Matemática Contemporânea, Vol 30, 145-168

ON COMPLETE RICCI-FLAT METRICS ON OPEN KÄHLER MANIFOLDS

Bianca Santoro

1 Introduction

In 1978, Yau [Y] proved the Calabi Conjecture, by showing existence and uniqueness of Kähler metrics with prescribed Ricci curvature on compact complex manifolds. Here the complex manifolds in question are already supposed to admit a Kähler metric whose Ricci form satisfies the natural conditions arising from Chern-Weil theory. The remarkable paper [Y] also establishes several related results of fundamental importance in the study of complex manifolds.

The results are proved by reducing them to questions in non-linear partial differential equations of Monge-Ampère type. Once this is done, the questions are treated by a continuity method involving intrincated *a priori* estimates. These methods, in fact, are interesting in themselves.

Following the work by Yau, Tian and Yau [TY1] settled a non-compact version of Calabi's Conjecture on quasi-projective manifolds that can be compactified by adding a smooth, ample divisor. Again, the desired metric was constructed by translating the geometric problem into a complex Monge-Ampère equation. Unfortunately, however, the solution to the Monge-Ampère equation obtained in [TY1] is only shown to be bounded at infinity. An interesting question that arises naturally is then to work out the asymptotic behavior of these completes metrics near the divisor (regarded as "infinity"). This question is, in fact, posed by Tian and Yau in [TY1]. Clearly the knowledge of full asymptotic expansion of these metrics immediately yields a more accurate picture of their behavior near infinity.

In a subsequent work ([TY2]), Tian and Yau extended their result for the case where the divisor has multiplicity greater than one, and is allowed to have orbifold singularities. A generalization that was done independently by Bando and Kobayashi [BK]. Later, Joyce [J] provided the sharp asymptotics for the decay of the solutions provided in [TY2] thus clarifying, in this case, the geometry associated to this metric near infinity.

The problem of computing the asymptotic expasion of the solutions to the Monge-Ampère equations obtained in [TY1] however remained open, since the case where the divisor has multiplicity exactly 1 contains special difficulties resisting to the method of Joyce [J].

In the author's thesis ([S1], [S2]), this question is treated in detail and the corresponding asymptotics of the solution to the Monge-Ampère equation in [TY1] is obtained in two steps. First, it is considered a sequence of complete Kähler metrics with special approximating properties on M. Whereas these metrics are only "approximate solutions", they have the advantage of being given by *explicit formulas.* The second step consists then of using these metrics as ambient metrics to solve the Monge-Ampère equation in [TY1]. Thanks to the approximating properties of the mentioned metrics, we shall be able to completely describe the decay of the solutions at infinity.

The present text has two purposes. First it partially extends the result in [S2], by considering a larger class of quasi-projective manifolds, which is described in Section 6. Basically the definition of this class consists of relaxing the condition that the divisor "at infinity" should be ample. This will allow us to encompass, for example, manifolds with cylindrical ends as those considered by [Kov], cf. Section 3.

On the other hand, this text is also intended to provide a quick introductory reference to the study of Ricci-flat metrics on open manifolds. For reasons of space, we shall not report on the analogous problem for Kähler-Einstein metrics except for a brief comment in Section 4. We point out however that the analysis of the differential operators involved in each case is very different.

It is assumed that the reader is familiar with some basic facts in Riemannian and Complex Geometry, as well as with the analysis of elliptic partial differential equations.

2 Background: Calabi Conjecture on Compact Manifolds

In order to understand the main problem considered in this work, we shall start from its original motivation.

Let M be a compact, complex manifold of complex dimension n, and consider g, a hermitian metric defined on M. Note that g is a complex-valued sesquilinear form acting on $TM \times TM$, and can therefore be written as

$$g = S - 2\sqrt{-1}\omega_g,$$

where S and $-\omega$ are real bilinear forms.

If $(z_1, \ldots, z_n)$ are local coordinates around a point $x \in M$, we can write the metric g as $\sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Then, it is easy to see that in these coordinates

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

The form ω_g is a real $2-$form of type $(1,1)$, and is called the *fundamental form* of the metric g.

Definition 2.1 *We say that a hermitian metric on a complex manifold is* Kähler *if its associated fundamental form ω_g is closed,* ie*, $d\omega_g = 0$. A complex manifold equipped with a Kähler metric is called a* Kähler manifold.

The reader will find in the literature a number of equivalent definitions for a Kähler metric. We will keep this choice for convenience of the exposition. Also, note that, on a Kähler manifold, the form ω_g is uniquely determined by the metric g, and vice-versa.

Let $R(g) = R_{i\bar{j}k\bar{l}}$ be the Riemann curvature tensor of the metric g written in the above mentioned coordinates. We define the *Ricci curvature tensor* of the metric g as being the trace of the Riemann curvature tensor. Its components in local coordinates can be written as

$$\mathrm{Ric}_{k\bar{l}} = \sum_{i,j=1}^{n} g^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log\det(g_{i\bar{j}}). \tag{1}$$

The *Ricci form* associated to g can then be defined by setting

$$\mathrm{Ric} = \sum_{i,j=1}^{n} Ric_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

in local coordinates.

Now, given a metric g, we can define a matrix-valued 2-form Ω by writing its expression in local coordinates, as follows

$$\Omega_i^j = \sum_{i,p=1}^{n} g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l. \tag{2}$$

This expression for Ω gives a well-defined matrix of $(1,1)$-forms, to be called the *curvature* of the metric g.

