subset B[x,r]$, where $r > 0$, then $G(x) \subset B[x,r]$ and hence cl(G) is a compact subgroup.*

Proof. If $f \in F$ then $d(f^{-1}(x), x) = d(f^{-1}(x), f^{-1}(f(x))) = d(x, f(x)) \leq r$. Let $g \in G$ be such that $g = f_1 \cdots f_n$ with $f_i \in F$ or $f_i^{-1} \in F$, $i \in \{1, \cdots, n\}$. We proceed by induction on n to show that $g(x) \in B[x,r]$. If $n = 1$, by the previous inequality, $g(x) \in B[x,r]$. Assume that the result is true for n. Let $g = f_1 \cdots f_n f_{n+1}$ and define $h = f_1 \cdots f_n$. Hence

$$
\begin{aligned}
d(g(x), x) = d(h(f_{n+1}(x)), x) &\leq \max\{d(h(f_{n+1}(x)), h(x)), d(h(x), x)\} = \qquad (8)\\
&= \max\{d(f_{n+1}(x), x), d(h(x), x)\} \leq r, \qquad (9)
\end{aligned}
$$

by the induction hypothesis, and for all $g \in G = \langle F \rangle$, $g(x) \in B[x,r]$. Therefore $G(x) \subset B[x,c]$ and cl$(G)(x)$ is compact. The properness of the action implies that cl(G) is a compact subgroup (see [14] and also [13]).

□

The next corollary states that if $G \subset I(M,d)$ has the algebraic property to be finitely generated then the closure of G has the topological property to be compact:

Corollary 4.1. *Let (M,d) a Heine-Borel ultrametric space. If $G \subset I(M,d)$ is finitely generated then cl(G) is a compact subgroup.*

Proof. We have $G = \langle F \rangle$, where $F = \{f_1, \cdots, f_n\}$. Hence for all $i \in \{1, \cdots, n\}$ we have $d(f_i(x), x) \leq \max\{d(f_1(x), x), \cdots, d(f_n(x), x)\} = r$. By the Lemma 4.1, cl$(G)$ is a compact subgroup.

□

Finally we have the following improvement of a theorem due to Gao and Kechris [13]:

Theorem 4.1. *Let (M,d) a Heine-Borel ultrametric space. $I(M,d) = \bigcup G_n$, where G_n are compact subgroups such that $G_n \subset G_{n+1}$.*

Proof. Let $x \in M$ and, for each $n \in \mathbb{N}$, we define

$$G_n = \{f \in I(M,d) : d(f(x),x) \leq n\}. \tag{10}$$

By Lemma 4.1 we have that $\langle G_n \rangle = G_n$ and $\mathrm{cl}(G_n)$ is compact. But it is easy to verify that G_n is closed in $I(M,d)$ with respect to the compact-open topology, showing that G_n is a compact subgroup. Clearly, we also have $G_n \subset G_{n+1}$ and $I(M,d) = \bigcup G_n$.

□

Theorem 4.1 also shows that, when (M,d) is Heine-Borel ultrametric space, $I(M,d)$ is an amenable group (cf. [17]).

5 $\Sigma(M)$ quasi-compact

Let (M,d) be a locally compact metric space and $C(M)$ the set of all continuous maps of M into itself. The following four lemmas are generalizations of some results of Manoussos-Strantzalos [9]. First we define

$$V[a,b] = \{f \in C(M) : f \text{ is onto}, a \leq \alpha(f) \leq \beta(f) \leq b\}. \tag{11}$$

where $a, b > 0$ and $\alpha(f)$, $\beta(f)$ are as in Equation (3). $V[a,b]$ is clearly an uniformly equicontinuous family of $C(M)$.

Lemma 5.1. *For all $a, b > 0$ and for all $V \subset V[a,b]$, let $V(x) = \{f(x) : f \in V\}$. Then the set*

$$K(V) = \{x \in M : V(x) \textit{ is relatively compact}\}, \tag{12}$$

is an open and closed subset of M.

