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MINIMAL SUBMANIFOLDS IN HIGHER CODIMENSION

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1 Introduction

In this series of lectures we will introduce methods for handling problems in Riemannian geometry involving curvature. These methods are especially effective in handling positive curvature, but they also motivate questions for minimal submanifolds in euclidean space. The theory of minimal hypersurfaces is particularly important for positive scalar curvature including questions in General Relativity (see [25] for a survey of this topic). Much of this paper concerns the second variation and variational existence questions in arbitrary codimension.

In Section 2 we introduce the basic ideas and consider questions involving sectional curvature and geodesics. We illustrate how lower estimates on the Morse index may be combined with existence theory to derive geometric conclusions. In addition to the case of closed geodesics and the fixed endpoint problem we also consider the free boundary problem for geodesics and some of its consequences.

In Section 3 we consider mainly the case of surfaces in arbitrary manifolds and show how the conditions of positive complex sectional curvature and PIC arise naturally from the second variation in complex form. We note that after these lectures were given, S. Brendle and the author ([4], [5]) were able to show that PIC is preserved by the Ricci flow and as a consequence to show that positive pointwise quarter-pinched manifolds are diffeomorphic to spherical space

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forms. The topological sphere theorem was proven under the PIC condition using minimal surface theory by Micallef and Moore [19]. We describe their work here.

In Section 4 we consider the variational theory for the volume among lagrangian surfaces in Kähler and symplectic manifolds. We pose the existence question for special lagrangian submanifolds of Calabi-Yau manifolds and for minimal lagrangian submanifolds in Kähler-Einstein manifolds. We describe some of the known results mainly for two dimensional lagrangian surfaces in two complex dimensional manifolds.

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2 The variational problems and the 1-dimensional case

Let (M^n, g) be a Riemannian manifold and let $\Sigma^k \subset M^n$ be a submanifold. We will denote by D the Levi-Civita connection of M which is characterized by the conditions

1. $D_X g = 0$;
2. $D_X Y - D_Y X = [X, Y]$;

for any vector fields X, Y. Let $\langle R(X,Y)Z, W\rangle$ be the Riemann curvature tensor, and

$$Ric(X,Y) = \sum_{i=1}^{n} \langle R(X, e_i)Y, e_i\rangle$$

be the Ricci tensor, where $\{e_1, \ldots, e_n\}$ is an arbitrary orthonormal frame. The scalar curvature is then defined by $R = \sum_{i=1}^{n} Ric(e_i, e_i)$.

Let ∇ denote the induced connection on Σ so that

$$\nabla_X Y = (D_X Y)^T$$

with $(\cdot)^T$ denoting the projection of a vector to the tangent space of Σ. Note that ∇ is then the Levi-Civita connection of (Σ, g) with respect to the induced metric. The second fundamental form h of Σ is then given by

$$h(X, Y) = (D_X Y)^N.$$

The mean curvature vector of $\Sigma \subset M$ is defined by

$$\vec{H} = Tr(h) = \sum_{i=1}^{k} h(e_i, e_i).$$

Let us now introduce the variational theory for the volume functional. This is defined through deformations of the submanifold. Let X be a vector field on the ambient space M, and let F_t be the flow generated by X. Define

$$\begin{aligned}\delta\Sigma(X) &= \frac{d}{dt} Vol(F_t(\Sigma))\Big|_{t=0}, \\ \delta^2\Sigma(X, X) &= \frac{d^2}{dt^2} Vol(F_t(\Sigma))\Big|_{t=0}.\end{aligned}$$

The first and second variation formulas express these in terms of the geometry of Σ and M. Precisely we have

First Variation Formula:

$$\delta\Sigma(X) = \int_\Sigma div_\Sigma(X) d\mu,$$

where

$$div_\Sigma(X) = \sum_{i=1}^{k} \langle D_{e_i} X, e_i \rangle.$$

Note that if Σ is smooth, we have

$$\begin{aligned}div_\Sigma(X) &= div(X^T) + \langle D_{e_i} X^N, e_i \rangle \\ &= div(X^T) - \langle \vec{H}, X \rangle.\end{aligned}$$

Second Variation Formula:

$$\begin{aligned}
&\delta^2\Sigma(X,X)\\
&=\int_\Sigma\Big(\sum_{i=1}^k|D_{e_i}^\perp X|^2+div_\Sigma(D_XX)-\sum_{i=1}^k R^M(e_i,X,e_i,X)\\
&\qquad+\Big(\sum_{i=1}^k\langle D_{e_i}X,e_i\rangle\Big)^2-\sum_{i,j=1}^k\langle D_{e_i}X,e_j\rangle\langle D_{e_j}X,e_i\rangle\Big)d\mu
\end{aligned}$$

where $(\cdot)^\perp$ denotes projection to the normal space of Σ. For simplicity, we can assume that Σ is smooth, $X^T=0$, and $\vec{H}=0$; that is, we are assuming that Σ is a smooth critical point of the volume and that X is a normal vector field. The condition that Σ is smooth can be a very serious condition, but under the assumption that Σ is smooth, there is no loss of generality in taking X to be normal if it is normal on $\partial\Sigma$ (or of compact support). In this case we have

$$\begin{aligned}
&\delta^2\Sigma(X,X)\\
&=\int_\Sigma\Big(|D^\perp X|^2-\sum_{i=1}^k R^M(e_i,X,e_i,X)-|\langle h,X\rangle|^2\Big)d\mu\\
&\qquad+\int_{\partial\Sigma}\langle D_XX,\nu\rangle d\sigma,
\end{aligned}$$

where ν is the outer conormal vector; that is, ν is the outer unit normal vector to $\partial\Sigma$ tangent to Σ.