Following Chern-Weil Theory, we want to look at the following expression

$$\det\left(\mathrm{Id} + \frac{t\sqrt{-1}}{2\pi}\Omega\right) = 1 + t\phi_1(g) + t^2\phi_2(g) + \dots,$$

where each $\phi_i(g)$ denotes the i-th homogeneous component of the left-hand side, considered as a polynomial in the variable t.

Each of the forms $\phi_i(g)$ is a (i,i)-form, and is called the *i-th Chern form* of the metric g. It is a fact (see for example [We] for further explanations) that the cohomology class represented by each $\phi_i(g)$ is independent on the metric g, and hence it is a topological invariant of the manifold M. These cohomology classes are called the *Chern classes* of M and they are going to be denoted by $c_i(M)$.

Remark: We can define more generally the curvature $\Omega(E)$ of a hermitian metric h on a general complex vector bundle E on a complex manifold M.

Let $\nabla = \nabla(h)$ be a connection on a vector bundle $E \to M$. Then the *curvature* $\Omega_E(\nabla)$ is defined to be the element $\Omega \in \Omega^2(M, \mathrm{End}(E, E))$ such that the $\mathbb{C}$-linear mapping

$$\Omega : \Gamma(M, E) \to \Omega^2(M, E)$$

has the following representation with respect to a frame f:

$$\Omega(f) = \Omega(\nabla, f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

Here, $\Gamma(M, E)$ is the set of sections of the vector bundle E, $\Omega^2(M, E)$ is the set of $E-$valued 2-forms, and $\theta(f)$ is the connection matrix associated with ∇ and f (with respect to f, we can write $\nabla = d + \theta(f)$).

Similarly, one defines the Chern class $c_i(M, E)$ of a vector bundle and these are also independent on the choice of the connection. In fact, we use the expression "Chern classes $c_i(M)$ of the manifold M" meaning the Chern classes $c_i(M, TM)$ of the tangent bundle of M.

We will restrict our attention to the first Chern class $c_1(M)$ of the manifold M. Note that the form $\phi_1(g)$ represents the class $c_1(M)$ (by definition), and that $\phi_1(M)$ is simply the trace of the curvature form:

$$\phi_1(g) = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n} \Omega_i^i = \frac{\sqrt{-1}}{2\pi} \sum_{i,p=1}^{n} g^{i\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l. \tag{3}$$

On the other hand, notice that the right-hand side of (3) is equal to $\frac{\sqrt{-1}}{2\pi}\mathrm{Ric}_{k\bar{l}}$, in view of (1). Therefore, we conclude that the Ricci form of a Kähler metric represents the first Chern class of the manifold M.A natural question that arises is: given a Kähler class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ in a compact, complex manifold M, and any $(1,1)$-form Ω representing $c_1(M)$, is that possible to find a metric g on M such that $\mathrm{Ric}(g) = \Omega$?

This question was addressed to by Calabi in 1960, and it was answered by Yau [Y] almost 20 years later.

Theorem 2.1 (Yau, 1978) *If the manifold M is compact and Kähler , then there exists a unique (up to constant) Kähler metric g on M satisfying $Ric(g) = \Omega$.*

This theorem has a large number of applications in different areas of Mathematics and Physics. Its proof is based on translating the geometric statement into a non-linear partial differential equation, as follows.

First, fix a Kähler form $\omega \in [\omega]$ representing the previously given Kähler class in $H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$. In local coordinates, we can write ω as $\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

The $(1,1)$-form Ω is a representative for $c_1(M)$, and we have seen that $\mathrm{Ric}(\omega)$ represents the same cohomology class as Ω. Therefore, since $\mathrm{Ric}(\omega)$ is also a $(1,1)$-form, we have that, due to the famous $\partial\bar{\partial}$-Lemma, there exists a function f on M such that

$$\mathrm{Ric}(\omega) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f,$$

where f is uniquely determined after imposing the normalization

$$\int_M \left(e^f - 1\right)\omega^n = 0. \tag{4}$$

Notice that f is fixed once we have fixed ω and Ω.

Again by the $\partial\bar{\partial}$-Lemma, any other $(1,1)-$form in the same cohomology class $[\omega]$ will be written as $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$, for some function $\phi \in C^\infty(M,\mathbb{R})$.

Therefore, our goal is to find a representative $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$ of the class $[\omega]$ that satisfies

$$\mathrm{Ric}\left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right) = \Omega = \mathrm{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f. \tag{5}$$

Rewriting (5) in local coordinates, we have

$$-\partial\bar{\partial}\log\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial\bar{z}_j}\right) = -\partial\bar{\partial}\log\det\left(g_{i\bar{j}}\right) - \partial\bar{\partial} f,$$

or

$$\partial\bar{\partial}\log\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial\bar{z}_j}\right)}{\det\left(g_{i\bar{j}}\right)} = \partial\bar{\partial} f. \tag{6}$$

Notice that, despite of the fact that this is an expression given in local coordinates, the term at the right-hand side of (6) is defined globally. Therefore, we obtain an equation well-defined on all of M. In turn, this equation gives rise to the following (global) equation

$$\left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right)^n = e^f \omega^n. \tag{7}$$

We shall also require positivity of the resulting Kähler form: $\left(\omega + \partial\bar{\partial}\phi\right) > 0$ on M. This equation is a non-linear partial differential equation of Monge-Ampère type, that is going to be referred to from now on as the Complex Monge-Ampère Equation.