Proof. The fact that $K(V)$ is open, is a consequence of the uniform equicontinuity of $V \subset C(M)$. In fact, take $x \in K(V)$ and for $y \in \mathrm{cl}(V(x))$ let $\varepsilon_y > 0$ be such that $B(y, \varepsilon_y)$ is relatively compact. We see that $\{B(y,\varepsilon_y)\}_{y \in cl(V(x))}$ is a covering of the compact set $\mathrm{cl}(V(x))$. Hence there are $y_i \in \mathrm{cl}(V(x))$,

$i \in \{1, \cdots, m\}$, such that $\{B(y_i, \varepsilon_{y_i})\}_{i\in\{1,\cdots,m\}}$ is a finite subcovering. If we take $\delta < b^{-1} \inf d(u,v)$, where $u \in \mathrm{cl}(V(x))$ and $v \in M \setminus \bigcup_{i\in\{1,\cdots,m\}} B(y_i, \varepsilon_{y_i})$, then $V(z) \subset \bigcup_{i\in\{1,\cdots,m\}} B(y_i, \varepsilon_{y_i})$, for all $z \in B(x,\delta)$ and therefore $B(x,\delta) \subset K(V)$. We prove now that $K(V)$ is closed. For all $N \subset M$, we define $B(N,\varepsilon) = \bigcup_{x\in N} B(x,\varepsilon)$. Take $x \in \mathrm{cl}(K(V))$ and let $\varepsilon > 0$ be such that $B(x, \frac{5b}{a}\varepsilon)$ is relatively compact, and $y \in K(V) \cap B(x,\varepsilon)$. Hence $\mathrm{cl}(V(y)) \subset B(V(B(x,\varepsilon)), a\varepsilon) \subset V(B(x, 2\varepsilon))$, because for each y, z such that $d(y,x) < \varepsilon$ and $d(z, f(y)) < a\varepsilon$, for some $f \in V$, then $d(f^{-1}(z), x) \leq d(f^{-1}(z), y) + d(y,x) \leq a^{-1} d(z, f(y)) + d(y,x) < 2\varepsilon$ and therefore $f^{-1}(z) \in B(x, 2\varepsilon)$, so $z \in V(B(x,2\varepsilon))$. By the compactness of $\mathrm{cl}(V(y))$ we can get a finite subset $F \subset V$ such that $\mathrm{cl}(V(y)) \subset F(B(x,2\varepsilon))$. We show that $V(x)$ is contained in the relatively compact set $F(B(x, \frac{5b}{a}\varepsilon))$. Let $f \in V$ and $g \in F$ such that $f(y) \in g(B(x,2\varepsilon)) \subset B(g(x), 2b\varepsilon)$. Hence

$$\begin{aligned} d(f(x), g(y)) &\leq d(f(x), f(y)) + d(f(y), g(x)) + d(g(x), g(y)) & (13)\\ &\leq bd(x,y) + d(f(y), g(x)) + bd(x,y) \leq 4b\varepsilon & (14) \end{aligned}$$

and therefore $f(x) \in B(g(y), 4b\varepsilon) \subset g(B(y, \frac{4b}{a}\varepsilon)) \subset g(B(x, \frac{5b}{a}\varepsilon)) \subset F(B(x, \frac{5b}{a}\varepsilon))$. Thus $x \in K(V)$ and the proof is concluded.

□

For the next lemmas, we assume that the space $\Sigma(M)$ of the connected components of M is quasi-compact (compact but not necessarily Hausdorff) with respect to quotient topology. Under this assumption M is separable and therefore is an increasing countable union of compact subsets, so that we can apply the Arzelá-Ascoli theorem (cf. [8]).

Lemma 5.2. *There is a neighborhood V of the identity in $H(M,d)$ that is relatively compact in $C(M)$ with respect to the compact-open topology.*

Proof. For all $x \in M$, let $r_x > 0$ be such that $B(x, 2r_x)$ is relatively compact and take $y_x \in M$ with $x \neq y_x$, and $r_x \leq \frac{1}{4} d(x, y_x)$. Define

$$V_x = \{f \in H(M,d) : d(f(x), x) \leq r_x \ \text{and} \ d(f(y_x), y_x) \leq r_x\}. \qquad (15)$$