The nature of the Plateau problem depends very much on the class of competitors one allows in the problem. An important aspect of this choice is the boundary condition that one considers. We discuss here three boundary conditions which commonly arise.

1) Plateau boundary condition:

This is the most classical boundary condition, physically representng the soap film problem. Let $\Gamma^{k-1}\subset M^n$ be an oriented boundary, and choose Σ^k to be a minimum (or critical point) for the volume functional among all Σ_0 with $\partial\Sigma_0=\Gamma$. If we assume that Σ is smooth, by the first variation formula we have

$$\int_\Sigma\langle X,\vec{H}\rangle d\mu=0$$

for every vector field X satisfying $X = 0$ on $\partial\Sigma = \Gamma$. Therefore we find that $\vec{H} = 0$, and Σ is a minimal submanifold. The details of this problem depend on the class of competing submanifolds; for example, one might require Σ to be orientable or of a fixed topological type.

2) Homology condition:

Given $\alpha \in H_k(M, \mathbb{Z})$, choose Σ^k to be a minimum (or critical point) for the volume functional among all integral cycles representing α. The first variation formula again implies $\vec{H} = 0$. It is actually possible to solve the homotopy problem in certain cases, particularly when $k = 1, 2$.

3) Free boundary condition:

Let $N^l \subset M$ be a given submanifold. Choose Σ^k to be a minimum (or critical point) for the volume functional among all Σ_0 with $\partial\Sigma_0 \subset N$. The first variation formula implies $\delta\Sigma(X) = 0$ for all X with $F_t(N) \subset N$. It follows that $\vec{H} = 0$ and

$$\int_{\partial\Sigma} \langle X, \nu \rangle ds = 0$$

for all X tangent to N along $\partial\Sigma$. Therefore Σ satisfies the *free boundary condition* $\nu \perp TN$.

Morse index and eigenvalues of δ^2:

Assume Σ is smooth and compact. If X has compact support and is normal to Σ, integration by parts yields

$$\delta^2\Sigma(X, X) = -\int_{\Sigma} \langle \mathcal{L}X, X \rangle d\mu,$$

where

$$\mathcal{L}X = \Delta^{\perp}X + \sum_{i=1}^{k} R^M(e_i, X)e_i + \sum_{i,j} \langle h_{ij}, X \rangle h_{ij}$$

is the Jacobi operator (here $\Delta^{\perp}$ denotes the Laplace operator on normal vector fields with respect to the induced normal connection). This is an elliptic operator on normal sections which represents the linearization of the nonlinear operator $\vec{H}$ at Σ.

The boundary conditions which correspond to the problems above are:

1. *Plateau* $\Rightarrow$ *Dirichlet*;

2. *Homology condition* $\Rightarrow$ *No boundary condition*;

3. *Free boundary* $\Rightarrow \{$ X tangent to N and $(D_\nu X - D_X \nu)^{TN} = 0\}$

where $(\cdot)^{TN}$ denotes the projection of a vector to the tangent space of N. Each of these is an elliptic boundary condition and so the spectrum of $\mathcal{L}$ is discrete with the following behavior in all three cases:

$$\lambda_1 \leq \lambda_2 \leq \ldots \lambda_m \to \infty.$$

Furthermore, there is an L^2 orthonormal basis of eigenfunctions. We define the (Morse) **index** to be the number of negative eigenvalues counted with multiplicity. We say that Σ is **stable** if the index is zero.

A key result in the application of the variational theory to Riemannian geometry is a lower bound on the index under suitable geometric conditions.

Geometric Index Estimates :

In this subsection we will assume $k = 1$.

E1) Bonnet-Myers:

If $K(M) \geq \kappa > 0$ (sectional curvature) and γ is a geodesic with length $L(\gamma) > \frac{\pi}{\sqrt{\kappa}}$, then

$$Ind(\gamma) \geq n - 1.$$

We should note that this corresponds to a Dirichlet boundary condition. It is saying that any geodesic which is sufficiently long has to have relatively high index. If $Ric \geq (n-1)\kappa$, then one can prove $Ind(\gamma) \geq 1$.

E2) Synge:

Suppose $K(M) > 0$, n is even, and M is orientable. Then $Ind(\gamma) \geq 1$ for any closed geodesic γ.

