DOUBLY PERIODIC MINIMAL TORI WITH PARALLEL ENDS

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Abstract

Let $\mathcal{K}$ be the space of properly embedded minimal tori in quotients of $\mathbb{R}^3$ by two independent translations, with any fixed (even) of parallel ends. After an appropriate normalization, we prove that $\mathcal{K}$ is a 3-dimensional real analytic manifold that reduces to finite coverings of the examples defined by Karcher, Meeks and Rosenberg in [5, 6, 10]. The degenerate limits of surfaces in $\mathcal{K}$ are the catenoid, the helicoid, the Riemann minimal examples and the simply and doubly periodic Scherk minimal surfaces.

1 Introduction

In 1988, Karcher [5] defined a 1-parameter family of minimal tori in quotients of $\mathbb{R}^3$ by two independent translations. Each of these surfaces, called *toroidal halfplane layer* and denoted in Section 3 by $M_{\theta,0,0}$, $\theta \in (0, \frac{\pi}{2})$, has four parallel Scherk-type ends, is invariant by reflection symmetries in three orthogonal planes and contains four parallel straight lines through the ends, see Figure 2 left. Thanks to this richness of symmetries, he gave explicitly the Weierstrass representation of these surfaces in terms of elliptic functions on a 1-parameter family of rectangular tori. Inside a brief remark in his paper and later in another work [6], Karcher exposed two distinct 1-parameter deformations of each $M_{\theta,0,0}$ by losing some of their symmetries (denoted by $M_{\theta,\alpha,0}, M_{\theta,0,\beta}$ in Section 3).

In 1989, Meeks and Rosenberg [10] developed a general theory for doubly periodic minimal surfaces with finite topology in the quotient, and used a completely different

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approach to find again the examples $M_{\theta,0,\beta}$ (before [13], it was not clear that Meeks and Rosenberg's examples were the same as Karcher's). In fact, it is not difficult to produce a 3-parameter family of examples $M_{\theta,\alpha,\beta}$ containing all the above examples, see Section 3. We will refer to the surfaces $M_{\theta,\alpha,\beta}$ and their k-sheeted coverings, $k \in \mathbb{N}$, as *KMR examples*.

Hauswirth and Traizet [2] proved that the moduli space of all properly embedded doubly periodic minimal surfaces with a given fixed finite topology in the quotient and parallel (resp. nonparallel) ends is a real analytic manifold of dimension 3 (resp. 1) around a *nondegenerate* surface, after identifying by translations, homotheties and rotations. Since each $M_{\theta,\alpha,\beta}$ will be nondegenerate (see Section 3), we get a local uniqueness around $M_{\theta,\alpha,\beta}$. Pérez, Rodríguez and Traizet obtain in [13] the following global uniqueness result.

Theorem 1 *If M is a properly embedded doubly periodic minimal surface with genus one in the quotient and parallel ends, then M is a KMR example.*

Theorem 1 does not hold if we remove the hypothesis on the ends to be parallel, as demonstrate the 4-ended tori discovered by Hoffman, Karcher and Wei in [3]. Also remark that any KMR example will admit an antiholomorphic involution without fixed points, so Theorem 1 also classifies all doubly periodic minimal Klein bottles with parallel ends.

In this paper, we are only going to sketch the proof of Theorem 1, which is explained in detail in [13]. The proof of Theorem 1 is a modified application of the machinery developed by Meeks, Pérez and Ros in their characterization of the Riemann minimal examples [9].

For $k \in \mathbb{N}$ fixed, one considers the space $\mathcal{S}$ of properly embedded doubly periodic minimal surfaces of genus one in the quotient and $4k$ parallel ends. The goal is to prove that $\mathcal{S}$ reduces to the space $\mathcal{K}$ of KMR examples. The argument is based on modeling $\mathcal{S}$ as an analytic subset in a complex manifold $\mathcal{W}$ of finite dimension (roughly, $\mathcal{W}$ consists of all admissible Weierstrass data for our problem). Then the procedure has three steps:

- *Properness:* We obtain uniform curvature estimates for a sequence of surfaces in $\mathcal{S}$ constrained to certain natural normalizations in terms of the period vector at the ends and of the flux of these surfaces (this flux will be defined in Section 4).

- *Openness:* Any surface in $\mathcal{S} - \mathcal{K}$ can be minimally deformed by moving its period at the ends and its flux. This step depends on the properness part and both together imply, assuming $\mathcal{S} - \mathcal{K} \neq \emptyset$ (the proof of Theorem 1 is by contradiction), that any period at the ends and flux can be achieved by examples in $\mathcal{S} - \mathcal{K}$.

- *Uniqueness around a boundary point of $\mathcal{S}$:* Only KMR examples can occur nearby a certain minimal surface outside $\mathcal{S}$ but obtained as a smooth limit of surfaces in $\mathcal{S}$. This property together with the last sentence in the openness point lead to the desired contradiction, thereby proving Theorem 1.

We consider the map C that associates to each $M \in \mathcal{S}$ two geometric invariants: its period at the ends and its flux along a nontrivial homology class with vanishing period vector, and prove that $C|_{\mathcal{S}-\mathcal{K}}$ is open and proper (recall we assumed $\mathcal{S} - \mathcal{K} \neq \emptyset$) by using curvature estimates as in the first step of the above procedure, together with a local uniqueness argument similar to the third step, performed around any singly periodic Scherk minimal surface considered as a point in $\partial\mathcal{S}$. We conclude the third step in our strategy with a local uniqueness result around the catenoid, also considered as a point of $\partial\mathcal{S}$.

The paper is organized as follows. In Section 2 we recall the necessary background to tackle our problem. Sections 3 and 4 are devoted to introduce briefly the 3-parameter family $\mathcal{K}$ of KMR examples, the complex manifold of admissible Weierstrass data $\mathcal{W}$, and natural mappings on $\mathcal{W}$. In Section 5 we sketch how to obtain the curvature estimates needed for the first point of our strategy. Sections 6 and 7 deal with the local uniqueness around the singly periodic Scherk minimal surfaces and the catenoid, respectively. The second point of our above strategy (openness) is the goal of Section 8, and finally Section 9 contains the proof of Theorem 1. We refer the interested reader to [13] for a detailed proof of the statements we are going to use in order to obtain Theorem 1.

2 Preliminaries.

Let $\widetilde{M} \subset \mathbb{R}^3$ be a connected orientable[1] properly embedded minimal surface, invariant by a rank 2 lattice $\mathcal{P}$ generated by two linearly independent translations T_1, T_2 (we will shorten by calling $\widetilde{M}$ a *doubly periodic minimal surface*). $\widetilde{M}$ induces a properly embedded minimal surface $M = \widetilde{M}/\mathcal{P}$ in the complete flat 3-manifold $\mathbb{R}^3/\mathcal{P} = \mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ is a 2-dimensional torus. Reciprocally, if $M \subset \mathbb{T} \times \mathbb{R}$ is a properly embedded nonflat minimal surface, then its lift $\widetilde{M} \subset \mathbb{R}^3$ is a connected doubly periodic minimal surface by the Strong Halfspace Theorem [4]. Existence and classification theorems in this setting are usually tackled by considering the quotient surfaces in $\mathbb{T} \times \mathbb{R}$. An important result by Meeks and Rosenberg [10] insures that a properly embedded minimal surface $M \subset \mathbb{T} \times \mathbb{R}$ has finite topology if and only if it has finite total curvature, and in this case M has an even number of ends, each one asymptotic to a flat annulus (Scherk-type end). Later, Meeks [8] proved that any properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ has a finite number of ends, so the finiteness of its total curvature is equivalent to the finiteness of its genus.

When normalized so that the lattice of periods $\mathcal{P}$ is horizontal, we distinguish two types of ends, depending on whether the well defined third coordinate function on M tends to ∞ (*top* end) or to $-\infty$ (*bottom* end) at the corresponding puncture. By separation properties, there are an even number of top (resp. bottom) ends. Because of embeddedness, top (resp. bottom) ends are always parallel each other. If the top ends are not parallel to the bottom ends, then there exists an algebraic obstruction on the period lattice, which must be *commensurable* as in the doubly periodic Scherk minimal surfaces. If the top ends are parallel to the bottom ends, then the cardinals of both families of ends coincide, therefore the total number of ends of M is a multiple of four. See [10] for details.

We will focus on the parallel ends setting, where the simplest possible topology

[1] From now on, all surfaces in the paper are supposed to be connected and orientable.

is a finitely punctured torus (properly embedded minimal planar domains in $\mathbb{T}\times\mathbb{R}$ must have nonparallel ends [10]; in fact Lazard-Holly and Meeks [7] proved that the doubly periodic Scherk minimal surfaces are the unique possible examples with genus zero). Theorem 1 gives a complete classification of all examples with genus one and parallel ends, after appropriate normalization.

Given $k\in\mathbb{N}$, let $\mathcal{S}$ be the space of all properly embedded minimal tori in $\mathbb{R}^3/\mathcal{P}=\mathbb{T}\times\mathbb{R}$ with $4k$ horizontal Scherk-type ends, where $\mathcal{P}$ is a rank 2 lattice generated by two independent translations (which depend on the surface), one of them being in the direction of the x_2-axis. Given $M\in\mathcal{S}$ and an oriented closed curve $\Gamma\subset M$, we denote respectively by P_Γ and F_Γ the period and flux vectors of M along Γ. By the Divergence Theorem, P_Γ, F_Γ only depend on the homology class of Γ in M. The period and flux vectors H,F at an end of M (i.e. the period and flux along a small loop around the puncture with the inward pointing conormal vector respect to the disk that contains the end) satisfy $F=H\wedge N_0$, where N_0 is the value of the Gauss map at the puncture. In our normalization, each of the period vectors at the ends of M is of the form $H=\pm(0,\pi a,0)$ with $a>0$. The end is called a *left end* if $F=(-\pi a,0,0)$, and a *right end* if $F=(\pi a,0,0)$. As M is embedded, each family of "sided" ends is naturally ordered by heights; in fact the maximum principle at infinity [11] implies that consecutive left (resp. right) ends are at positive distance. Furthermore, their limit normal vectors are opposite by a trivial separation argument.