We remark that, if ϕ is a solution to (7), $\omega + \partial\bar{\partial}\phi$ is the Kähler form of our target metric g, *ie*, $\mathrm{Ric}(g) = \Omega$. Therefore, in order to find metrics that are solutions to Calabi's problem, it suffices to determine a solution ϕ to (7).

The celebrated Yau's Theorem in [Y] determines a unique solution to (7) when f satisfies the integrability condition (4). The proof of this result is based on the continuity method, and we sketch here a brief outline of the proof.

The uniqueness part of Calabi Conjecture was proved by Calabi in the 50's. Let $\omega', \omega'' \in [\omega]$ be representatives of the Kähler class $[\omega]$ such that $\mathrm{Ric}(\omega') = \mathrm{Ric}(\omega'') = \Omega$. Without loss of generality, we may assume that $\omega'' = \omega$, and hence $\omega' = \omega + \partial\bar{\partial}u$.

Notice that

$$0 = \begin{array}{rcl} \frac{1}{\mathrm{Vol}_\omega(M)}\int_M u((\omega')^n - \omega^n) & = & \frac{1}{\mathrm{Vol}_\omega(M)}\int_M -u\partial\bar{\partial}u \wedge [(\omega')^{n-1} + (\omega')^{n-2}\wedge\omega \\ & & + \cdots + \omega^{n-1}]\,. \end{array} \tag{8}$$

However ω' is a Kähler form, so that $\omega' > 0$. We then conclude that the right-hand side of (8) is bounded from below by $\frac{1}{\mathrm{Vol}_\omega(M)}\int_M -u\partial\bar{\partial}u \wedge w^{n-1}$. Therefore,

$$\begin{array}{rcl} 0 \geq \frac{1}{\mathrm{Vol}_\omega(M)}\int_M -u\partial\bar{\partial}u \wedge w^{n-1} & = & \frac{1}{n\mathrm{Vol}_\omega(M)}\int_M |\partial u|^2 w^n \\ & = & \frac{1}{2n\mathrm{Vol}_\omega(M)}\int_M |\nabla u|^2 w^n, \end{array} \tag{9}$$

implying that $|\nabla u| = 0$, hence u is constant, proving the uniqueness of solution to (7).

Let us now consider the existence of solution to (7). Define, for all $s \in [0,1]$, $f_s = sf + cs$, where the constant c_s is defined by the requirement that f_s satisfies the integrability condition $\int_M [e^{f_s} - 1]\omega^n = 0$.

Consider the family of equations

$$\left(\omega + \partial\bar{\partial}u_s\right)^n = e^{f_s}\omega^n. \tag{10}$$

We already prove that the solution u_s to (10) is unique, if it exists.

Let $A = \{s \in [0,1];\ (10) \text{ is solvable for all } t \leq s\}$. Since $A \neq \emptyset$ $(0 \in A)$, we just need to show that A is open and closed.

Openness: Let $s \in A$, and let t be close to s. We want to show that $t \in A$. In order to do so, let $\omega_s = \omega + \partial\bar{\partial}u_s$, for u_s a solution to (10). We define the operator $\Psi = \Psi_s$ by

$$\Psi : X \to Y; \qquad\qquad \Psi(g) = \log\left(\frac{(\omega_s + \partial\bar{\partial}g)^n}{w_s^n}\right),$$

where X and Y are subsets (not subspaces) of $C^{2,1/2}(M.\mathbb{R})$ and $C^{0,1/2}(M.\mathbb{R})$ satisfying some extra non-linear conditions.

The linearization of Ψ about $g = 0$ is simply the metric laplacian with respect to the metric ω_s. By the Implicit Function Theorem, the invertibility of the laplacian (a classical result that can be found in [GT], for example) establishes the claim.

Closedness: The proof that A is closed is a deep result, involving complicated *a priori* estimates. A reference for this proof is Yau's paper itself [Y], or for a more detailed proof, the books [T] and [A].

Yau's Theorem provided a satisfatory answer to the problem of finding Ricci-flat metrics when the underlying manifold M is compact.

Calabi's Problem, though, has a very natural generalization for the case of a special class of open manifolds. This is the subject of the next section.

3 Noncompact version of the Calabi-Yau Problem - Existence

To discuss the Calabi Conjecture on open manifolds, some minor modifications in the original problem are needed. Namely, we need to understand which line bundle associated to an open manifold should play the role of the canonical line bundle K_M for a compact manifold on the Calabi-Yau problem.

Suppose that $\overline{M}$ is a compact, Kähler manifold, and let D be a smooth divisor in $\overline{M}$. We are now interested in constructing complete Kähler metrics with prescribed Ricci curvature on the open manifold M, defined as the complement of the divisor D in $\overline{M}$.

If g' is a metric defined on $\overline{M}$, then the metric $\det(g')$ (given locally by $\det(g') = \det(g'_{i\bar{j}})dz_1 d\bar{z}_1 \dots dz_n d\bar{z}_n)$ is a metric defined on the canonical line bundle $K_{\overline{M}}$ of $\overline{M}$.

Consider the line bundle L_D associated to D, let S be a defining section of D in L_D, and finally, let h define a hermitian metric on L_D. Let us write h, in the previous choice of local coordinates, as a positive function a.