E3) Frankel:

Let M be compact with nonempty boundary. Suppose $K(M) \geq 0$ and the boundary ∂M is p-convex, for some $1 \leq p \leq n-1$. Then any geodesic segment γ, orthogonal to ∂M at the ends, satisfies

$$Ind(\gamma) \geq n - p$$

where the index is taken for the free boundary condition relative to $N = \partial M$. Recall that ∂M is **p-convex** if the sum of its lowest p principal curvatures (with respect to the inner normal) is positive. If $p = 1$ we say ∂M is **convex**, while if $p = n-1$ we say that ∂M is **mean convex**. The proofs of the first two of these are well known and appear in standard differential geometry texts (see M. doCarmo [7]). We give the proof of the third.

Proof of E3: Let $\gamma(s)$, $0 \leq s \leq l$, be a geodesic segment with $\gamma(0), \gamma(l) \in \partial M$, and $\gamma'(0), \gamma'(l) \perp \partial M$. Given $v \in T_{\gamma(0)}M$ with $\langle v, \gamma'(0)\rangle = 0$, we may parallel transport it along γ to get $V(s)$. Note that $V(0), V(l)$ are tangent to ∂M, since they are orthogonal to γ. Let $\mathcal{V}$ be the $(n-1)$-dimensional space of such normal parallel vector fields. Any such vector field is a valid variation, and we have

$$\delta^2\gamma(V,V) = \int_0^l -R^M(\gamma', V, \gamma', V)ds - h_{\gamma(l)}(V,V) - h_{\gamma(0)}(V,V),$$

where $V \in \mathcal{V}$ and h is the second fundamental form of ∂M with respect to the inner normal. Let $\{V_1, \ldots, V_{n-1}\}$ be an orthonormal basis of $\mathcal{V}$ formed by eigenvectors of the quadratic form $\delta^2\gamma$ restricted to $\mathcal{V}$. Assume its eigenvalues are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. Using the curvature condition and the p-convexity assumption we have

$$\sum_{i=n-p}^{n-1} \delta^2\gamma(V_i, V_i) = \sum_{i=n-p}^{n-1} \lambda_i < 0.$$

It follows that $\lambda_{n-p} < 0$. Therefore $\delta^2\gamma$ is negative definite on the span of $V_1, \ldots, V_{n-p}$, and hence $Ind(\gamma) \geq n - p$.

□

Remark: The condition $K(M) \geq 0$ may be replaced by the requirement $\sum_{i=1}^{p} R(e_i, u, e_i, u) \geq 0$ for any unit vector u and any orthonormal set $\{e_1, \ldots, e_p\}$ orthogonal to u. If $p = n-1$, this is equivalent to requiring $Ric(M) \geq 0$.

Role of existence theory: In order to give applications of these index estimates we need to combine them with the existence theory. This theory is relatively standard for $k = 1$.

Application 1:

i) Suppose M is complete and $Ric(M) \geq (n-1)\kappa > 0$. Then M is compact and $diam(M) \leq \frac{\pi}{\sqrt{\kappa}}$.

Proof: Let $p, q \in M$. Then there exists a minimizing geodesic γ from p to q. This geodesic solves the Plateau problem for one-dimensional curves, and its existence follows from the Hopf-Rinow theorem. Since $Ind(\gamma) = 0$, it follows from (E1) that $L(\gamma) \leq \frac{\pi}{\sqrt{\kappa}}$. Since the points were arbitrary this gives the estimate on the diameter.

ii) Assume M is simply connected, and $1 < K(M)$. If $inj(M) > \frac{\pi}{2}$, then M is a homotopy sphere.

Proof: Let $C^0(S^1, M)$ denote the space of all continuous maps from the circle S^1 to M. There is a topological result which says that

$$\pi_{p-1}(C^0(S^1, M)) = \pi_p(M),$$

for all $p \geq 2$. To understand this result one can consider a map from $\mathbb{S}^p$ to M which does not extend to a continuous map of B^{p+1}. The sphere $\mathbb{S}^p$ may be thought of as being swept out by a family of circles parametrized on S^{p-1} (think of the case $p = 2$). Mapping this into M then gives a map from $\pi_p(M)$ to $\pi_{p-1}(C^0(S^1, M))$ which turns out to be an isomorphism.

Since $inj(M) > \frac{\pi}{2}$, any closed geodesic γ satisfies $L(\gamma) > \pi$. It follows from (E1) that $Ind(\gamma) \geq n-1$. We also have that, if $\pi_p(M) \neq 0$, then there exists a closed geodesic γ with $Ind(\gamma) \leq p-1$. Therefore we must have $p \geq n$. Hence

$\pi_1(M) = \cdots = \pi_{n-1}(M) = 0$, and M is a homotopy sphere.

Remark: The bound assumed on the injectivity radius was proven under the assumption of 1/4-pinched curvature by W. Klingenberg (see [7] for discussion). That is, under the assumption that $1 < K(M) \leq 4$ it was shown that $inj(M) > \frac{\pi}{2}$. This was the main step in the proof of the topological sphere theorem.