We will denote by $\widetilde{M}\subset\mathbb{R}^3$ the doubly periodic minimal surface obtained by lifting M. Since points in $\widetilde{M}$ homologous by $\mathcal{P}$ have the same normal vector, the stereographically projected Gauss map $g:\widetilde{M}\to\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ descends to M. As M has finite total curvature, g extends meromorphically to the conformal torus $\mathbb{M}$ obtained after attaching the ends to M, with values $0,\infty$ at the punctures. As $\mathcal{P}$ is nonhorizontal, the third coordinate function x_3 of $\widetilde{M}$ is multivalued on M but the *height differential* $dh=\frac{\partial x_3}{\partial z}dz$ defines a univalent meromorphic differential on M (here z is a holomorphic coordinate). Since M has finite total curvature and horizontal ends, dh extends to a holomorphic differential on $\mathbb{M}$. The next statement collects some elementary properties of the surfaces in $\mathcal{S}$. Given $v\in\mathcal{P}-\{\vec{0}\}$, $\widetilde{M}/v$

will stand for the singly periodic minimal surface obtained as the quotient of $\widetilde{M}$ by the translation of vector v; and $\Pi \subset \mathbb{R}^3$ will be a horizontal plane.

Proposition 1 *Given $M \in \mathcal{S}$ with Gauss map g, it holds:*

1. *$g : \mathbb{M} \to \overline{\mathbb{C}}$ has degree $2k$, total branching number $4k$, does not take vertical directions on M and is unbranched at the ends.*

2. *The period lattice $\mathcal{P}$ of $\widetilde{M}$ is generated by the period vectors at the ends, $H = \pm(0, \pi a, 0)$ with $a > 0$, and a nonhorizontal vector $T \in \mathbb{R}^3$, this last one being the period vector along a closed curve $\gamma_1 \subset M$ such that $[\gamma_1] \neq 0$ in the homology group $H_1(\mathbb{M}, \mathbb{Z})$.*

3. *Let $\mathcal{E}$ be the set of Scherk-type ends of $\widetilde{M}/H$. Then $(\widetilde{M}/H) \cup \mathcal{E}$ is conformally $\mathbb{C}^* = \mathbb{C} - \{0\}$, and the height differential writes as $dh = c\frac{dz}{z}$ in $\mathbb{C}^*$, with $c \in \mathbb{R}^* = \mathbb{R} - \{0\}$.*

4. *If Π/H is not asymptotic to an end in $\mathcal{E}$, then $(\widetilde{M} \cap \Pi)/H$ is transversal and connected. The period vector along $(\widetilde{M} \cap \Pi)/H$ either vanishes or equals $\pm H$.*

5. *We divide $\mathcal{E}$ in right ends and left ends, depending on whether the flux vector at the corresponding end (with the inward pointing conormal vector) is $(a, 0, 0)$ or $(-a, 0, 0)$, respectively. If Π/H is asymptotic to an end in $\mathcal{E}$, then $(\widetilde{M} \cap \Pi)/H$ consists of one properly embedded arc whose ends diverge to the same end in $\mathcal{E}$, or of two properly embedded arcs traveling from one left end to one right end in $\mathcal{E}$.*

6. *There exists an embedded closed curve $\gamma_2 \subset M$ such that $\{[\gamma_1], [\gamma_2]\}$ is basis of $H_1(\mathbb{M}, \mathbb{Z})$ and $P_{\gamma_2} = \vec{0}$. Up to orientation, γ_2 represents the unique nontrivial homology class in $H_1(\mathbb{M}, \mathbb{Z})$ with associated period zero and an embedded representative.*

7. *Let $[\gamma] \in H_1(M, \mathbb{Z})$ be a homology class with an embedded representative that generates the homology group of $(\widetilde{M}/H) \cup \mathcal{E}$. Then the second and third components $(F_\gamma)_2, (F_\gamma)_3$ of the flux of M along any representative $\gamma \in [\gamma]$ do not depend on $[\gamma]$ (up to orientation), and $(F_\gamma)_3 \neq 0$.*

Next we describe all possible limits of surfaces in $\mathcal{S}$ with the additional assumption of having uniform curvature bounds. For all $n \in \mathbb{N}$, let $M_n \in \mathcal{S}$ and $\widetilde{M}_n$ denotes its lift to $\mathbb{R}^3$. This is, $M_n = \widetilde{M}_n/\mathcal{P}_n$, where $\mathcal{P}_n = \mathrm{Span}\{H_n, T_n\}$ satisfies statement *2* of Proposition 1 for M_n. We will denote by K_Σ the Gaussian curvature function of any surface Σ.

Proposition 2 *Let $\{\widetilde{M}_n\}_n$ be a sequence in the above conditions. Suppose that for all n, $\widetilde{M}_n$ passes through the origin of $\mathbb{R}^3$ and $|K_{\widetilde{M}_n}(\vec{0})| = 1$ is a maximum value of $|K_{\widetilde{M}_n}|$. Then (after passing to a subsequence), $\widetilde{M}_n$ converges uniformly on compact subsets of $\mathbb{R}^3$ with multiplicity 1 to a properly embedded minimal surface $\widetilde{M}_\infty$ in one of the following cases:*

(i) *$\widetilde{M}_\infty$ is a vertical catenoid with flux $(0, 0, 2\pi)$. In this case, both $\{H_n\}_n, \{T_n\}_n$ are unbounded for any choice of T_n as above.*

(ii) *$\widetilde{M}_\infty$ is a vertical helicoid with period vector $(0, 0, 2\pi m)$ for some $m \in \mathbb{N}$. Now $\{H_n\}_n$ is unbounded and there exists a choice of T_n for which $\{T_n\}_n \to (0, 0, 2\pi m)$ as $n \to \infty$.*

(iii) *$\widetilde{M}_\infty$ is a Riemann minimal example with horizontal ends. Moreover, $\{H_n\}_n$ is unbounded and certain choice of $\{T_n\}_n$ converges to the period vector of $\widetilde{M}_\infty$.*

(iv) *$\widetilde{M}_\infty$ is a singly periodic Scherk minimal surface, two of whose ends are horizontal. Furthermore, any choice of $\{T_n\}_n$ is unbounded, $\{H_n\}_n$ converges to the period vector $H_\infty = (0, a, 0)$ of $\widetilde{M}_\infty$ (with $a > 0$), and $\widetilde{M}_\infty/H_\infty$ has genus zero.*

(v) *$\widetilde{M}_\infty$ is a doubly periodic Scherk minimal surface. In this case, $\{H_n\}_n$, $\{T_n\}_n$ converge respectively to period vectors H_∞, T_∞ of $\widetilde{M}_\infty$, and $\widetilde{M_\infty}/\{H_\infty, T_\infty\}$ has genus zero with at least two horizontal ends and exactly two nonhorizontal ends.*

(vi) *$\widetilde{M}_\infty$ is a doubly periodic minimal surface invariant by a rank 2 lattice $\mathcal{P}_\infty$, $M_\infty = \widetilde{M}_\infty/\mathcal{P}_\infty$ has genus one and $4k$ horizontal Scherk-type ends, and $\{H_n\}_n \to H_\infty$, $\{T_n\}_n \to T_\infty$, where H_∞, T_∞ are defined by Proposition 1 applied to M_∞.*

From now on, we will consider one more normalization on the surfaces in $\mathcal{S}$: Given $M \in \mathcal{S}$, Proposition 1 gives a nontrivial homology class in $H_1(\mathbb{M}, \mathbb{Z})$ with an embedded representative $\gamma_2 \subset M$ such that $P_{\gamma_2} = \vec{0}$ and $(F_{\gamma_2})_3 > 0$. In the sequel, we will always normalize our surfaces so that $(F_{\gamma_2})_3 = 2\pi$, which can be achieved after an homothety. Note that this normalization is independent of the homology class of γ_2 in $H_1(M, \mathbb{Z})$ (up to orientation), see item *7* of Proposition 1.
We label by $\widetilde{\mathcal{S}}$ the set of *marked surfaces* $(M, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma_2])$ where

1. M is a surface in $\mathcal{S}$ whose period lattice is generated by $H, T \in \mathbb{R}^3$, where $H = (0, a, 0)$, $T = (T_1, T_2, T_3)$ and $a, T_3 > 0$;

2. $\{p_1, \ldots, p_{2k}\} = g^{-1}(0)$, $\{q_1, \ldots, q_{2k}\} = g^{-1}(\infty)$ and the ordered lists $(p_1, q_1, \ldots, p_k, q_k)$, $(p_{k+1}, q_{k+1}, \ldots, p_{2k}, q_{2k})$ are the two families of "sided" ends of M, both ordered by increasing heights in the quotient;

3. $[\gamma_2] \in H_1(M, \mathbb{Z})$ is the homology class of an embedded closed curve $\gamma_2 \subset M$ satisfying $P_{\gamma_2} = \vec{0}$, $(F_{\gamma_2})_3 = 2\pi$. We additionally impose that γ_2 lifts to a curve contained in a fundamental domain of the doubly periodic lifting of M lying between two horizontal planes $\Pi, \Pi + T$.

We will identify in $\widetilde{\mathcal{S}}$ two marked surfaces that differ by a translation that preserves both orientation, the "sided" ordering of their lists of ends and the associated homology classes. The same geometric surface in $\mathcal{S}$ can be viewed as a finite number of different marked surfaces in $\widetilde{\mathcal{S}}$. We will simply denote as $M \in \widetilde{\mathcal{S}}$ the marked surfaces unless it leads to confusion.