With the preceding notations, the line bundle given by $K_{\overline{M}} \otimes -L_D$ has a metric defined locally by $\det(g'_{i\bar{j}})a^{-1}$. Indeed the reader will note that this expression makes sense globally on M. In turn, the metric $\det(g'_{i\bar{j}})a^{-1}$ can be written as $\det(g'_{i\bar{j}})a^{-1} = \det(\frac{g'_{i\bar{j}}}{b})$, where $b^n = a$. In particular, we have a new metric g defined on M (and also on $\overline{M}$) which is given in local coordinates by $g_{i\bar{j}} = \frac{g'_{i\bar{j}}}{b}$. Naturally the Ricci form of the metric g is a representative of the first Chern class $c_1(-K_{\overline{M}} \otimes -L_D)$. On the other hand, we would like the resulting metric g to be complete on the open manifold M. Strictly speaking, this will never happen since g is also a metric on the closure $\overline{M}$. Nonetheless, this construction suggests a natural way to try to obtain complete metrics. Namely we let the metric h conveniently degenerate on the divisor D. This implies that the function a will vanish on D and thus that the metric g will become unbounded near D. So we may hope to find complete metrics on M by this procedure. Note also that the class of the Ricci form of g is not affected by the "degeneration" of h.

Summarizing what precedes, to generalize Calabi's Conjecture to open manifolds, we begin by fixing a representative $\Omega \in c_1(-K_{\overline{M}} \otimes -L_D)$. From our previous discussion, the Ricci form of a Kähler metric defined on M is a representative of $c_1(-K_{\overline{M}} \otimes -L_D)$. Now we want to study the converse problem, namely:

Question: Fixed a Kähler class $[\omega]$ in the manifold M, pick any representative Ω of the first Chern class $c_1(-K_{\overline{M}} \otimes -L_D)$. Can we construct a complete Kähler metric g on M such that $\mathrm{Ric}(g) = \Omega$?

In order to state the existence result by Tian and Yau, we need the following definitions, that can also be found in [TY1].

Definition 3.1 *Let (X, g) be a complete, Riemannian manifold, and let K, α, β be nonnegative numbers. The manifold (X, g) is of (K, α, β)-*polynomial growth *if*

- *The sectional curvature of (X, g) is bounded by K;*
- *For some fixed point $x_0 \in X$, there is a constant C such that $Vol_g(B_R(x_0)) \leq CR^\alpha$ for all $R > 0$,and*
- *$Vol_g(B_1(x)) \geq C^{-1}(1 + d_g(x_0, x))^{-\beta}$,*

where $B_R(x)$ is the geodesic ball of radius R around x and Vol_g, d_g denote, respectively, the volume and the distance function associated to the metric g.

Definition 3.2 *Let (M, g) be a complete Kähler manifold. We say that (M, g) is of* quasifinite geometry of order $\ell + \delta$ *if there exist $r > 0$, $r_1 > r_2 > 0$ such that for all $x \in M$, there exists a holomorphic map*

$$\phi_x : U_x \subset (\mathbb{C}^n, 0) \to B_r(x)$$

such that

- *$\phi_x(0) = x$, and $B_{r_2} \subset U_x \subset B_{r_1}$, where $B_t = \{z \in \mathbb{C}^n; |z| < t\}$;*
- *The pullback metric $\phi_x^\star g$ is a Kähler metric on U_x such that the metric tensor of $\phi_x^\star g$ and its derivatives up to order ℓ are bounded and δ-Hölder-continuously bounded with respect with the natural coordinate system on $\mathbb{C}^n$.*

We note that if (M, g) is a Kähler manifold such that its sectional curvature and covariant derivative of the scalar curvature are bounded, then (M, g) is of quasi-finite geometry of order $2 + 1/2$ (as in [TY1]).

Theorem 3.1 (Tian, Yau, 1990) *Let (M, g) be a complete Kähler manifold of quasi-finite geometry of order $2+1/2$ and with $(K, 2, \beta)$-polynomial growth. Let f be a smooth function satisfying the integrability condition $\int_M (e^f - 1)\omega_g = 0$, and for some constant C,*

- $\sup\{|\nabla_g f|, |\Delta_g f|\} \leq C$*;*
- $|f(x)| \leq \frac{C}{(1+\rho(x))^\beta}$*,*

where $\rho(x) = d_g(x_0, x)$ for a fixed point $x \in M$.Then there exists a bounded, smooth solution u to the Monge-Ampère equation

$$\left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = e^f\omega^n, \tag{11}$$
$$\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0 \qquad \text{on } M.$$

Sketch of Proof: The idea of the proof of this theorem is based on a perturbation method. The function f is replaced by a sequence $\{f_m\}$ of compactly supported functions that converge uniformly to f, and we consider the sequence of modified Monge-Ampère Equations

$$\left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_m\right)^n = e^{f_m}\omega^n, \tag{12}$$
$$\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_m > 0 \qquad \text{on } M.$$

The strategy of solving (11) is to show that, for each m, (12) admits a solution u_m, and that a subsequence of $\{u_m\}$ converges uniformly to a solution u of (11).

And, in order to show solvability of (12), Tian and Yau describe the solution u_m as the uniform limit of solutions $u_{m,\varepsilon}$ to

$$\begin{aligned}\left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_{m,\varepsilon}\right)^n &= e^{f_m+\varepsilon u_{m,\varepsilon}}\omega^n, \\ \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_{m,\varepsilon} &> 0 \qquad \text{on } M.\end{aligned} \tag{13}$$

The solvability of (13) is due to Cheng and Yau [CY1]. By standart elliptic theory, the main step in showing that $u_{m,\varepsilon} \to u_m$ is the uniform C^0 estimate for the solutions $u_{m,\varepsilon}$ (some extra work will give the $C^{2,1/2}$ uniform estimates, which are sufficient to guarantee the result). In order to derive such estimates, a key step is the derivation of weighted Sobolev inequalities. Further detail in the proof of this theorem can be found in the paper [TY1].