Application 2:

Suppose M^n is compact, orientable, with $K(M) > 0$. If n is even, then M is simply connected.

Proof: This is an application of the variational theory. We will argue by contradiction. Suppose $\pi_1(M) \neq 0$, then there exists a closed geodesic γ_c minimizing length in every nontrivial free homotopy class $c \in \pi_1(M)$. Therefore $Ind(\gamma_c) = 0$, which is in contradiction with (E2).

Application 3:

Let $(M, \partial M)$ be compact, $K(M) \geq 0$, so that ∂M is p-convex. Then $\pi_i(M, \partial M) = 0$ for all $1 \leq i \leq n - p$. (see Lawson [13] for an application of the case p=n-1 , Mercuri-Noronha [17], and J.Sha [27], H.Wu [30]).

Proof: We will use the following existence theorem which is analogous to the one above: if $\pi_i(M, \partial M) \neq 0$, then there exists a geodesic arc γ with end points on ∂M of index (for the free boundary problem) at most $i - 1$. It follows from (E3) that $n - p \leq i - 1$. Hence $\pi_i(M, \partial M) = 0$ for all $1 \leq i \leq n - p$.

The third application used an existence theorem which is not completely standard. We now state that result.

Theorem 2.1. *If $i \geq 2$, and $\alpha \in \pi_{i-1}\Big(C^0([0,1], (M, \partial M))\Big)$, then*

$$L = \inf_{f \in \alpha} \max\{L(f(t) \cap int(M)) : t \in S^{i-1}\}$$

is a critical value which is realized by a union $\gamma_1 \cup \cdots \cup \gamma_s$ of geodesic arcs in M with free boundary condition at ∂M and with total (free) index at most $i - 1$.

Remark: In the previous theorem, easy examples show that we must allow competing curves to have segments lying inside ∂M, but we only count length interior to M (thus we think of the metric being zero on ∂M). If we instead count the total length, we have an "obstacle" problem and critical curves would not satisfy a free boundary condition, but instead would meet ∂M tangentially and would contain segments in ∂M.

Problem: Generalize the previous theorem to submanifolds of higher dimension. For example the critical points for the area functional in $\pi_2(M, \partial M)$ should be represented by genus 0 minimal surfaces with free boundary condition. The work of F. Almgren [1] gives a very general setting for this problem.

Index estimates in special manifolds:

We now digress to discuss index estimates which have been obtained for special ambient manifolds using the geometry of the ambient space. In a famous paper of the late 60's, J. Simons [28] considered minimal submanifolds in manifolds with some special structure. He proved that if $\Sigma^k \subset S^n$ is a minimal submanifold, then $Ind(\Sigma) \geq 1$. If $\Sigma^k \subset \mathbb{C}P^n$ is stable ($Ind(\Sigma) = 0$), then Σ is holomorphic (or anti-holomorphic). There is a generalization by Lawson-Simons [14] which allows singular Σ.

The idea of the proofs of these theorems is to make use of special ambient vector fields. For instance, if $\Sigma^k \subset S^n \subset \mathbb{R}^{n+1}$, one considers the tangential projections $E_1, \ldots, E_{n+1}$ to S^n of an orthonormal basis $e_1, \ldots, e_{n+1}$ for $\mathbb{R}^{n+1}$. Then, after some computation one proves that

$$\sum_{i=1}^{n+1} \delta^2 \Sigma(E_i, E_i) < 0,$$

which implies $Ind(\Sigma) \geq 1$.

Extensions to free boundary problem:

There is an extension of this idea to the free boundary problem by A. Fraser [8] in case the manifold is the standard ball and $k \geq 1$. Moore and Schulte [22] used a similar idea in the case $k = 2$ for much more general M. It was observed by Fraser [10] that their result can easily be generalized to arbitrary k. We describe that result here.

Theorem 2.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let $\Sigma^k \subset \Omega \subset \mathbb{R}^n$ be a minimal submanifold satisfying the free boundary condition. If $\partial\Omega$ is p-convex, and $k \leq n - p$, then $Ind(\Sigma) \geq 1$.*

Proof: Let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathbb{R}^n$ and let $V_i = e_i^{\perp}$ be the projection normal to Σ. Since the V_i are perpendicular to Σ they are tangential to $\partial\Omega$ and hence are valid variations for the free boundary problem. We then have

$$\delta^2\Sigma(V_i, V_i) = \int_{\Sigma}(|D^{\perp}V_i|^2 - |D^T V_i|^2)\, d\mu - \int_{\partial\Sigma} h_{\partial\Omega}(V_i, V_i)\, d\sigma$$

where $h_{\partial\Omega}$ denotes the second fundamental form of $\partial\Omega$ with respect to the inner unit normal. Since $\sum_{i=1}^n |D^{\perp}V_i|^2$ and $\sum_{i=1}^n |D^T V_i|^2$ are independent of the orthonormal basis $e_1, \ldots, e_n$, we may assume that $e_1, \ldots, e_k$ are tangent to Σ at a point. If $v_1, \ldots, v_k$ form a local tangent orthonormal basis near this point with $v_i = e_i$ at the point, we have $V_i = e_i - \sum_{\alpha=1}^k (e_i \cdot v_\alpha)v_\alpha$ and

$$DV_i = -\sum_{\alpha=1}^{k}[(e_i \cdot D^{\perp}v_\alpha)v_\alpha + (e_i \cdot v_\alpha)D^{\perp}v_\alpha]$$

at the point. It follows that at the point

$$\begin{aligned}\sum_i |D^{\perp}V_i|^2 &= |h_\Sigma|^2,\\ \sum_i |D^T V_i|^2 &= |h_\Sigma|^2.\end{aligned}$$