Consider $M \in \widetilde{\mathcal{S}}$ with Gauss map g and height differential dh. An elementary calculation gives the periods P_{p_j}, P_{q_j} and fluxes F_{p_j}, F_{q_j} at the ends of M as follows:

$$P_{p_j} + i\, F_{p_j} = \pi \mathrm{Res}_{p_j} \frac{dh}{g}\,(i, -1, 0), \qquad P_{q_j} + i\, F_{q_j} = -\pi \mathrm{Res}_{q_j}(g\, dh)(i, 1, 0),$$

where Res_A denotes the residue of the corresponding meromorphic differential at a point A. The fact that P_{p_j}, P_{q_j} point to the x_2-axis translates into $\mathrm{Res}_{p_j} \frac{dh}{g}$, $\mathrm{Res}_{q_j}(g\, dh) \in \mathbb{R}$. By definition of the ordering of the ends of M as a marked surface, we have that

$$\mathrm{Res}_{p_j}\frac{dh}{g} = -\mathrm{Res}_{q_j}(g\,dh) = \left\{\begin{array}{rl} a & (1 \le j \le k) \\ -a & (k+1 \le j \le 2k) \end{array}\right. \tag{1}$$

for certain $a \in \mathbb{R}^*$ (the case $a > 0$ corresponds to $p_1, \dots, p_k, q_1, \dots, q_k$ being right ends of M). Recall that $P_{\gamma_2} = \vec{0}$ and $(F_{\gamma_2})_3 = 2\pi$. Thus,

$$\overline{\int_{\gamma_2}\frac{dh}{g}} = \int_{\gamma_2} g\,dh \qquad \text{and} \qquad \int_{\gamma_2} dh = 2\pi i. \tag{2}$$

3 KMR examples.

We dedicate this Section to introduce briefly the 3-parameter family of KMR examples $\mathcal{K} \subset \mathcal{S}$ to which the uniqueness Theorem 1 applies. A more detailed study of $\mathcal{K}$ can be found in [14]. The analysis of the space of KMR examples $\mathcal{K}$ with $4k$ parallel ends can be obviously reduced to the case $k = 1$ by taking k-sheeted coverings. So assume from now on that $k = 1$; this is, $\mathcal{K} = \{M_{\theta,\alpha,\beta}\}_{\theta,\alpha,\beta}$. Each $M_{\theta,\alpha,\beta}$ is determined by the 4 branch values of its Gauss map, which consist of two pairs of antipodal points $D, D', D'' = -D, D''' = -D'$ in the sphere $\mathbb{S}^2$. Since the Gauss map of any surface in $\mathcal{S}$ is unbranched at the ends, D, D', D'', D''' must be different from the North and South Poles. Let us introduce some notation in order to understand who are θ, α, β:

1. e denotes the equator in $\mathbb{S}^2$ that contains D, D', D'', D''';

2. $\alpha \in [0, \frac{\pi}{2}]$ is the angle between e and the equator $\mathbb{S}^2 \cap \{x_1 = 0\}$;

3. $P \in \mathbb{S}^2$ is the point that bisects the angle $\theta \in (0, \frac{\pi}{2})$ between D and D', and it also corresponds to the image of the North Pole through the composition of a rotation by angle $\beta \in [0, \frac{\pi}{2}]$, $(\alpha, \beta) \neq (0, \theta)$, around the x_1-axis with a rotation by angle α around the x_2-axis, see Figure 1 left.

We will call a *spherical configuration* to any set $\{D, D', D'', D'''\}$ as above. The spherical configuration associated to $(\theta, 0, 0)$ projects stereographically into the four

Figure 1: Left: Spherical configuration of $M_{\theta,\alpha,\beta}$. Right: Behavior of the pole $A(\alpha,\beta)$ of g in the dark shaded rectangle $\mathcal{R}$ (here, $0<\beta_1<\theta<\beta_2<\frac{\pi}{2}$, $\alpha\in(0,\frac{\pi}{2})$ and $\beta\in[0,\frac{\pi}{2}]$). The remaining ends of $M_{\theta,\alpha,\beta}$ move in the light shaded rectangles as α,β vary.

roots of the polynomial $(z^2+\lambda^2)(z^2+\frac{1}{\lambda^2})$ where $\lambda=\lambda(\theta)=\cot\frac{\theta}{2}$. Therefore, the underlying conformal compactification of the potential surface $M_{\theta,0,0}$ is the rectangular torus $\Sigma_\theta=\left\{(z,w)\in\overline{\mathbb{C}}^2 \mid w^2=(z^2+\lambda^2)(z^2+\frac{1}{\lambda^2})\right\}$, and its extended Gauss map is the z-projection $(z,w)\in\Sigma_\theta\mapsto z\in\overline{\mathbb{C}}$ on Σ_θ. Note that the spherical configuration for angles (θ,α,β) differs from the one associated to $(\theta,0,0)$ in a Möbius transformation φ. Thus the compactification of any $M_{\theta,\alpha,\beta}$, which is the branched covering of $\mathbb{S}^2$ through the z-map, is Σ_θ as well. Furthermore, the composition of the Gauss map of $M_{\theta,0,0}$ with φ gives the Gauss map g of the potential example $M_{\theta,\alpha,\beta}$, and its ends are $\{A,A',A'',A'''\}=g^{-1}(\{0,\infty\})$. As the height differential dh of $M_{\theta,\alpha,\beta}$ is a holomorphic 1-form on Σ_θ, we have $dh=\mu\frac{dz}{w}$ for certain $\mu=\mu(\theta)\in\mathbb{C}^*$. We choose $\mu\in\mathbb{R}^*$.

Σ_θ can be represented as a quotient of the ξ-plane $\mathbb{C}$ by two orthogonal translations. In this ξ-plane model of Σ_θ we may see the ends A,A',A'',A''' as functions of $(\alpha,\beta)\in[0,\frac{\pi}{2}]^2-\{(0,\theta)\}$. $A(\alpha,\beta)$ moves on the dark shaded rectangle $\mathcal{R}$ in Figure 1 right. The behavior of the remaining three ends of $M_{\theta,\alpha,\beta}$ on the ξ-plane model can be deduced from $A(\alpha,\beta)$ by using the isometry group $\mathrm{Iso}(\theta,\alpha,\beta)$ of the induced metric $ds^2=\frac{1}{4}\left(|g|+\frac{1}{|g|}\right)^2|dh|^2$, which we now investigate.

First note that the identity in $\mathbb{S}^2$ lifts via g to two different isometries of ds^2, namely the identity in Σ_θ and the deck transformation $\mathcal{D}(z,w)=(z,-w)$, both restricted to $\Sigma_\theta - g^{-1}(\{0,\infty\})$. $\mathcal{D}$ corresponds in the ξ-plane to the $180°$-rotation about any of the branch points of the z-projection. The antipodal map $\aleph$ on $\mathbb{S}^2$ also leaves invariant both the spherical configuration of $M_{\theta,\alpha,\beta}$ and the set $\{(0,0,\pm 1)\}$, and the equality $\left[w\left(\frac{-1}{\overline{z}}\right)\right]^2 = \frac{\overline{[w(z)]^2}}{\overline{z}^4}$ for any $(z,w)\in\Sigma_\theta$ implies that $\aleph$ lifts through g to two isometries of ds^2, which we call $\mathcal{E}$ and $\mathcal{F} = \mathcal{D}\circ\mathcal{E}$. Both $\mathcal{E},\mathcal{F}$ are antiholomorphic involutions of Σ_θ without fixed points. The remaining ends of $M_{\theta,\beta,\alpha}$ in terms of $A = A(\alpha,\beta)$ are (up to relabeling)

$$A'' = \mathcal{D}(A),\ A''' = \mathcal{E}(A),\ A' = \mathcal{D}(A'''). \tag{3}$$

The period P_A and flux F_A of $M_{\theta,\alpha,\beta}$ at the end A with $g(A)=\infty$ are given by

$$P_A = \pi\mu\big(i\,E(\theta,\alpha,\beta),0\big), \qquad F_A = \pi\mu\big(E(\theta,\alpha,\beta),0\big), \tag{4}$$

where we have used the identification $\mathbb{R}^3 \equiv \mathbb{C}\times\mathbb{R}$ by $(a,b,c)\equiv(a+ib,c)$, and $E(\theta,\alpha,\beta) = [\cos^2\alpha + \csc^2\theta(\sin\alpha\cos\beta - i\sin\beta)^2]^{-1/2}$ (we have chosen a branch of w for computing (4), which only affects the result up to sign). The periods and fluxes at A', A'', A''' can be easily obtained using (3) and (4).

Concerning the period problem in homology, let γ_1,γ_2 be the simple closed curves in Σ_θ obtained respectively as quotients of the horizontal and vertical lines in the ξ-plane passing through D, D''' and through the right vertical edge of $\partial\mathcal{R}$ (see Figure 1 right). Clearly $\{[\gamma_1],[\gamma_2]\}$ is a basis of $H_1(\Sigma_\theta,\mathbb{Z})$. Straightforward computations give us

$$F_{\gamma_1} = -F_A \qquad \text{and} \qquad P_{\gamma_2} = \vec{0}.$$

Since $P_{\gamma_2} = \vec{0}$, we can take γ_2 as the embedded closed curve appearing in item *6* of Proposition 1. As dh is holomorphic and nontrivial on Σ_θ, $P_{\gamma_2} = \vec{0}$ also implies that $\int_{\gamma_2} dh \in i\mathbb{R}^*$, from where P_{γ_1} has nonvanishing third component. In particular, P_A and P_{γ_1} are linearly independent. Therefore, $M_{\theta,\alpha,\beta}$ is a complete immersed doubly periodic minimal torus with four horizontal embedded Scherk-type ends and period lattice generated by P_A, P_{γ_1}. Moreover, the maximum principle allows us to prove that $M_{\theta,\alpha,\beta}$ is in fact embedded, since for fixed θ the heights of the ends of $M_{\theta,\alpha,\beta}$ depend continuously on (α,β) in the connected set $[0,\frac{\pi}{2}]^2 - \{(0,\theta)\}$ and $M_{\theta,0,0}$ is embedded (it is the toroidal halfplane layer defined by Karcher in [5], that decomposes in 16 congruent disjoint pieces, each one being the conjugate surface of certain Jenkins-Serrin graph).