□

This theorem, though, does not provide enough information about the behavior of the solution u close to the divisor. This is an interesting question to be studied, with a number of applications. In order to illustrate the applications, let us quickly discuss how do they enter into the construction of an example due to Kovalev [Kov] of a Riemannian manifold having special holonomy group G_2. Here we remind the reader that examples of Riemannian manifolds with holonomy G_2 are hard to construct though these manifolds play an important role in Physics.

Application: Manifolds with Special Holonomy G_2 (Kovalev, [Kov]).

The manifold in question has dimension 7 and will be obtained by means of a connected-sum construction.

The strategy is to glue appropriately two copies of $S^1 \times W$, where W is a 6-manifold with Holonomy group $S\mathcal{U}(3)$, a maximal subgroup of G_2.

An example of such W would be a complete Kähler manifold with zero Ricci curvature and asymptotically cylindrical end, diffeomorphic to $D \times \mathbb{R}_+ \times S^1$, where D is a simply-connected, compact Kähler manifold with $c_1(D) = 0$.

Hence, a main step in Kovalev's construction is to find a Ricci-flat metric on W that is asymptotically close to a *product* Ricci-flat metric on $D \times \mathbb{R}_+ \times S^1$.

Once this is done, it is possible to show ([Kov]) that this metric has holonomy group $S\mathcal{U}(3)$.

The Riemannian product of W and S^1 yields open 7-manifolds with same holonomy group as W. Finally, in order to obtain the manifold with holonomy group G_2, it is necessary to glue (in a non-standart way) two copies of $S^1 \times W$, interchanging the S^1-factors (recall that the end of W fibers over $D \times \mathbb{R}_+$) in order to avoid infinite fundamental group, a necessary condition for a manifold to admit a metric with holonomy group G_2.

The existence of a Ricci-flat metric in this context is attributed to Tian and Yau ([TY1]), but the case dealt with in the paper [TY1] is different, and doesn't apply directly to the case above. The asymptotics of this problem can be (partially) derived from the results in Section 6 of this paper, and in a future work, we will present a detailed proof of the existence result in the context of complete Kähler manifold with zero Ricci curvature and asymptotically cylindrical end, as well as its detailed asymptotics.

4 Construction of complete Kähler metrics with special approximating properties

The goal of this section is to discuss recent developments on the study of the behavior of complete Ricci-flat metrics at infinity.

In the authors' thesis ([S1], [S2]), the strategy to study the asymptotics of Tian-Yau solution to (11) is divided in two main steps.

The first of the is the inductive construction of an explicitly given sequence of complete Kähler metrics on the manifold M with special approximating properties.The other step is to use these metrics as ambient metrics, and study the solution to (11) on M.

To state the main results of [S2], let us consider a compact, complex manifold $\overline{M}$ of complex dimension n. Let D be an *admissible* divisor in $\overline{M}$, ie, a divisor satisfying the following conditions:

- Sing $\overline{M} \subset D$.

- D is smooth in $\overline{M} \setminus$ Sing $\overline{M}$.
- For every $x \in$ Sing $\overline{M}$, the corresponding local uniformization $\Pi_x : \tilde{\mathcal{U}}_x \to \mathcal{U}_x$, with $\tilde{\mathcal{U}}_x \in \mathbb{C}^n$, is such that $\Pi_x^{-1}(D)$ is smooth in $\tilde{\mathcal{U}}_x$.

Let Ω be a smooth, closed $(1,1)$-form in the cohomology class $c_1(K_{\overline{M}}^{-1} \otimes L_D^{-1})$. Let S be a defining section of D on L_D and let M be the open manifold $M = \overline{M} \setminus D$. Consider a hermitian metric $||.||$ on L_D.

Theorem 4.1 **[S2]** *Let M, Ω and D be as above. Then, for every $\varepsilon > 0$, there exists an explicitly given complete Kähler metric g_ε such that*

$$Ric(g_\varepsilon) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f_\varepsilon \qquad \text{on } M, \tag{14}$$

where f_ε is a smooth function on M that decays to the order of $O(||S||^\varepsilon)$. Furthermore, the Riemann curvature tensor $R(g_\varepsilon)$ of the metric g_ε decays to the order of $O((-n \log ||S||^2)^{-\frac{1}{n}})$.

Note that the metrics given above are explicitly given, and its construction is an interesting result in itself.

To prove Theorem 4.1, we start by fixing an orbifold hermitian metric $||.||$ on L_D such that its curvature form $\tilde{\omega}$ is positive definite along D. Here we are using the assumption that D is ample. In Section 6, we discuss how we can weaken this assumption on the divisor, providing a generalization on the statements in [S2].

Yau's Theorem for compact manifolds implies that, by rescaling $||.||$ by an appropriate factor, we may assume that $\tilde{\omega}$, when restricted to the infinity D, defines a metric g_D such that $\mathrm{Ric}(g_D) = \Omega|_D$. Denote by $||.||_\phi = e^{-\phi/2}||.||$ the rescaling of the metric $||.||$, where ϕ is any smooth function on $\overline{M}$.