The trace then reduces to a boundary term:

$$\sum_{i=1}^{n} \delta^2\Sigma(V_i, V_i) = -\int_{\partial\Sigma} \sum_{i=1}^{n} h_{\partial\Omega}(V_i, V_i) d\sigma.$$

Since $\sum_{i=1}^n h_{\partial\Omega}(V_i, V_i)$ does not depend on the basis, we may assume $e_1, \ldots, e_{n-k}$ are orthogonal to Σ at a point. Thus $V_i = 0$ for $i = n - k + 1, \ldots, n$ at a point.

If $n-k \geq p$, it follows from p-convexity that $\sum_{i=1}^{n-k} h_{\partial\Omega}(e_i, e_i) > 0$. Thus we have $\sum_{i=1}^{n} \delta^2\Sigma(V_i, V_i) < 0$ and $Ind(\Sigma) \geq 1$ if $k \leq n-p$.

□

Remarks: 1) This result should extend to singular Σ (integral currents), and as a consequence of the work of Almgren [1] it implies $H_k(\Omega, \partial\Omega, \mathbb{Z}) = 0$ for all $1 \leq k \leq n-p$. This was done in the case of B^n in [8].

2) We remark that Mercuri-Noronha [17] obtained these topological conclusions by other methods (Morse theory for height functions).

3 Two dimensional surfaces in arbitrary codimension

We now discuss methods which are special to the case $k = 2$. Recall that if Σ is oriented then the induced metric on Σ determines a complex structure on Σ making it into a Riemann surface. It is natural to exploit this special structure by complexifying the second variation quadratic form. We will denote by $T^{\mathbb{C}}M$ the complexification of TM. We can extend the notions of inner product $\langle X, Y\rangle$ and curvature $R(X, Y, Z, W)$ so that they are complex linear in each slot. Given X, Y satisfying $\langle X, \overline{Y}\rangle = 0$, $\langle X, \overline{X}\rangle = \langle Y, \overline{Y}\rangle = 1$, the complex sectional curvature of the plane spanned by X, Y is defined by

$$K(X, Y) = R(X, Y, \overline{X}, \overline{Y}) \in \mathbb{R}.$$

We say that $X \in T_p^{\mathbb{C}}M$ is *isotropic* if $\langle X, X\rangle = 0$. Equivalently, X is isotropic if and only if $\langle ReX, ImX\rangle = 0$ and $|ReX|^2 = |ImX|^2$. A 2-plane π is *isotropic* if all vectors in it are isotropic. An isotropic plane Π is determined by an orthonormal set $\{e_1, e_2, e_3, e_4\}$ of real vectors, where

$$\Pi = span\{e_1 + ie_2, e_3 + ie_4\}.$$

The complexification leads to two distinct positivity conditions. We say that M has *positive complex sectional curvatures* if the curvature of each complex plane is positive at all points of M. A more important and weaker condition is the PIC condition. We say that M has *positive isotropic curvature* (PIC) if $K(\Pi) > 0$ for every isotropic 2-plane Π. An important special case is when $K(\Pi) \geq \kappa > 0$.

The complex sectional curvature comes out naturally in the stability analysis for surfaces as we describe below. The relation between complex or isotropic curvature and surfaces is analogous to the relation between sectional curvature and geodesics. In real terms the PIC condition says that for any orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ we have the inequality

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2\,R_{1234} > 0.$$

The PIC condition was defined by M. Micallef and J.D. Moore [19] and used to prove a strong generalization of the topological sphere theorem. Here are a few facts to illustrate the importance of the PIC condition. The first two of these are done in [19].

1. Positive pointwise $\frac{1}{4}$-pinched curvature ($0 < \kappa(p) < K_p(M) \leq 4\kappa(p)$ for some positive and continuous function κ on M) $\Rightarrow$ PIC;

2. Positive curvature operator (acting on 2-forms) $\Rightarrow$ PIC;

3. PIC manifolds can have large fundamental group: M_1,M_2 have PIC metrics $\Rightarrow M_1 \# M_2$ has a PIC metric (see [20]).

Example (M. Micallef, M. Wang [20]): The manifolds $(S^{n-1}\times S^1)\#\cdots\#(S^{n-1}\times S^1)$ all have (PIC) metrics.