Remark 1 *In order to see $M_{\theta,\alpha,\beta}$ in $\mathcal{S}$, we must possibly rotate it by a suitable angle around the x_3-axis, sinceP_A does not necessarily point to the x_2-axis, and rescale to have $\int_{\gamma_2} dh = 2\pi i$ (this last equation allows us to obtain the precise value of μ).*

We have defined the 3-parametric family of examples $\mathcal{K} = \{M_{\theta,\alpha,\beta}\} \subset \mathcal{S}$, with (θ, α, β) varying in $\mathcal{I} = \left\{(\theta, \alpha, \beta) \in (0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}]^2 \mid (\alpha, \beta) \neq (0, \theta)\right\}$. Clearly this definition can be extended to larger ranges in (θ, α, β), but such an extension only produces symmetric images of these surfaces with respect to certain planes orthogonal to the x_1, x_2 or x_3-axes. Nevertheless, some of these geometrically equivalent surfaces are considered as distinct points in the space $\widetilde{\mathcal{S}}$ defined in Section 2.

The next Lemma states that $\mathcal{K}$ is *self-conjugate*, in the sense that the conjugate surface of a KMR example is another KMR example.

Lemma 1 *Given $(\theta, \alpha, \beta) \in \mathcal{I}$, the conjugate surface $M^*_{\theta,\alpha,\beta}$ of $M_{\theta,\alpha,\beta}$ coincides with $M_{\frac{\pi}{2}-\theta,\alpha,\frac{\pi}{2}-\beta}$ up to a symmetry in a plane orthogonal to the x_2-axis (after normalizations).*

Looking at its spherical configuration, we realize that $M_{\theta,\frac{\pi}{2},\beta}$ does not depend on $\beta \in [0, \frac{\pi}{2}]$. Thus $M_{\theta,\alpha,\beta}$ is self-conjugate when $(\theta, \alpha, \beta) \in \left(\{\frac{\pi}{4}\} \times (0, \frac{\pi}{2}] \times \{\frac{\pi}{4}\}\right) \cup \left(\{\frac{\pi}{4}\} \times \{\frac{\pi}{2}\} \times [0, \frac{\pi}{2}]\right)$.

As the branch values of the Gauss map of any $M_{\theta,\alpha,\beta}$ lie on a spherical equator, a result by Montiel and Ros [12] insures that the space of bounded Jacobi functions on $M_{\theta,\alpha,\beta}$ is 3-dimensional (they reduce to the linear functions of the Gauss map), a condition usually referred in literature as the *nondegeneracy* of $M_{\theta,\alpha,\beta}$. This nondegeneracy can be interpreted by means of an Implicit Function Theorem argument to obtain that around $M_{\theta,\alpha,\beta}$, the space $\mathcal{S}$ is a 3-dimensional real analytic manifold (Hauswirth and Traizet [2]); in particular, the only elements in $\mathcal{S}$ around a KMR example are themselves KMR. This local uniqueness result will be extended in the large by Theorem 1 in this paper.

Figure 2: Left: The toroidal halfplane layer $M_{\frac{\pi}{4},0,0}$. Right: The surface $M_{\frac{\pi}{4},0,\frac{\pi}{8}}$.

We finish this section by summarizing some additional properties of the KMR examples, that can be checked using their Weierstrass representation. For details, see [14].

Proposition 3

1. *For any $\theta \in (0, \frac{\pi}{2})$, $M_{\theta,0,0}$ admits 3 reflection symmetries S_1, S_2, S_3 in orthogonal planes and contains a straight line parallel to the x_1-axis, that induces a $180°$-rotation symmetry R_D. The isometry group* $\mathrm{Iso}(\theta, 0, 0)$ *of $M_{\theta,0,0}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, with generators S_1, S_2, S_3, R_D (see Figure 2 left).*

2. *For any $(\theta, \alpha) \in (0, \frac{\pi}{2})^2$, $M_{\theta,\alpha,0}$ is invariant by a reflection S_2 in a plane orthogonal to the x_2-axis and by a $180°$-rotation R_2 around a line parallel to the x_2-axis that cuts the surface orthogonally (Figure 3 left).* $\mathrm{Iso}(\theta, \alpha, 0) =$ *Span*$\{S_2, R_2, \mathcal{D}\} \cong (\mathbb{Z}/2\mathbb{Z})^3$*. Furthermore,* $\mathrm{Iso}(\theta, \frac{\pi}{2}, 0) = \mathrm{Iso}(\theta, 0, 0)$ *(Figure 3 right).*

3. *For any $(\theta, \beta) \in (0, \frac{\pi}{2})^2 - \{(\theta, \theta)\}$, $M_{\theta,0,\beta}$ is invariant by a reflection S_1 in a plane orthogonal to the x_1-axis, by a $180°$-rotation symmetry R_D around a straight line parallel to the x_1-axis contained in the surface, and by a $180°$-rotation R_1 around another line parallel to the x_1-axis that cuts the surface orthogonally.* $\mathrm{Iso}(\theta, 0, \beta) =$ *Span*$\{S_1, R_D, R_1\} \cong (\mathbb{Z}/2\mathbb{Z})^3$ *(Figure 2 right). Furthermore,* $\mathrm{Iso}(\theta, 0, \frac{\pi}{2}) = \mathrm{Iso}(\theta, 0, 0)$.

4. *For any $(\theta, \alpha) \in (0, \frac{\pi}{2})^2$, $M_{\theta,\alpha,\frac{\pi}{2}}$ is invariant by a $180°$-rotation S_3 around a*

Figure 3: Left: The surface $M_{\frac{\pi}{4},\frac{\pi}{4},0}$. Right: The surface $M_{\frac{\pi}{4},\frac{\pi}{2},0}$.

straight line parallel to the x_2-axis contained in the surface, and by the composition R_3 of a reflexion symmetry across a plane orthogonal to the x_2-axis with a translation by half a horizontal period. In this case, $\mathrm{Iso}(\theta, \alpha, \frac{\pi}{2}) =$ *Span*$\{S_3, R_3, \mathcal{D}\} \cong (\mathbb{Z}/2\mathbb{Z})^3$.

5. *For any* $(\theta, \alpha, \beta) \in (0, \frac{\pi}{2})^3$, $\mathrm{Iso}(\theta, \alpha, \beta)) =$ *Span*$\{\mathcal{D}, \mathcal{E}\} \cong (\mathbb{Z}/2\mathbb{Z})^2$.

6. *When* $(\theta, \alpha, \beta) \to (0, 0, 0)$, $M_{\theta,\alpha,\beta}$ *converges smoothly to two vertical catenoids, both with flux* $(0, 0, 2\pi)$.

7. *Let* $\theta_0 \in (0, \frac{\pi}{2})$. *When* $(\theta, \alpha, \beta) \to (\theta_0, 0, \theta_0)$, $M_{\theta,\alpha,\beta}$ *converges to a Riemann minimal example with two horizontal ends, vertical part of its flux* 2π *and branch values of its Gauss map at* $0, \infty, i \tan\theta_0, -i \cot\theta_0 \in \overline{\mathbb{C}}$.

8. *When* $(\theta, \alpha, \beta) \to (\frac{\pi}{2}, 0, \frac{\pi}{2})$, $M_{\theta,\alpha,\beta}$ *converges (after blowing up) to two vertical helicoids.*

9. *Let* $(\alpha_0, \beta_0) \in [\theta, \frac{\pi}{2}]^2 - \{(0, 0)\}$. *When* $(\theta, \alpha, \beta) \to (0, \alpha_0, \beta_0)$, $M_{\theta,\alpha,0}$ *converges to two singly periodic Scherk minimal surfaces, each one with two horizontal ends and two ends forming angle* $\arccos(\cos\alpha_0 \cos\beta_0)$ *with the horizontal.*

10. *Let* $(\alpha_0, \beta_0) \in [\theta, \frac{\pi}{2}]^2 - \{(0, \frac{\pi}{2})\}$. *When* $(\theta, \alpha, \beta) \to (\frac{\pi}{2}, \alpha_0, \beta_0)$, $M_{\theta,\alpha,0}$ *converges (after blowing up) to two doubly periodic Scherk minimal surfaces, each one with two horizontal ends and two ends forming angle* $\arccos(\cos\alpha_0 \sin\beta_0)$ *with the horizontal.*

4 The moduli space $\mathcal{W}$ of Weierstrass representations.

We will call $\mathcal{W}$ to the space of lists $(\mathbb{M}, g, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma])$, where $g : \mathbb{M} \to \overline{\mathbb{C}}$ is a meromorphic degree $2k$ function defined on a torus $\mathbb{M}$, which is unbranched at its zeros $\{p_1, \ldots, p_{2k}\}$ and poles $\{q_1, \ldots, q_{2k}\}$, and $[\gamma]$ is a homology class in $g^{-1}(\mathbb{C}^*)$ with $[\gamma] \neq 0$ in $H_1(\mathbb{M}, \mathbb{Z})$. Note that there is an infinite subset of $\mathcal{W}$ associated to the same map g, which is discrete with the topology defined in [13]. We will simply denote by g the elements of $\mathcal{W}$, which will be referred to as *marked meromorphic maps*.