Then V_x is a neighborhood of the identity in $H(M,d)$ with respect to the compact-open topology. By the triangular inequality, for all $f \in V_x$, we have $d(f(x), f(y_x)) \leq 2r_x + d(x, y_x) \leq \frac{3}{2}d(x, y_x)$ and $d(x, y_x) \leq 2r_x + d(f(x), f(y_x))$, which implies that $\frac{1}{2}d(x, y_x) \leq d(x, y_x) - 2r_x \leq d(f(x), f(y_x))$. Hence $\frac{1}{2} \leq \lambda(f) \leq \frac{3}{2}$. Therefore $V_x \subset V[\frac{1}{2}, \frac{3}{2}]$ and $x \in K(V_x)$. By Lemma 5.1, $K(V_x)$ is an open and closed subset of M and therefore it is the union of connected components of M. Hence $q(K(V_x))$ is an open subset of $\Sigma(M)$, where $q : M \to \Sigma(M)$ is the canonical projection, and $\{q(K(V_x))\}_{x\in M}$ is an open covering of $\Sigma(M)$. Since $\Sigma(M)$ is quasi-compact, there are $x_i \in M$, $i \in \{1, \cdots, m\}$ such that $\{q(K(V_{x_i}))\}_{i\in\{1,\cdots,m\}}$ is a finite open subcovering. We define $V = \bigcap_{i\in\{1,\cdots,m\}} V_{x_i}$ so that V is also a neighborhood of the identity in $H(M,d)$ with respect to the compact-open topology. Therefore, $V(x) \subset V_{x_i}(x)$ is relatively compact for all $x \in M$. Since V is clearly an equicontinuous family, by Arzelá-Ascoli theorem, V is relatively compact in $C(M)$.

□

The preceding lemma shows that there is a neighborhood of the identity in $H(M,d)$ which is relatively compact in $C(M)$ with respect to the compact-open topology. For our purpose, namely to prove that $H(M,d)$ is locally compact, it sufficient to show that this neighborhood is also closed and therefore its closure is contained in $H(M,d)$. The following two lemmas are used to prove what is the most delicate part of the proof: the limit of a sequence in this neighborhood is onto:

Lemma 5.3. *Let V be the neighborhood of the identity as in Lemma 5.2. If a sequence $(f_n)_{n\in\mathbb{N}} \subset V$ is such that $f_n \to f$ with respect to the compact-open topology, then its image $f(M)$ is an open and closed subset of M.*

Proof. By Lemma 5.1, it is sufficient to show that $f(M) = K(L)$, where $L = \{f_n^{-1}, n \in \mathbb{N}\}$, because $L^{-1} \subset V \subset V[\frac{1}{2}, \frac{3}{2}]$. One one hand, since $d(x, f_n^{-1}(x)) \leq 2d(f_n(x), f(x))$ and $f_n(x) \to f(x)$, we have $f_n^{-1}(f(x)) \to x$ and $f(x) \in K(L)$ for all $x \in M$, because M is locally compact. On the other hand, if $y \in K(L)$

then $F(y)$ is relatively compact in M, so there are $x \in M$ and a subsequence $(f_{n_k}^{-1}(y))_{k\in\mathbb{N}}$ such that $f_{n_k}^{-1}(y) \to x$. Hence we have $d(f(x), y) \leq d(f(x), f_{n_k}(x)) + d(f_{n_k}(x), y) \leq d(f(x), f_{n_k}(x)) + \frac{3}{2} d(x, f_{n_k}^{-1}(y))$ and, since $f_{n_k}(x) \to f(x)$, we have $y = f(x)$.

□

If M is connected then $f(M) = M$ and therefore f is onto. But, as the following lemma shows, this is also true in the more general situation when $\Sigma(M)$ is quasi-compact with respect to the quotient topology:

Lemma 5.4. *Let V be the neighborhood of the identity as in Lemma 5.2. If a sequence $(f_n)_{n\in\mathbb{N}} \subset V$ is such that $f_n \to f$ with respect to the compact-open topology, then f is onto.*