It is a natural question to classify the manifolds which admit PIC metrics. There are no obstructions in dimensions 2 and 3 since in these dimensions every Riemannian metric is PIC. In dimension 4, there is a partial classification due to R. Hamilton [12] which uses the Ricci flow. One might conjecture that any compact PIC manifold is finitely covered by a manifold which is diffeomorphic to a connected sum of copies of $S^{n-1} \times S^1$.

Let us now return to minimal surfaces and index estimates. The idea here is to prove results which are analogous to the ones for geodesics. It turns out the diameter is not the right thing to estimate. The following conjecture is an analogue to the theorem of Bonnet-Myers.

Conjecture E1': Suppose Σ^2 is a stable minimal disk in M with $K(\pi) \geq \kappa > 0$ for all isotropic planes π. Then there is a constant c depending only on n such that

$$d(x, \partial\Sigma) \leq \frac{c}{\sqrt{\kappa}},$$

for every $x \in \Sigma$.

The following theorem is an analogue of E2 for geodesics.

Theorem 3.1. *(Micallef-Moore) Let M^n be a (PIC) manifold, and suppose $\Sigma^2 \subset M$ is a minimal surface diffeomorphic to S^2. Then*

$$Ind(\Sigma) \geq [\frac{n-2}{2}]$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof: (Outline) In the case of geodesics we used parallel vector fields in an essential way. For $k \geq 2$ there will not exist any such normal vector fields in general. The idea for $k = 2$ is to use "holomorphic" variations in place of parallel ones.

First we complexify the quadratic form $\delta^2\Sigma$. If X is a complex normal vector field, then

$$\delta^2\Sigma(X, \overline{X}) = \delta^2\Sigma(ReX, ReX) + \delta^2\Sigma(ImX, ImX)$$

which is simply the sum of the second variation for two real variations.

Assume Σ^2 is orientable and choose a complex structure on Σ such that $u : \Sigma \to M$ is conformal. Then write the normal covariant derivative $D^\perp X = \partial^\perp X + \overline{\partial}^\perp X$ in terms of its $(1,0)$ and $(0,1)$ parts. Integrate by parts to get rid of the $|\partial^\perp X|^2$ term and we obtain (see [19] for details)

$$\delta^2\Sigma(X,\overline{X}) = -\int_\Sigma \Big(R(X,\frac{\partial u}{\partial z},\overline{X},\overline{\frac{\partial u}{\partial z}}) + |D^T_{\frac{\partial}{\partial z}}X|^2 - |D^\perp_{\frac{\partial}{\partial \overline{z}}}X|^2\Big)dxdy,$$

where X is a complex normal vector field.

Now we want to solve the Cauchy-Riemann equations $D^\perp_{\frac{\partial}{\partial \overline{z}}}X = 0$. Since the almost complex structure defined by the normal connection is integrable over a surface, we may think of the complexified normal bundle as a holomorphic $(n-2)$-plane bundle E over the Riemann surface Σ. In the case that $\Sigma \approx S^2$, it follows that $E = \bigoplus_{j=1}^{n-2} E_j$, where $E_1, \dots, E_{n-2}$ are line bundles. The first Chern class $c_1(E_j)$ is nonnegative if and only if E_j admits a holomorphic section. Since $E^* \approx E$ (using the pairing $\langle , \rangle$), it follows that at least half of E_j satisfy $c_1(E_j) \geq 0$.

Note that if X_1, X_2 are holomorphic, then $\langle X_1, X_2\rangle = const$ since it is a holomorphic function on Σ.

By linear algebra there exists an isotropic space of dimension at least $p = [\frac{n-2}{2}]$ consisting of holomorphic sections. For each vector X in this space, either the real or the imaginary part of X has $\delta^2\Sigma < 0$. It follows that there exists a p-dimensional space of real deformations on which $\delta^2\Sigma < 0$. We have thus proved $Ind(\Sigma) \geq [\frac{n-2}{2}]$.

□

It is now natural to ask what happens for $genus(\Sigma) \geq 1$. The following example shows that the same result is no longer true.

Example: The product metric on $RP^3 \times S^1$ is PIC, and since RP^3 has a minimizing closed geodesic, it follows that this geodesic crossed with the S^1 factor is a stable minimal torus. Thus PIC manifolds may contain minimal tori T^2 with $Ind(T^2) = 0$.

We have the following theorem, due to A. Fraser [9] in the genus 1 case, and to Fraser-Wolfson [11] if $genus(\Sigma) > 1$.

Theorem 3.2. *If Σ is a minimal surface in a PIC manifold M^n with $genus(\Sigma) \geq$*

1, *then a finite cover of* Σ *is unstable; that is, there is a finite cover* $\hat{\Sigma}$ *with* $Ind(\hat{\Sigma}) \geq 1$.

Proof: (Outline) Construct a finite cover $\hat{\Sigma}$ and a map $F : \hat{\Sigma} \to S^2$ with $|dF| < \varepsilon$, ε small. Consider the holomorphic vector bundle $E \to \hat{\Sigma}$ arising from the complexified normal bundle, and let $\xi = F^*(\xi_0)$, where ξ_0 is a sufficiently positive line bundle over S^2.