A local chart for $\mathcal{W}$ around any $g \in \mathcal{W}$ can be obtained by the list $(\sigma_1(g), \ldots, \sigma_{4k}(g))$, where $\sigma_i(g)$ is the value of the symmetric elementary polynomial of degree i on the unordered list of $4k$ (not necessarily distinct) branch values of g, $1 \leq i \leq 4k$. These symmetric elementary polynomials can be considered as globally defined holomorphic functions $\sigma_i : \mathcal{W} \to \mathbb{C}$, $1 \leq i \leq 4k$. Also, the map $(\sigma_1, \ldots, \sigma_{4k}) : \mathcal{W} \to \mathbb{C}^{4k}$ is a local diffeomorphism, hence $\mathcal{W}$ can be seen as an open submanifold of $\mathbb{C}^{4k}$.

Given a marked meromorphic map $g = (\mathbb{M}, g, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma]) \in \mathcal{W}$, there exists a unique holomorphic 1-form $\phi = \phi(g)$ on $\mathbb{M}$ such that $\int_\gamma \phi = 2\pi i$. The pair (g, ϕ) must be seen as the Weierstrass data of a potential surface in the setting of Theorem 1. We will say that $g \in \mathcal{W}$ *closes periods* when there exists $a \in \mathbb{R}^*$ such that (1) and the first equation in (2) hold with $dh = \phi$ and $\gamma_2 = \gamma$.

Lemma 2 *If $g \in \mathcal{W}$ closes periods, then (g, ϕ) is the Weierstrass data of a properly immersed minimal surface $M \subset \mathbb{T} \times \mathbb{R}$ for a certain flat torus $\mathbb{T}$, with total curvature $8k\pi$ and $4k$ horizontal Scherk-type ends. Furthermore, the fluxes at the ends $p_1, q_1, \ldots, p_k, q_k$ are equal to $(\pi a, 0, 0)$ and opposite to the fluxes at $p_{k+1}, q_{k+1}, \ldots, p_{2k}, q_{2k}$ (here $a \in \mathbb{R}^*$ comes from equation (1)).*

As the sum of the residues of a meromorphic differential on a compact Riemann surface equals zero, then it suffices to impose (1) for $1 \leq j \leq 2k - 1$. Provided that the first equation in (2) also holds for g with $\gamma_2 = \gamma$, the horizontal component of the flux of the corresponding immersed minimal surface M in Lemma 2 along γ is given by

$$F(\gamma) = i\int_\gamma g\phi \in \mathbb{C} \equiv \mathbb{R}^2.$$

Definition 1 We define the *ligature map* $L : \mathcal{W} \to \mathbb{C}^{4k}$ as the holomorphic map that associates to each marked meromorphic map $g \in \mathcal{W}$ the $4k$-tuple

$$L(g) = \left(\operatorname{Res}_{p_1}\frac{\phi}{g}, \dots, \operatorname{Res}_{p_{2k-1}}\frac{\phi}{g}, \operatorname{Res}_{q_1}(g\phi), \dots, \operatorname{Res}_{q_{2k-1}}(g\phi), \int_\gamma \frac{\phi}{g}, \int_\gamma g\phi\right).$$

We consider the subset of marked meromorphic maps that close periods $\mathcal{M} = \cup_{\substack{a\in\mathbb{R}^* \\ b\in\mathbb{C}}}\mathcal{M}(a,b)$, where for any $a \in \mathbb{R}^*$ and $b \in \mathbb{C}$, $\mathcal{M}(a,b) = \{g \in \mathcal{W} \mid L(g) = L_{(a,b)}\}$ and

$$L_{(a,b)} = \Big(\underbrace{a, \dots, a}_{1\le j\le k}, \underbrace{-a, \dots, -a}_{k+1\le j\le 2k-1}, \underbrace{-a, \dots, -a}_{2k\le j\le 3k-1}, \underbrace{a, \dots, a}_{3k\le j\le 4k-2}, b, \overline{b}\Big) \in \mathbb{C}^{4k}.$$

The holomorphicity of L leads easily to the following result.

Proposition 4 *$\mathcal{M}(a,b)$ is an analytic subvariety*[2] *of $\mathcal{W}$.*

Moreover, if we consider $\mathcal{M}$ with the restricted topology from the one of $\mathcal{W}$, then the canonical injection $J : \widetilde{\mathcal{S}} \to \mathcal{M}$ defined below is an embedding,

$$(M, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma_2]) \mapsto J(M) = (g^{-1}(\overline{\mathbb{C}}), g, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma_2]),$$

with g being the Gauss map of M. Thus we can seen $\widetilde{\mathcal{S}}$ as a subset of $\mathcal{M}$, which can be shown to be open and closed in $\mathcal{M}$. As a simultaneously open and closed subset of an analytic subvariety is also an analytic subvariety, we conclude that $\widetilde{\mathcal{S}} \cap \mathcal{M}(a,b)$ is an analytic subvariety of $\mathcal{W}$.

The following lemma exhibits a property of $\mathcal{W}$ inherited from $\mathbb{C}$ through the symmetric polynomials $\sigma_i(g)$ defined above.

Lemma 3 *The only compact analytic subvarieties of $\mathcal{W}$ are finite subsets.*

Definition 2 The value of the ligature map L at a marked surface $M \in \widetilde{\mathcal{S}}$ is determined by two numbers $a \in \mathbb{R}^*, b \in \mathbb{C}$ so that $\operatorname{Res}_{p_1}\frac{dh}{g} = a$ and $F_{\gamma_2} = (i\overline{b}, 2\pi)$. We define the *classifying map* $C : \widetilde{\mathcal{S}} \to \mathbb{R}^* \times \mathbb{C}$ by $C(M) = (a,b)$.

[2]Recall that a subset V of a complex manifold N is said to be an *analytic subvariety* if for any $p \in N$ there exists a neighborhood U of p in N and a finite number of holomorphic functions $f_1, \dots, f_r$ on U such that $U \cap V = \{q \in U \mid f_i(q) = 0,\ 1 \le i \le r\}$.

Remark 2 *Let $M \in \mathcal{S}$ be a geometric surface, seen as two marked surfaces $M_1, M_2 \in \widetilde{\mathcal{S}}$ with associated homology classes $[\gamma_2(M_1)], [\gamma_2(M_2)] \in H_1(M, \mathbb{Z})$ such that $[\gamma_2(M_1)] = [\gamma_2(M_2)]$ in $H_1(\mathbb{M}, \mathbb{Z})$ (here $\mathbb{M}$ is the compactification of M). Then, $\gamma_2(M_1) \cup \gamma_2(M_2)$ bounds an even number of ends whose periods add up to zero, and the components of C at M_1, M_2 satisfy $a(M_1) = \pm a(M_2) \in \mathbb{R}^*$ and $b(M_1) = b(M_2) + t\pi a(M_1)$ with $t \in \mathbb{Z}$ even.*

5 Properness.

The following result is a crucial curvature estimate in terms of the classifying map C.

Proposition 5 *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence of marked surfaces. Suppose that $C(M_n) = (a_n, b_n) \in \mathbb{R}^* \times \mathbb{C}$ satisfies*

(i) $\{a_n\}_n$ is bounded away from zero.

(ii) $\{|b_n|\}_n$ is bounded by above.

Then, the sequence of Gaussian curvatures $\{K_{M_n}\}_n$ is uniformly bounded.

Sketch of the proof. By contradiction, assume that $\lambda_n := \max_{M_n} \sqrt{|K_{M_n}|} \to \infty$ as $n \to \infty$. Let $\Sigma_n = \lambda_n M_n \subset \mathbb{R}^3/\lambda_n \mathcal{P}_n$, where $\mathcal{P}_n = \text{Span}\{H_n, T_n\}$ is the rank 2 lattice associated to M_n and H_n is the period vector at the ends of M_n (up to sign). Let us also call $\widetilde{\Sigma}_n$ to the lifting of Σ_n to $\mathbb{R}^3$. After translation of $\widetilde{\Sigma}_n$ to have maximum absolute Gauss curvature one at the origin, a subsequence of $\{\widetilde{\Sigma}_n\}_n$ converges smoothly to a properly embedded minimal surface $\mathcal{H}_1 \subset \mathbb{R}^3$, which must lie in one of the six possibilities in Proposition 2. Since a_n is bounded away from zero and $\lambda_n \to \infty$, then both the period vector $\lambda_n H_n$ at the ends of Σ_n and the vertical part of the flux of Σ_n along a compact horizontal section, which is $2\pi\lambda_n$, diverge. Then $\mathcal{H}_1$ must be a vertical helicoid with period vector $T = \lim\, \lambda_n T_n = (0, 0, 2\pi m)$ for certain $m \in \mathbb{N}$. We can also produce a second helicoid $\mathcal{H}_2$ as a limit of suitable translations of the $\widetilde{\Sigma}_n$, and we additionally prove that outside these two partial limits, no other interesting geometry can appear as limits of the Σ_n. Since both $\mathcal{H}_1, \mathcal{H}_2$ are limits of translations of the $\widetilde{\Sigma}_n$, the period vector of $\mathcal{H}_2$ is again $T = \lim\ \lambda_n T_n$.

Next we construct an embedded closed curve $\Gamma_n \subset M_n$ joining the two forming helicoids. Let $\pi_n : \mathbb{R}^3/\lambda_n\mathcal{P}_n \to \{x_3 = 0\}/\lambda_n\mathcal{P}_n$ be the linear projection in the direction of T_n and consider two disjoint round disks $\mathcal{D}_1(n), \mathcal{D}_2(n) \subset \{x_3 = 0\}/\lambda_n\mathcal{P}_n \subset \mathbb{R}^3/\lambda_n\mathcal{P}_n$, with common radius r_n such that the annular component $\mathcal{H}_i(n) = \Sigma_n \cap \pi_n^{-1}(\mathcal{D}_i(n))$ is arbitrarily close to a translated copy of the forming helicoid $\mathcal{H}_i/T$ minus neighborhoods of its ends, $i = 1, 2$. After passing to a subsequence, we can also choose r_n so that

1. $r_n \to \infty$ and $\frac{r_n}{\lambda_n} \to 0$ as $n \to \infty$.