Proof. Let $y \in M$. We denote by S_y the connected component of y and, for all $n \in \mathbb{N}$, we let S_n be the component of $f_n^{-1}(y)$. Since each f_n is a homeomorphism, we have $f_n(S_n) = S_y$ for all $n \in \mathbb{N}$. First we assume that $(S_n)_{n\in\mathbb{N}}$ has a constant subnet $(S_{n_k} = S_0)_{k\in\mathbb{N}}$, for some $S_0 \in \Sigma(M)$. Hence $f_{n_k}(S_0) = S_y$, for all $k \in \mathbb{N}$. Let $x_0 \in S_0$. Hence $f_{n_k}(x_0) \in S_y$ and, since $f_{n_k}(x_0) \to f(x_0)$, we have $f(x_0) \in S_y$. By Lemma 5.3, $S_y \subset f(M)$ and therefore $y \in f(M)$. Now we suppose that $(S_n)_{n\in\mathbb{N}}$ has no constant subnet. Hence, with the above notation, we can apply the following lemma that was proved in Manoussos-Strantzalos [9]:

Lemma 5.5. *If $(S_n)_{n\in\mathbb{N}}$ has no constant subnet then there are $x_0 \in M$, a subsequence $(S_{n_k})_{k\in\mathbb{N}}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in S_{n_k}$ and $x_k \to x_0$.*

By Lemma 5.5, there are $x_0 \in M$, a subsequence $(S_{n_k})_{k\in\mathbb{N}}$ and a subsequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in S_{n_k} = f_{n_k}^{-1}(S_y)$ and $x_k = f_{n_k}^{-1}(y_k) \to x_0$, where $y_k \in S_y$. Hence

$$
\begin{aligned}
d(f(x_0), y_k) &\leq d(f(x_0), f_{n_k}(x_0)) + d(f_{n_k}(x_0), y_k) && (16)\\
&\leq d(f(x_0), f_{n_k}(x_0)) + \frac{3}{2} d(x_0, f_{n_k}^{-1}(y_k)) && (17)
\end{aligned}
$$

and, since $f_{n_k}(x_0) \to f(x_0)$, we have $y_k \to f(x_0)$ and therefore $f(x_0) \in S_y$. By Lemma 5.3, $S_y \subset f(M)$ and therefore f is onto.

□

The following is a generalization of a result due to Manoussos and Strantzalos [9] for the group of homotheties:

Theorem 5.1. *Let (M, d) be a locally compact metric space such that the space of connected components $\Sigma(M)$ is quasi-compact with respect to the quotient topology. Then $H(M, d)$ is locally compact with respect to the compact-open topology.*

Proof. Let $V = \bigcap_{i\in\{1,\cdots,m\}} V_{x_i} \subset C(M)$ be the relatively compact neighborhood of the identity as in Lemma 5.2. It is sufficient to show that V is closed with respect to the compact-open topology. Let $(f_n)_{n\in\mathbb{N}} \subset V$ be a sequence such that $f_n \to f$. By Lemma 5.4, f is onto. Arguing as in Lemma 3.1, we have that $d(f(x), f(y)) = \lambda(f)d(x, y)$, where $\lambda(f) = \lim \lambda(f_n)$. Clearly, we also have $d(f(x_i), x_i) \leq r_{x_i}$ and $d(f(y_{x_i}), y_{x_i}) \leq r_{x_i}$, for all $i \in \{1, \cdots, m\}$ and therefore $f \in V$. Hence V is compact with respect to the compact-open topology.

□

As a immediately corollary, we have a generalization of the classical theorem of van Dantzig and van der Waerden for the group of homotheties $H(M, d)$.

Corollary 5.1. *Let (M, d) be a connected and locally compact metric space. Then $H(M, d)$ is locally compact with respect to the compact-open topology.*

The following corollary is a immediately consequence of Theorem 2.1.

Corollary 5.2. *Let (M, d) be a locally compact metric space such that the space of connected components $\Sigma(M)$ is quasi-compact with respect to the quotient topology. Then $I(M, d)$ is a Lie group if, and only if, $H(M, d)$ is a Lie group.*