Consider $E \otimes \xi$ and apply Riemann-Roch to find a holomorphic section X_1. Apply a controlled element $l \in \xi^*$ to get a section $X = l(X_1)$ which is "almost holomorphic" in the sense that

$$\int_{\hat{\Sigma}} |D^{\perp}_{\frac{\partial}{\partial \bar{z}}} X|^2 dxdy < \varepsilon_1 \int_{\hat{\Sigma}} |X|^2 d\mu.$$

Since M is PIC, it then follows from the second variation formula that $Ind(\hat{\Sigma}) \geq 1$ since either the real or imaginary part of X_1 yields negative second variation for ϵ_1 sufficiently small.

□

E3'. Free boundary problem. The following result of [10] gives the same estimate as in the Micallef-Moore theorem.

Theorem 3.3. *Let* $\Sigma^2 \subset \Omega$ *be a minimal surface topologically equivalent to the unit disk such that* $\partial\Sigma \subset \partial\Omega$. *If* $\partial\Omega$ *is 2-convex and* Ω *has nonnegative isotropic curvature , then*

$$Ind(\Sigma) \geq [\frac{n-2}{2}].$$

The following result of [10] gives a better estimate (sharp in $\mathbb{R}^n$), although under a less interesting geometric condition.

Theorem 3.4. *Let* $\Sigma \subset \Omega$ *be as in the previous theorem. If* $\partial\Omega$ *is p-convex and* Ω *has nonnegative complex sectional curvature, then*

$$Ind(\Sigma) \geq n - p - 1.$$

There is a key new ingredient which is used in the proofs of these theorems to handle the boundary condition. The classical Riemann-Hilbert boundary value problem concerns holomorphic functions in the disk which are real on the boundary. As a vector-valued generalization of this it is shown in [8] that there is an $(n-2)$-dimensional space of holomorphic sections which are real on $\partial\Sigma$. We may choose a basis $V_1, \ldots, V_{n-2}$ such that $V_j\Big|_{\partial\Sigma}$, $1 \le j \le n-2$, form an orthonormal basis for $(T\Sigma)^\perp$ at each point of $\partial\Sigma$. The reason this is possible is that (V_i, V_j) is constant, since it is a holomorphic function real on the boundary. The existence result then follows from taking linear combinations.

Remark. It is not known whether the Micallef-Moore index estimate or Theorem 3.3 is sharp in the PIC case.

Existence Theory and Applications

We will now discuss how index estimates imply geometric and topological conclusions. The next result is the main application of the Micallef-Moore theorem.

Theorem 3.5 (Micallef-Moore). *Let M^n be a compact PIC manifold. Then*

$$\pi_i(M) = 0$$

for all $i = 2, \ldots, [\frac{n}{2}]$.

Corollary 3.6. *Let M^n be a simply connected compact PIC manifold. Then M^n is a homotopy sphere.*

Note that this implies M is homeomorphic to S^n if $n \ge 4$.

Proof: First

$$\pi_i(M) \approx \pi_{i-2}\Big(C^0(S^2, M)\Big)$$

for $i \ge 2$ by similar reasoning as in the S^1 case.

It follows from an important paper on existence theory due to Sacks and Uhlenbeck [24] that if $\pi_i(M) \neq 0$, then there exists a minimal sphere Σ^2 with

$Ind(\Sigma) \leq i-2$. Hence $[\frac{n}{2}]-1 \leq i-2$ or $i \geq [\frac{n}{2}]+1$. This shows that $\pi_i(M)=0$ for $i \leq [\frac{n}{2}]$ as claimed.

□

Results on the topology of PIC manifolds.

The next two conjectures are based on the known examples and results in the $n=4$ case as well as results on the fundamental group which we will describe.

Conjecture 1: If M^n is a compact PIC manifold, then a finite covering $\hat{M}$ of M is diffeomeorphic either to S^n or to a connected sum of copies of $S^{n-1}\times S^1$.

Conjecture 2: If M^n is a compact PIC manifold with infinite fundamental group, then $\pi_1(M)$ is "virtually free" in the sense that there exists a free subgroup of finite index in $\pi_1(M)$.

The following result supports Conjecture 2.

Theorem 3.7. *([9], [11]) $\pi_1(M)$ does not contain any surface group; that is a subgroup isomorphic to the fundamental group of a surface of genus at least* 1.

Proof: If $\Sigma \to M$ induces an injective homomorphism on π_1, then we can make Σ minimal. Arrange a covering

$$\hat{\Sigma} \to \hat{M} \to M$$

which is stable and with $F : \hat{\Sigma} \to S^2$, $|dF| < \varepsilon$. The argument given above then implies that $\hat{\Sigma}$ is unstable, a contradiction.

□

The next application of [10] is parallel to the Micallef-Moore theorem.