We define

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial\bar{\partial}(-\log ||S||_\phi^2)^{\frac{n+1}{n}}. \tag{15}$$

The key step on the proof of Theorem 4.1 is its local version, as follows.

Proposition 4.1 *Let $\overline{M}$ be a compact Kähler manifold of complex dimension n, and let D be an admissible divisor in $\overline{M}$. Consider also a form $\Omega \in c_1(-K_{\overline{M}} - L_D)$, where L_D is the line bundle induced by D.*

Then there exist sequences of neighborhoods $\{V_m\}_{m\in\mathbb{N}}$ of D along with complete Kähler metrics ω_m on $(V_m \setminus D, \partial(V_m \setminus D))$ (as defined in 15) such that

$$Ric(\omega_m) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f_m \text{ on } V_m \setminus D \tag{16}$$

where f_m are smooth functions on $M = \overline{M} \setminus D$. Furthermore each f_m decays to the order of $O(||S||^m)$. In addition the curvature tensors $R(g_m)$ of the metrics g_m decay at least to the order of $(-n \log ||S||^2_m)^{\frac{-1}{n}}$ near the divisor.

Some further work ([S2]) allow us to extend each of these metrics accordingly to the whole manifold.

Let us say a few words about the proof of Proposition 4.1. Fefferman, in his paper [F], developed inductively an n-th order approximation to a complete Kähler-Einstein metric on strictly pseudoconvex domains on $\mathbb{C}^n$ with smooth boundary, and he suggested that higher order approximations could be obtained by considering log terms in the formal expansion of the solution to a certain complex Monge-Ampère equation. This idea was taken up by Lee and Melrose in [LM], where they constructed the full asymptotic expansion of the solution to the Monge-Ampère equation considered by Fefferman, determining completely the form of the singularity and improving the regularity of the existence result of Cheng and Yau [CY1].

Tian and Yau [TY3] showed the existence of Kähler-Einstein metrics under certain conditions on the divisor, and Wu [Wu] developed the asymptotic expansion to the Cheng-Yau metric on a quasi-projective manifold (also assuming some conditions on the divisor), as the parallel part to the work of Lee and Melrose [LM].

As mentioned in the Introduction, the Kähler-Einstein condition used in these works makes the underlying analysis very special and widely developed by many people, and does not apply to the Ricci-flat case. Yet the proof of

Proposition 4.1 is inspired on the inductive methods of Fefferman and Lee-Melrose, applied to the context of complete Ricci-flat metrics quasi-projective manifolds. The full detailed version of it can be found in [S1], [S2].

Idea of the proof of Proposition 4.1:

First, consider the function

$$f_\phi(x) = -\log ||S||^2 - \log(\frac{\omega_\phi^n}{\omega'^n}) - \Psi,$$

where ω' is any Kähler form in $\overline{M}$, and Ψ is related to ω' by $\Omega = \mathrm{Ric}(g') - \tilde{\omega} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi$.

When $\phi = 0$, it is not hard to see that the function $f_0(x)$ will converge uniformly to a constant if and only if $\mathrm{Ric}(g_D) = \Omega|_D$. Without loss, we can assume that this constant is zero.

It is possible to extend f_0 smoothly to be zero along the divisor. Hence, there exists a $\delta_0 > 0$ such that, in the neighborhood $V_0 := \{x \in \overline{M}; ||S(x)|| < \delta_0\}$, f_0 can be written as

$$f_0 = S \cdot u_1 + \overline{S} \cdot \bar{u}_1,$$

where u_1 is a C^∞ local section in $\Gamma(V_0, L_D^{-1})$.

Our goal now would be to construct a function ϕ_1 of the form $S\cdot\theta_1 + \overline{S}\cdot\overline{\theta}_1$, so that the corresponding $f_{\phi_1} = f_1$ vanishes at order 2 along D, and then proceed inductively to higher order. Unfortunately, there is an obstruction to higher order approximation that lies in the kernel of the laplacian on L_D^{-1} restricted to D. In order to deal with this difficulty, we must introduce $(-\log ||S||^2)$-terms in the expansion of ϕ_1.

Following the techniques in [TY2], we construct inductively in [S2] a sequence of hermitian metrics $\{||.||_m\}_{m>0}$ on L_D such that, for any $m > 0$, there exists a $\delta_m > 0$ satisfying:

1. The corresponding Kähler form ω_m associated to $||.||_m$ (as defined in (15)) is positive definite in $V_m := \{x \in \overline{M}; ||S(x)|| < \delta_m\}$.

2. The function f_m associated to ω_m can be expanded in V_m as

$$f_m = \sum_{k \geq m+1} \sum_{\ell=0}^{\ell_k} u_{k\ell} (-\log ||S||_m^2)^\ell, \tag{17}$$

where $u_{k\ell}$ are smooth functions on $\overline{V}_m$ that vanish to order k on D. In particular the function $u_{k\ell}$ can be written as

$$u_{k\ell} = \sum_{i+j=k} S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij},$$

for $\theta_{ij} \in \Gamma(V_m, L_D^{-i} \bigotimes \overline{L}_D^{-j})$.

This construction is very technical, and it is based on studying the kernel of a second-order differential operator. A detailed exposition of this result can be found in [S2].

5 Asymptotics of the Monge-Ampère Equation on quasi-projective manifolds

In this section, we want to conclude our second step: considering the metrics given by Theorem 4.1 as ambient metrics, to study the asymptotics of the solution to the Complex Monge-Ampère Equation.