2. The normal direction to Σ_n along the helix-type curves in the boundary of $\mathcal{H}_i(n)$ makes an angle less than $\frac{1}{n}$ with the vertical, $i = 1, 2$.

With these choices the extended Gauss map applies $\overline{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ in the spherical disks centered at the North and South Poles of $\mathbb{S}^2$ with radius $\frac{1}{n}$. Let $\mathcal{F}_1(n), \mathcal{F}_2(n)$ be the two components of $\overline{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$. Each $\mathcal{F}_i(n)$ is a closed annuli with k left and k right ends of Σ_n. Coming back to the original scale, we consider a embedded closed curve $\Gamma_n \subset M_n$ formed by four consecutive arcs $L_1(n)^{-1} * \beta_1(n) * L_2(n) * \beta_2(n)$ as follows: $L_1(n), L_2(n)$ are liftings of the distance minimizing horizontal segment $L(n)$ from $\frac{1}{\lambda_n}\partial\mathcal{D}_1(n)$ to $\frac{1}{\lambda_n}\partial\mathcal{D}_2(n)$, lying in consecutive sheets by the covering f_n obtained by projecting in the direction of T_n, $f_n : \frac{1}{\lambda_n}[\mathcal{F}_1(n) \cup \mathcal{F}_2(n)] \to \frac{\{x_3=0\}\cup\{\infty\}}{\mathcal{P}_n} - \frac{1}{\lambda_n}[\mathcal{D}_1(n) \cup \mathcal{D}_2(n)]$; and each $\beta_i(n) \subset \frac{1}{\lambda_n}\mathcal{H}_i(n)$ joins $L_1(n)$ with $L_2(n)$, see Figure 4.

Let g_n be the complex Gauss map of M_n. Since Γ_n is embedded, not trivial in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$ and has period zero, Proposition 1 implies that Γ_n can be oriented so that $[\Gamma_n] = [\gamma_2(n)]$ in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$, where $[\gamma_2(n)]$ is the last component of the marked surface $M_n \in \widetilde{\mathcal{S}}$ (recall that $C(M_n) = (a_n, b_n)$, where $F_{\gamma_2(n)} = (F(\gamma_2(n)), 2\pi) = (i\overline{b_n}, 2\pi)$). By Remark 2, there exits an even $t(n) \in \mathbb{Z}$ so that

$$F(\Gamma_n) = i\overline{b_n} + t(n)\pi a_n. \tag{5}$$

Figure 4: Front and top views of two helicoids forming with the direction of $L(n)$ tending to the direction of the x_2-axis.

Since both Γ_n, $\gamma_2(n)$ can be chosen in the same fundamental domain of the doubly periodic lifting $\widetilde{M}_n$ of M_n lying between two horizontal planes $\Pi, \Pi+T_n$, the embeddedness of both curves insures that $\{t(n)\}_n$ is bounded. Passing to a subsequence, we can suppose that $t = t(n)$ does not depend on n. By item *7* of Proposition 1, $(F_{\Gamma_n})_3 = (F_{\gamma_2(n)})_3 = 2\pi$. This property implies that the lengths of $L_1(n), L_2(n)$ diverge to ∞ as $n \to \infty$, so $F(\Gamma_n) \to \infty$. As b_n is bounded, then equation (5) says that both $\frac{F(\Gamma_n)}{|F(\Gamma_n)|}$, $\frac{t\pi a_n}{|F(\Gamma_n)|}$ converge to the same limit $e^{i\theta}$, $\theta \in [0, 2\pi)$, from where $t \neq 0$ and $a_n \to \infty$. Since $a_n \in \mathbb{R}^*$, we also have $\theta = 0$ or π. In particular, the direction of the segment $L(n)$ tends to the direction of the x_2-axis.

Now consider another embedded closed curve $\Gamma_n^* \subset M_n$ constructed similarly as Γ_n, i.e. $\Gamma_n^* = L_1^*(n)^{-1} * \beta_1^*(n) * L_2^*(n) * \beta_2^*(n)$ where $L_1^*(n), L_2^*(n)$ are liftings in consecutive sheets of the length minimizing horizontal segment $L^*(n)$ from $\frac{1}{\lambda_n}\partial\mathcal{D}_2(n)$ to $\frac{1}{\lambda_n}\partial\mathcal{D}_1(n) + H_n$, and each $\beta_i^*(n) \subset \frac{1}{\lambda_n}\mathcal{H}_i(n)$ joins $L_1^*(n)$ with $L_2^*(n)$, $i = 1, 2$. We orient Γ_n^* in such a way that Γ_n, Γ_n^* share their orientations along $\beta_1(n) \cap \beta_1^*(n)$. Thus $[\Gamma_n] = -[\Gamma_n^*]$ in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$. As above, we have that after passing to a subsequence, $F(\Gamma_n^*) = -i\overline{b_n} + t^*\pi a_n$ for certain nonzero even integer t^*.

Reasoning geometrically with fluxes, it is not difficult to check that $\lim \frac{F(\Gamma_n)+F(\Gamma_n^*)}{\pi a_n} = \pm 2$ and that t, t^* have the same sign. But then

$$\pm 2 = \lim \frac{F(\Gamma_n) + F(\Gamma_n^*)}{\pi a_n} = \lim \frac{-2i\overline{b_n} + (t + t^*)\pi a_n}{\pi a_n} = t + t^*,$$

which contradicts that both t, t^* are nonzero even integers with the same sign.

□

Remark 3 *The KMR examples $M_{\theta,0,\frac{\pi}{2}}$ with $\theta \nearrow \frac{\pi}{2}$ contain two helicoids forming with axes joined horizontally by a line parallel to the period vector at the ends, so their curvatures to blow-up. This says us that we cannot remove the hypothesis (ii) in Proposition 5.*

6 Uniqueness around the singly periodic Scherk surfaces.

In this Section we will prove that if $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ degenerates in a singly periodic Scherk minimal surface (case *(iv)* of Proposition 2), then $L(M_n)$ tends to a tuple in $\mathbb{C}^{4k}$. In particular, the classifying map $C : \widetilde{\mathcal{S}} \to \mathbb{R}^* \times \mathbb{C}$ cannot be proper. In order to overcome this lack of properness, we will prove that only KMR examples can occur in $\widetilde{\mathcal{S}}$ nearby the singly periodic Scherk limit. This will be essential when proving that the restriction of C to $\mathcal{S} - \mathcal{K}$ is proper (Theorem 5).

A deep analysis of the possible limits given by Proposition 2 for a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ with certain constraints of the values of $C(M_n)$ gives the following result.

Proposition 6 *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence with $\{C(M_n)\}_n \to (a,b) \in \mathbb{R}^* \times \mathbb{C}$, $\{H_n\}_n \to H_\infty = (0,\pi a,0)$ and $\{T_n\}_n \to \infty$ (for any choice of T_n as in Proposition 2). Then, for n large, the geometric surface M_n is close to $2k$ translated images of arbitrarily large compact regions of a singly periodic Scherk minimal surface of genus zero with two horizontal ends, together with $2k$ annular regions $C_n(1),\dots,C_n(2k)$ each one containing two distinct simple branch points of the Gauss map of M_n. Moreover, there exists a nonhorizontal plane $\Pi \subset \mathbb{R}^3$ such that any annulus $C_n(j)$ is a graph over the intersection of Π/H_n with a certain horizontal slab, $j = 1,\dots,2k$.*

Let S_ρ be the singly periodic Scherk minimal surface that appears as a limit in Proposition 6, $\rho \in (0,1]$. Since the period vector H_∞ of S_ρ points to the x_2-axis, the values of its Gauss map at the ends are $0, \infty, \rho, -\frac{1}{\rho}$, so we can parametrize S_ρ

by the Weierstrass data

$$g(z) = z, \quad dh = c\frac{dz}{(z-\rho)(\rho z+1)}, \quad z \in \overline{\mathbb{C}} - \Big\{0, \infty, \rho, -\frac{1}{\rho}\Big\}, \tag{6}$$

where $c \in \mathbb{R}^*$ is determined by the equation $2\pi i = \int_\Gamma \phi$, Γ being the horizontal level section of S_ρ at large positive height.