6 Quasi-metric spaces

Let $d : M \times M \to [0, \infty)$ satisfying the distance axioms except perhaps that $d(x, y)$ is not necessary equal to $d(y, x)$ for all $x, y \in M$. If the topologies generated by the forward metric balls $B_+(x, r) = \{y \in M : d(x, y) < r\}$ and generated by the backward metric balls $B_-(x, r) = \{y \in M : d(y, x) < r\}$ are the same, we call (M, d) a *quasi-metric space* and d a *quasi-distance*. The group of homotheties $H(M, d)$ and the group of isometries $I(M, d)$ are defined in the same way as for metric spaces. An interesting example of such spaces are the Finsler manifolds where its Finsler function F is positively homogeneous but not necessary absolutely homogeneous. The notation of the forward and backward metric balls follows the book of D. Bao, S.S. Chern and Z. Shen [1]. The following corollary is an extension of Theorem 5.1 for quasi-metric spaces:

Corollary 6.1. *Let (M, d) be a locally compact quasi-metric space and suppose that its space of connected components $\Sigma(M)$ is quasi-compact with respect to the quotient topology. Then $H(M, d)$ is locally compact with respect to the compact-open topology.*

Proof. If (M, d) is a quasi-metric space then the function $\bar{d} : M \times M \to [0, \infty)$, defined by $\bar{d}(x, y) = d(x, y) + d(y, x)$, is a distance and it is straightforward to verify that the topology of $(M, \bar{d})$ is the same as the topology of (M, d). So, $(M, \bar{d})$ is a locally compact metric space and its space of connected components $\Sigma(M)$ is quasi-compact with respect to the quotient topology. It also straightforward to verify that $H(M, d)$ is a closed subgroup of $H(M, \bar{d})$ with respect to the compact-open topology and hence the proof is concluded.

□

7 Applications

The following theorems are immediately consequences of Corollary 5.2 and its version for quasi-metric spaces.

7.1 Finsler manifolds

Let (M, d) be a connected Finsler manifold (cf. [1]), where its Finsler function F is positively homogeneous but not necessary absolutely homogeneous, i.e., $F(x, av) = aF(x, v)$, for all $v \in T_x(M)$, only if $a \geq 0$. We define, for all $x, y \in M$, $d(x, y)$ as the greatest lower bound of the length of all smooth curves joining x to y. As we noted in the previous section, (M, d) is a quasi-metric space but d is not necessary a distance [1]. Recently, Shaoqiang Deng and Zixin Hou (cf. [5]) proved the following generalization of a classical result of Riemannian geometry due to Myers and Steenrod [11]: if (M, d) is a connected Finsler manifold then its group of isometries is a Lie group with respect to the compact-open topology. Thus we have the following result:

Theorem 7.1. *If (M, d) is a connected Finsler manifold, then $H(M, d)$ is a Lie group.*

7.2 Singular spaces

Let (M, d) be a length space, that is, a metric space where the distance between every two points in it is realized as the infimum of lengths of curves joining them. Alexandroff generalized the concept of sectional curvature of a Riemannian manifold to such spaces using the conclusion of Topogonov's comparison theorem as the definition (cf. [2]). A length space which is locally compact, of finite Hausdorff dimension and whose curvature is nowhere $-\infty$ is called an Alexandroff space. It was proved by Fukaya-Yamaguchi [6] that $I(M, d)$ is a Lie group if (M, d) is an Alexandroff space. Thus we have the following result:

Theorem 7.2. *If (M, d) is an Alexandroff space, then $H(M, d)$ is a Lie group.*

Finally, let (M, d) be a Hadamard space, that is, a complete simply connected length space with nonpositive curvature. For such space (M, d) the ideal boundary $M(\infty)$ is defined as the set of equivalence classes of rays in (M, d) with the natural topology. The space $M(\infty)$ has a natural metric Td called Tits metric, which measures the deviation of (M, d) from flatness (cf. [2]).

It was proved by Yamaguchi [16] that if (M, d) is locally compact, geodesically complete Hadamard space and $(M(\infty), Td)$ is compact, then $I(M, d)$ and $I(M(\infty), Td)$ are Lie groups. Thus we have the following result:

Theorem 7.3. *If (M, d) is a locally compact, geodesically complete Hadamard space and $(M(\infty), Td)$ is compact, then $H(M, d)$ is a Lie group.*

We remark that the compactness of $(M(\infty), Td)$ is equivalent to the hypothesis that the Tits topology coincides with the natural topology.

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Instituto de Matemática
Universidade Estadual de Campinas
Cx. Postal 6065
13.081-970 Campinas-SP, Brasil