Theorem 5.1 **[S2]** *For each $\varepsilon > 0$, let g_ε and f_ε be given by Theorem 4.1. As before, let S be a defining section for the divisor D.*

Consider the solution u_ε to the problem

$$\begin{cases} \left(\omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon\right)^n = e^{f_\varepsilon}\omega_{g_\varepsilon}^n, & \\ \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon > 0, & u_\varepsilon \in C^\infty(M, \mathbb{R}). \end{cases} \tag{18}$$

Then the solution u_ε decays as $O(||S||^\varepsilon)$ near the divisor.

Theorem 5.1 shows that the metrics ω_{g_ε} given by Theorem 4.1 are in fact a very good description of what happens with the actual solution $\omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon$ at infinity.

We need to observe that, in fact, the manifold constructed in Theorem 4.1 actually satisfy the conditions on Tian-Yau's Theorem (see [S1].

Clearly, it suffices to prove the asymptotic assertions on u_ε for a small tubular neighborhood of D in $\overline{M}$. Therefore, we will consider the equation

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_m\right)^n = e^{f_m}\omega_m^n \qquad \text{on } V_m \setminus D, \tag{19}$$

where f_m $||.||_m$ and ω_m are given by Proposition 4.1 and in the discussion that follows this proposition. We remind the reader that $|f_m|_{g_m}$ is of order of $O(||S||_m^m)$.

The proof of Theorem 5.1 is based on the very careful construction of a barrier function, that we state here for completeness.

Lemma 5.1 *On the neighborhood* $V_m \setminus D = \{0 < ||S||_m < \delta_m\}$, *we have*

$$\begin{aligned}
&\left\{\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(C\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^iS^j\bar{\theta}_{ij}\right](-n\log(||S||_m^2))^k\right)\right\}^n = \\
&= \omega_m^n\left[1 + C(-n\log(||S||_m^2))^{k-\frac{n+1}{n}}\left\{ij(-n\log(||S||_m^2))^2\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^iS^j\bar{\theta}_{ij}\right] - \right.\right. \\
&-(-n\log(||S||_m^2))\left[(k(i+j)+j(n-1))\,S^i\overline{S}^j\theta_{ij} + (k(i+j)+i(n-1))\,\overline{S}^iS^j\bar{\theta}_{ij}\right] \\
&\left.\left. + k(k-n)\right\} + O(||S||_m^{i+j+1})\right],
\end{aligned} \tag{20}$$

where θ_{ij} *is a* C^∞ *local section of* $L_D^{-i} \otimes \overline{L}_D^{-j}$ *on* V_m.

Using Lemma 5.1 and the maximum principle for the complex Monge-Ampère operator, we can show:

Proposition 5.1 *Let* u_m *be a solution to the Monge-Ampère equation (18). If* $u_m(x)$ *converges uniformly to zero as* x *approaches the divisor, then there exists a constant* $C = C(m)$ *such that*

$$|u_m(x)| \le C||S||_m^{m+1} \qquad \text{on } V_m \setminus D. \tag{21}$$

In order to complete the proof of Theorem 5.1, we need to show that, for a fixed $m > 2$, the solution $u_m(x)$ converges uniformly to zero as x approaches the divisor D.

The idea of this proof is to use again the maximum principle, with the new extra ingredient: as pointed out before, in [TY1], the solution u_m to the Monge-Ampère equation (11) is obtained as the uniform limit, as ε goes to *zero*, of solutions $u_{m,\varepsilon}$ of the perturbed Monge-Ampère equations

$$\begin{cases} \left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = e^{f_m+\varepsilon u}\omega_m^n, \\ \omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0, \qquad u \in C^\infty(M, \mathbb{R}). \end{cases} \tag{22}$$

It turns out ([S2], Proposition 5.2) that it is possible to construct a barrier function that is *uniform* on ε, so that the estimates will carry out to the limiting function u_m.

Finally, we remark that, by the work of Cheng and Yau [CY1], $u_{m,\varepsilon}$ vanishes uniformly as we approach D. Therefore we can apply the maximum principle to conclude that $u_m(x)$ converges uniformly to zero as x converges to D.

This concludes the proof of Theorem 5.1.

6 Further Developments

So far, we have been considering the problem of constructing Ricci-flat metrics on quasi-projective manifolds M that can be compactified by adding a smooth and *ample* divisor D.

The goal of this section is to show that the construction in Theorem 4.1 can be generalized to the case where the divisor D is a *semi-ample* divisor. The definition of a semi-ample line bundle/divisor is given a few paragraphs below.

Even though we still need to assume a few further conditions on D, this class of manifolds is quite large, including in particular the asymptotically cylindrical manifolds considered by Kovalev ([Kov]), described at the end of Section 3.

In a subsequent work, we expect to extend the existence results by Tian and Yau (Theorem 3.1) to asymptotically cylindrical manifolds, as well as to the case where D is allowed to have normal crossings. In those cases, we also expect to be able to determine the asymptotic behavior of the solutions.

Consider a compact, complex manifold $\overline{M}$ of dimension n, and let D be a smooth divisor on $\overline{M}$.