Next we give a local chart for $\mathcal{W}$ around S_ρ. Let $D(*, \varepsilon) \subset \mathbb{C}$ be a small disk of radius $\varepsilon > 0$ centered at $* = \rho, -\frac{1}{\rho}$. Given k unordered couples of points $a_{2i-1}, b_{2i-1} \in D(\rho, \varepsilon)$ with $a_{2i-1} \neq b_{2i-1}$ and another k couples $a_{2i}, b_{2i} \in D(-\frac{1}{\rho}, \varepsilon)$ with $a_{2i} \neq b_{2i}$, $1 \leq i \leq k$, the usual cut-and-paste process gives us a torus $\mathbb{M}$ obtained by gluing $2k$ copies $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2, \ldots, \overline{\mathbb{C}}_{2k}$ of $\overline{\mathbb{C}}$. Let g be the $2k$-degree meromorphic map defined on $\mathbb{M}$ which corresponds to the natural z-map on each copy of $\overline{\mathbb{C}}$. Note that the brach values of such a g are the a_j, b_j. We denote by 0_j and ∞_j the zero and pole of g in the copy $\overline{\mathbb{C}}_j$ of $\overline{\mathbb{C}}$ and by $[\gamma]$ the nontrivial homology class in $H_1(\mathbb{M} - \{0_j, \infty_j\}_j, \mathbb{Z})$ associated to the circle $\{|z| = 1\}$ in $\overline{\mathbb{C}}_1$ with the anticlockwise orientation. Thus we can define a map

$$(a_1, b_1, \ldots, a_{2k}, b_{2k}) \mapsto g = (\mathbb{M}, g, 0_1, \ldots, 0_{2k}, \infty_1, \ldots, \infty_{2k}, [\gamma]) \in \mathcal{W} \tag{7}$$

which is not injective (one can exchange a_i by b_i obtaining the same g). Therefore we consider the arithmetic and geometric means of the couples a_j, b_j

$$x_i = \frac{1}{2}(a_i + b_i), \qquad y_i = \sqrt{a_i b_i}.$$

Note that (x_i, y_i) lies in a neighborhood of (ρ, ρ) or $(-\frac{1}{\rho}, \frac{1}{\rho})$, and that $a_i \neq b_i$ if and only if $x_i^2 \neq y_i^2$. Given $\varepsilon > 0$, we label $\mathcal{U}(\varepsilon) = \Big[D(\rho, \varepsilon) \times D(\rho, \varepsilon) \times D(\frac{-1}{\rho}, \varepsilon) \times D(\frac{1}{\rho}, \varepsilon)\Big]^k$ and $\mathcal{A} = \{(x_1, y_1, \ldots, x_{2k}, y_{2k}) \mid x_i^2 = y_i^2 \text{ for some } i = 1 \ldots, 2k\}$. Clearly $\mathcal{A}$ is an analytic subvariety of $\mathbb{C}^{4k}$. It can be shown that for $\varepsilon > 0$ small, the correspondence $\mathbf{z} = (x_1, y_1, \ldots, x_{2k}, y_{2k}) \in \mathcal{U}(\varepsilon) - \mathcal{A} \overset{\aleph}{\mapsto} g = (\mathbb{M}, g, 0_1, \ldots, 0_{2k}, \infty_1, \ldots, \infty_{2k}, [\gamma]) \in \mathcal{W}$ defines a local chart for $\mathcal{W}$.

Remark 4

(i) If a marked meromorphic map $g = \aleph(\mathbf{z})$ produces a marked surface M, then the ordered list $(0_1, \ldots, 0_{2k}, \infty_1, \ldots, \infty_{2k})$ does not necessarily coincide with

the ordering on the ends of $M \in \widetilde{\mathcal{S}}$, but this different notation will not affect to the arguments that follow.

(ii) Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence in the hypothesis of Proposition 6. Then $\{M_n\}_n$ converges uniformly to a singly periodic Scherk minimal surface S_ρ parametrized as in (6) for certain $\rho \in (0,1]$. After exchanging the homology class of the marked surface M_n by $[\Gamma_n] \in H_1(M_n, \mathbb{Z})$, we can see the same geometric surface M_n as a new marked surface M'_n inside the domain of the chart $\aleph$ for n large enough. Also note that if $C(M_n) = (a_n, b_n)$, then $C(M'_n) = (a_n, b_n + t_n \pi a_n)$ for some even integer t_n.

When $\mathbf{z} \in \mathcal{A}$, the continuous extension of the above cut-and-paste process gives a Riemann surface with nodes, each node occurring between copies $\overline{\mathbb{C}}_{j-1}, \overline{\mathbb{C}}_j$ where $a_j = b_j$, consisting of l spheres S_i joined by node points $P_i, Q_i \in S_i$ (here $Q_i = P_{i+1}$ and the subindexes are cyclic). Moreover, g degenerates in l nonconstant meromorphic maps $g(i) : S_i \to \overline{\mathbb{C}}$ such that $\sum_i \deg(g(i)) = 2k$ and $g(i)(\{P_i, Q_i\}) \subset \{\rho, -\frac{1}{\rho}\}$, and ϕ degenerates in the l unique meromorphic differentials $\phi(i)$ on S_i, such that $\phi(i)$ has exactly two simple poles at P_i, Q_i with residues 1 at P_i and -1 at Q_i (these residues are determined by the equation $\int_{|z|=1} \phi = 2\pi i$). Both g and ϕ depend holomorphically on all parameters (including at points of $\mathcal{A}$). Using this fact and that $\mathcal{U}(\varepsilon) \cap \mathcal{A}$ is an analytic subvariety of $\mathcal{U}(\varepsilon)$, we can prove that L extends holomorphically to $\mathcal{U}(\varepsilon)$. Finally, after direct computations, the Inverse Function Theorem insures that L is a biholomorphism in a neighborhood of $(\rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho})^k \in \mathbb{C}^{4k}$.

Theorem 2 *There exists $\varepsilon > 0$ small such that the ligature map L extends holomorphically to $\mathcal{U}(\varepsilon)$, and $L : \mathcal{U}(\varepsilon) \to L(\mathcal{U}(\varepsilon))$ is a biholomorphism.*

7 Uniqueness around the catenoid.

When a sequence $\{M_n\}_n \in \widetilde{\mathcal{S}}$ degenerates in a vertical catenoid (case *(i)* of Proposition 2), the residues in the ligature map L diverge to ∞. In this Section we will modify L to have a well defined locally invertible extension through this boundary point of $\mathcal{W}$.

Similar arguments as the ones before Proposition 6 lead us to the following statement.

Proposition 7 *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence with $\{C(M_n) = (a_n, b_n)\}_n \to (\infty, 0)$. Then for n large, the geometric surface M_n is close to $2k$ translated images of arbitrarily large compact regions of a catenoid with flux $(0,0,2\pi)$, together with $2k$ regions $C_n(1), \ldots, C_n(2k)$. Each $C_n(j)$ is a twice punctured annulus with one left end, one right end of M_n and two distinct simple branch points of its Gauss map. Furthermore, $C_n(j)$ is a graph over its horizontal projection on $\{x_3 = 0\}/H_n$.*

Following the line of arguments in Section 6, we next show a local chart for $\mathcal{W}$ around the catenoid obtained as boundary point in Proposition 7. Given $i = 1, \ldots, k$, choose two distinct points a_{2i-1}, b_{2i-1} (resp. a_{2i}, b_{2i}) in a small punctured neighborhood of 0 (resp. of ∞) in $\overline{\mathbb{C}}$. These unordered couples produce a marked surface $g \in \mathcal{W}$ as in (7), by using a cut-and-paste construction. Since the roles of a_j and b_j are symmetric, their elementary symmetric functions are right parameters in this setting. We consider for each $1 \leq j \leq 2k$

$$\begin{array}{lll} x_j = \frac{1}{2}(a_j + b_j)\,, & y_j = a_j b_j & \text{if } j \text{ is odd;} \\ x_j = \frac{1}{2}\left(\frac{1}{a_j} + \frac{1}{b_j}\right), & y_j = \frac{1}{a_j b_j} & \text{if } j \text{ is even.} \end{array}$$

Note that all parameters x_j, y_j are close to 0, and that the conditions on a_j, b_j translate into $y_j \neq x_j^2$ and $y_j \neq 0$. Given $\varepsilon > 0$, we let

$$\begin{aligned} D(0,\varepsilon)^{4k} &= \{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{4k} \mid |x_j|, |y_j| < \varepsilon \text{ for all } j = 1 \ldots, 2k\}, \\ \mathcal{B} &= \{(\mathbf{x}, \mathbf{y}) \in D(0,\varepsilon)^{4k} \mid x_j^2 = y_j \text{ for some } j\}, \\ \widehat{\mathcal{B}} &= \{(\mathbf{x}, \mathbf{y}) \in D(0,\varepsilon)^{4k} \mid y_j = 0 \text{ for some } j\}, \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_{2k})$, $\mathbf{y} = (y_1, \ldots, y_{2k})$. $\mathcal{B} \cup \widehat{\mathcal{B}}$ is an analytic subvariety of $D(0,\varepsilon)^{4k}$ and the map $(\mathbf{x}, \mathbf{y}) \in D(0,\varepsilon)^{4k} - (\mathcal{B} \cup \widehat{\mathcal{B}}) \mapsto \chi(\mathbf{x}, \mathbf{y}) = g \in \mathcal{W}$ is a local chart for $\mathcal{W}$.

Remark 5 *Given a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ with $C(M_n) \to (\infty, 0)$, there exists another sequence of marked surfaces $\{M_n'\}_n$ inside the image of the chart χ such that* *for each n M_n, M'_n only differ in the homology class in the last component of the marked surface.*

Given $1 \leq j \leq 2k$, let Γ_j be the circle defined by $|z| = 1$ in the copy $\overline{\mathbb{C}}_j$ of $\overline{\mathbb{C}}$ oriented so that all these curves are homologous in $g^{-1}(\overline{\mathbb{C}})$ and $[\Gamma_1]$ is the last component of the marked meromorphic map g. Recall that $0_j, \infty_j$ denote respectively the $0, \infty$ in $\overline{\mathbb{C}}_j$ and that ϕ is defined as the unique holomorphic 1-form with $\int_\gamma \phi = 2\pi i$. Define for $1 \leq j \leq 2k$

$$A_j = \begin{cases} \int_{\Gamma_j} \frac{\phi}{g} & (j \text{ odd}) \\ \int_{\Gamma_{j+1}} g\phi & (j \text{ even}) \end{cases} \qquad B_j = \begin{cases} \operatorname{Res}_{0_{j-1}} \frac{\phi}{g} \cdot \operatorname{Res}_{0_j} \frac{\phi}{g} & (j \text{ odd}) \\ \operatorname{Res}_{\infty_{j-1}} (g\phi) \cdot \operatorname{Res}_{\infty_j} (g\phi) & (j \text{ even}) \end{cases}$$