Let $L_D \to \overline{M}$ be the line bundle associated to D, and for $m \in \mathbb{N}$ sufficiently large, consider the linear system

$$|mL_D| = H^0(\overline{M}, mL_D),$$

the set of sections of mL_D. Assume that L_D is *semi-ample*, *ie*, the (*a priori* biholomorphic) map given by the linear system

$$\begin{aligned} |mL_D| \ : \ & \overline{M} \to \overline{M}_{can} \subset \mathbb{CP}^{N_m} \\ & p \mapsto [S_0^{(m)}(p), \cdots, S_{N_m}^{(m)}(p)] \end{aligned}$$

is actually a holomorphic map for $m >> 1$. Here, $\{S_0^{(m)}(p), \cdots, S_{N_m}^{(m)}(p)\}$ form a basis for $H^0(\overline{M}, mL_D)$.

Assuming that L_D is semi-ample implies that this line bundle is also *semi-positive*, *ie*, that there exists a hermitian metric $||.||$ on L_D such that its curvature form $\tilde{\omega}$ is semi-positive definite along D ($\tilde{\omega} \geq 0$). Assume further that $\tilde{\omega}$ has constant rank equals to k, $0 \leq k < n-1$. This implies that the Kodaira dimension of L_D is equal to k, *ie*, $\dim(\overline{M}_{can}) = k$.

Consider the restriction of $|mL_D|$ to D, and let $D_{can} \subset \mathbb{CP}^{N_m}$ be its image under $|mL_D|$. We are interested in the fibration $D \to D_{can}$, and we will refer to the fibers of this fibrations as D_s, for $s \in D_{can}$.

As before, consider a $(1,1)$-form $\Omega \in c_1(-K_{\overline{M}} - L_D)$. Our goal is to construct a complete metric g on $M = \overline{M} \setminus D$ such that $\mathrm{Ric}(g) - \Omega = \partial\bar{\partial} f$, for a function f with sufficiently fast decay.

Let ω_F be a $(1,1)$- form on D such that

1. $\mathrm{Ric}(\omega_F) \in [\Omega|_D]$

2. $\omega_F|_{D_s}$ is independent of s, for all s in $\{s \in D_{can}; D_s \text{ is non-singular}\}$.

We can pick ω_F satisfying those two conditions, since we can use Yau's Theorem on D, and $D \to D_{can}$ is a regular fibration away from the singular fibers, so all fibers are cohomologous.

If the regular fibers of $D \to D_{can}$ were assumed to be Calabi-Yau manifolds ($c_1(D) = 0$ and simply-connected), we could define the so-called *semi-flat* metrics (metrics that are Ricci-flat on regular fibers), considered by Gross and Wilson in [GW], for the 2-dimensional case (fibers being $K3$ surfaces).

Notice that ω_F is not necessarily a Kähler form anymore. In the sequel we will define the extension of ω_F (in the same cohomology class as ω_F) to a neighborhood of $D \in \overline{M}$. With a small abuse of notation, this extension will still be denoted by ω_F.

We define, for k =rank($\tilde{\omega}$),

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi}\frac{k^{1+1/k}}{k+1}\partial\bar{\partial}(-\log ||S||^2_\phi)^{\frac{k+1}{k}}. \tag{23}$$

$$\omega_\phi = (-k\log ||S||_\phi)^{\frac{1}{k}}\tilde{\omega}_\phi + (-k\log ||S||_\phi)^{\frac{1-k}{k}}\frac{\sqrt{-1}}{2\pi}\partial\log ||S||^2_\phi \wedge \bar{\partial}\log ||S||^2_\phi, \tag{24}$$

where $||.||_\phi = e^{\phi/2}||.||$ denotes a rescaling of the original metric $||.||$ and $\tilde{\omega}_\phi$ is the curvature form of the rescaled metric $||.||_\phi$.

From (24), we can see that ω_ϕ will have rank k near D, since $\tilde{\omega}_\phi$ has rank k along D, being a simple rescaling of $\tilde{\omega}$.

Now, we define

$$\eta_\phi = \omega_\phi + (-k\log ||S||_\phi)^{\frac{1}{k}}\omega_F, \tag{25}$$

that will be locally a Kähler form due to the observations above.

Note that

$$\eta^n_\phi = C(k,n)\omega^k_\phi \wedge (-k\log ||S||_\phi)^{\frac{n-k}{k}}\omega^{n-k}_F. \tag{26}$$

Define, analogously to the "D ample" case,

$$f_\phi(x) = -\log(||S||)^2 - \log\left(\frac{\eta^n_\phi}{\omega'}\right) - \Psi,$$

where ω' is any fixed Kähler form on $\overline{M}$ and it is related to the function Ψ by $\mathrm{Ric}(\omega') - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi$.

The key point is that the difference of two functions f_{ϕ_1} and f_{ϕ_2} will be only based on the term $\frac{\omega_{\phi_1}}{\omega_{\phi_2}}$, since the part involving ω_F will cancel out on the computations. Hence, we are able to prove an analogous lemma to [S2], Lemma 2.2, that is the main technical lemma that allow us to complete the proof of Theorem 4.1. Hence, since the computations carry out very similarly to the "D ample" case, we will simply state the final result, that extends one of the main results in [S2].

Theorem 6.1 *Let M be a quasi-projective manifold that can be compactified by adding a smooth, semi-ample divisor D, that further satisfy the conditions discussed above. Then, for every $\varepsilon > 0$, there exists an explicitly given complete Kähler metric g_ε such that*

$$Ric(g_\varepsilon) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f_\varepsilon \qquad \text{on } M, \tag{27}$$

where f_ε is a smooth function on M that decays to the order of $O(||S||^\varepsilon)$.

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Mathematical Sciences Research Institute
17 Gauss Way, room 217
Berkeley, CA 94720, USA
E-mail: bsantoro@msri.org