In this definition and in the sequel we will adopt a cyclic convention on the subindexes, so when $j = 1$, $j - 1$ must be understood as $2k$. It follows that g closes periods if and only if there exist $a \in \mathbb{R}^*$, $b \in \mathbb{C}$ such that

$$\left.\begin{array}{ll} A_{2i-1} = b,\ A_{2i} = \overline{b} & \text{for all } i = 1, \ldots, k; \\ B_j = -a^2 & \text{for all } j = 1, \ldots, 2k. \end{array}\right\} \tag{8}$$

Each $(\mathbf{x}, \mathbf{y}) \in \mathcal{B}$ gives rise to a Riemann surface with nodes which consists of l spheres S_i joined by node points P_i, Q_i so that $P_i = Q_{i+1}$, l nonconstant meromorphic maps $g(i) : S_i \to \overline{\mathbb{C}}$ with $\sum_i \deg(g(i)) = 2k$ and $g(i)(\{P_i, Q_i\}) \subset \{0, \infty\}$, and l meromorphic differentials $\phi(i)$ on S_i with just two simple poles at P_i, Q_i and residues 1 at P_i, -1 at Q_i. On the other hand, each $(\mathbf{x}, \mathbf{y}) \in \widehat{\mathcal{B}} - \mathcal{B}$ produces a conformal torus $\mathbb{M}$, a single meromorphic degree $2k$ map $g : \mathbb{M} \to \overline{\mathbb{C}}$ with at least a double zero or pole and a holomorphic differential ϕ on $\mathbb{M}$ with $\int_{\Gamma_1} \phi = 2\pi i$. Both g and ϕ depend holomorphically on all parameters $(\mathbf{x}, \mathbf{y})$ in a neighborhood of $(\mathbf{0}, \mathbf{0}) = (0, \ldots, 0) \in D(0, \varepsilon)^{4k}$ (including at points of $\mathcal{B} \cup \widehat{\mathcal{B}}$). In this setting, we get a statement analogous to Theorem 2, but instead of the ligature map L, we now consider the map $\Theta = \left(A_1 \ldots, A_{2k}, \frac{1}{B_1}, \ldots, \frac{1}{B_{2k}}\right) : D(0, \varepsilon)^{4k} \to \mathbb{C}^{4k}$, which also can be used as a tool to distinguish when a marked meromorphic map closes periods (the proof of the following theorem is not as straightforward as the one of Theorem 2; we refer the reader to [13] for details).

Theorem 3 *There exists $\varepsilon > 0$ small such that Θ extends holomorphically to $D(0, \varepsilon)^{4k}$, and $\Theta : D(0, \varepsilon)^{4k} \to \Theta(D(0, \varepsilon)^{4k})$ is a biholomorphism.*

8 Openness.

Recall that $\mathcal{K} \subset \mathcal{S}$ represents the space of KMR examples with $4k$ ends. A direct consequence of its construction is that $\mathcal{K}$ is closed in $\mathcal{S}$. We also saw in Section 3 that $\mathcal{K}$ is open in $\mathcal{S}$, by the nondegeneracy of any KMR example. Both closeness and openness remain valid for the space $\widetilde{\mathcal{K}}$ of marked KMR examples inside $\widetilde{\mathcal{S}}$. Theorem 1 reduces to prove that $\widetilde{\mathcal{K}} = \widetilde{\mathcal{S}}$. Assume $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \neq \emptyset$ and we will reach to a contradiction in Section 9.

Theorem 4 *The classifying map $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \to \mathbb{R}^* \times \mathbb{C}$ is open.*

Proof. Given $M \in \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$, it suffices to see that C is open in a neighborhood of M in $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. Let $(a, b) = C(M) \in \mathbb{R}^* \times \mathbb{C}$ and $\mathcal{M}(a, b) = L^{-1}(L_{(a,b)}) \subset \mathcal{M}$. Since $\widetilde{\mathcal{K}}$ is open and closed in $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}(a, b) = \widetilde{\mathcal{S}} \cap \mathcal{M}(a, b)$ is an analytic subvariety of $\mathcal{W}$, we conclude that $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b) = (\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}) \cap \widetilde{\mathcal{S}}(a, b)$ is an analytic subvariety of $\mathcal{W}$.

Assertion 1 *$(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$ is compact.*

We next demonstrate Assertion 1. Given a sequence $\{M_n\}_n \subset (\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$, let us prove that a subsequence of $\{M_n\}_n$ converges in $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$. By Proposition 5, $\{K_{M_n}\}_n$ is uniformly bounded. It can be also checked that K_{M_n} cannot converge uniformly to zero so, after passing to a subsequence, suitable liftings of M_n converge smoothly to a properly embedded nonflat minimal surface $\widetilde{M}_\infty \subset \mathbb{R}^3$ in one of the six cases listed in Proposition 2.

As a_n is fixed a for all n, $\widetilde{M}_\infty$ cannot be in the cases *(i),(ii),(iii)* of Proposition 2. If the case *(iv)* holds, then any choice of the nonhorizontal period vector T_n of M_n must diverge to ∞. By Proposition 6 and Remark 4-*(ii)*, for n large enough we can see the geometric surface M_n as a new marked surface M_n' inside $\mathcal{U}(\varepsilon) \subset \mathbb{C}^{4k}$ appearing in Theorem 2. Note that $C(M_n') = (a, b + t\pi a)$ for some even fixed integer t. Since $L|_{\mathcal{U}(\varepsilon)}$ is a biholomorphism (Theorem 2), the space of tuples in $\mathcal{U}(\varepsilon)$ producing immersed minimal surfaces has three real freedom parameters. But $\widetilde{\mathcal{K}}$ has real dimension three and $C|_{\widetilde{\mathcal{K}}}$ takes values arbitrarily close to $(a, b+t\pi a)$, so $C(M_n')$ must coincide with the value of C at a certain KMR example $M_n'' \in \widetilde{\mathcal{K}}$. In particular

$L(M'_n) = L(M''_n)$, so $M'_n = M''_n$, which is a contradiction since $M'_n \in \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. Thus, $\widetilde{M}_\infty$ is not a singly periodic Scherk minimal surface.

Now assume that $\widetilde{M}_\infty$ lies in case *(v)* of Proposition 2, and let Γ be a component of the intersection of $\widetilde{M}_\infty$ with a horizontal plane $\{x_3 = c\}$ whose height does not coincide with the heights of the horizontal ends of $\widetilde{M}_\infty$. Since $\widetilde{M}_\infty$ has exactly two nonhorizontal ends, Γ is an embedded U-shaped curve with two almost parallel divergent ends, and if we denote by $\Pi \subset \mathbb{R}^3$ the plane passing through the origin parallel to the nonhorizontal ends of $\widetilde{M}_\infty$, then the conormal vector to $\widetilde{M}_\infty$ along each of the divergent branch of Γ becomes arbitrarily close to the upward pointing unit vector $\eta \in \Pi$ such that η is orthogonal to $\Pi \cap \{x_3 = c\}$. Since translated liftings of the M_n converge smoothly to $\widetilde{M}_\infty$, we deduce that M_n contains arbitrarily large arcs at constant height along which the conormal vector η_n is arbitrarily close to η. In particular, the integral of the third component of η_n along such arcs becomes arbitrarily large. As the conormal vector of M_n along any compact horizontal section misses the horizontal values (the Gauss map of M_n is never vertical), it follows that the vertical component of the flux of M_n along a compact horizontal section diverges to ∞. This contradicts our normalization on the surfaces of $\mathcal{S}$.

Thus $\widetilde{M}_\infty$ is in case *(vi)* of Proposition 2. As $\widetilde{\mathcal{K}}$ is open in $\widetilde{\mathcal{S}}$, then the quotient of $\widetilde{M}_\infty$ would actually be in $\widetilde{S} - \widetilde{K}$, and Assertion 1 follows.

We now finish the proof of Theorem 4. By Assertion 1 and Lemma 3, $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$ is a finite subset, hence we can find an open set $\mathcal{U}$ of $\mathcal{W}$ containing M such that $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b) \cap \mathcal{U} = \mathcal{M}(a, b) \cap \mathcal{U} = \{M\}$. In terms of the ligature map $L : \mathcal{W} \to \mathbb{C}^{4k}$, the last equality writes as $L^{-1}(L_{(a,b)}) \cap \mathcal{U} = \{M\}$. Since L is holomorphic, we can apply the Openness Theorem for finite holomorphic maps (see [1] page 667) to conclude that $L|_{\mathcal{U}}$ is an open map. Finally, the relationship between the ligature map L and the map C gives the existence of a neighborhood of M in $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$ where the restriction of C is open.

□

The same argument in the proof of Assertion 1 remains valid under the weaker hypothesis on $C(M_n)$ to converge to some $(a, b) \in \mathbb{R}^* \times \mathbb{C}$ instead of being constant on a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. This proves the validity of the following statement.

Theorem 5 *The classifying map $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \to \mathbb{R}^* \times \mathbb{C}$ is proper.*

9 The proof of Theorem 1.

Recall that we were assuming $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \neq \emptyset$. By Theorems 4 and 5, $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \to \mathbb{R}^* \times \mathbb{C}$ is an open and proper map. Thus, $C(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})$ is an open and closed subset of $\mathbb{R}^* \times \mathbb{C}$. Since $C(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})$ has points in both connected components of $\mathbb{R}^* \times \mathbb{C}$, we deduce that $C|_{\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}}$ is surjective. In particular, we can find a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$ such that $\{C(M_n)\}_n$ tends to $(\infty, 0)$ as n goes to infinity. Now the argument is similar to the one in the proof of Assertion 1 when we discarded the singly periodic Scherk limit, using Proposition 7, Remark 5 and Theorem 3 instead of Proposition 6, Remark 4-*(ii)* and Theorem 2.

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