Theorem 3.8. *Suppose $(M, \partial M)$ is compact, PIC, with ∂M 2-convex. Then $\pi_i(M, \partial M) = 0$ for $i = 2, \ldots, [\frac{n}{2}]$.*

Corollary 3.9. *If $\pi_1(M) = \pi_1(M, \partial M) = \{1\}$, then M is contractible.*

Proof: Fraser [8] extended Sacks and Uhlenbeck's existence result to 2-convex domains (∂M is a barrier for two-dimensional minimal surfaces) for relative homotopy

$$\pi_i(M, \partial M) = \pi_{i-2}\Big(C((D, \partial D), (M, \partial M))\Big).$$

If $\pi_i \neq 0$ then there exists Σ either a compact minimal sphere or a minimal disk with free boundary condition and such that $Ind(\Sigma) \leq i-2$. The result follows then from respectively the Micallef-Moore or the Fraser index estimate.

□

Stability of coverings.

The ideas developed above suggest a more general question about stability which we now briefly discuss.

Question: If $\Sigma^k \subset M^n$ is stable and $\hat{\Sigma} \to \Sigma$ is a covering (either in M or in a covering $\hat{M}$), is $\hat{\Sigma}$ stable?

The answer is no, not always. A counterexample is given by a geodesic $S^1 \subset RP^2$ of least length, which lifts to an equator of S^2. There are two general classes for which the result does hold:

1) If $\Sigma^{n-1} \subset M^n$ is a 2-sided hypersurface, the Jacobi operator $\mathcal{L}$ is a scalar operator and stability is equivalent to the existence of $u > 0$ with $\mathcal{L}(u) \leq 0$. These functions can be lifted to coverings, and therefore stability is preserved under coverings.

2) If Σ is calibrated, then $\hat{\Sigma}$ is calibrated. Recall that a k-form θ is a *calibration* if $d\theta = 0$ and $|\theta_p(\xi_p)| \leq 1$ for every simple k-vector ξ_p with $|\xi_p| = 1$. The submanifold Σ^k is *calibrated* by θ if $\theta\Big|_\Sigma$ is the volume form. The basic theorem of calibrations is the following.

Theorem 3.10. *If Σ is calibrated, then Σ minimizes volume for its boundary and relative homology class.*

Proof: If $\partial\Sigma = \partial\Sigma_1$ and $\Sigma - \Sigma_1 = \partial T^{k+1}$, then Stokes theorem implies

$$\int_\Sigma \theta = \int_{\Sigma_1} \theta,$$

since $d\theta = 0$.

The inequality $Vol(\Sigma) \leq Vol(\Sigma_1)$ follows from the definition of calibrations.

□

Generally we will say that a minimal submanifold Σ is *covering stable* if Σ as well as any covering $\hat{\Sigma}$ is stable. The results described above for PIC manifolds may be reformulated to say that there is no compact covering stable minimal surface in a PIC manifold.

Conjectures:

1) If $\Sigma^2 \subset \mathbb{R}^n$ is complete, of finite total curvature, and covering stable, then Σ is holomorphic with respect to a constant complex structure J on $\mathbb{R}^{2p}$ such that $\Sigma \subset \mathbb{R}^{2p} \subset \mathbb{R}^n$.

Remark. Under the assumption of stability, this is true if $genus(\Sigma) = 0$. It is also true in dimension 4 (see Micallef [18]), but there is a counterexample of $genus(\Sigma) = 1$ in dimension 21 due to Arezzo, Micallef, and Pirola [3].

2) Let $\Sigma^2 \subset T^n$, flat. If Σ is compact, oriented, covering stable, then Σ is holomorphic for a constant complex structure on T^{2p}, where $\Sigma \subset T^{2p} \subset T^n$.

Remark. This is true if $n = 4$ (Micallef [18]) and false with only the assumption of stability in high dimensions [2].

Theorems such as the above play a role analogous to the Bernstein Theorem in codimension 1. Note the parallel between these conjectures and results with those described for PIC manifolds. Micallef's result for $genus(\Sigma) = 0$ uses a very similar argument as that in [19].

4 Lagrangian submanifolds

Let g be a Riemannian metric on M^{2n}, and let $J : TM \to TM$ be a linear endomorphism satisfying $J^2 = -id$ and $g(JX, JY) = g(X, Y)$. In this case we say that (M^{2n}, g, J) is an *almost Hermitian* manifold. We can define a 2-form ω by $\omega(X, Y) = g(X, JY)$, and we say the manifold is *almost Kähler* if $d\omega = 0$. The manifold is *Kähler* if $DJ = 0$, where D denotes the Levi-Civita connection.

Flat case: An example of a Kähler manifold is given by the Euclidean space $(\mathbb{R}^{2n}, g, J)$ with

$$\begin{aligned} g &= dx_1^2 + \cdots + dx_n^2 + dy_1^2 + \cdots + dy_n^2, \\ J(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial y_i}, \\ J(\frac{\partial}{\partial y_i}) &= -\frac{\partial}{\partial x_i}, \end{aligned}$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ denote standard coordinates on $\mathbb{R}^{2n}$. In this example $\omega = \sum_{j=1}^n dx_j \wedge dy_j$ is the standard symplectic form.

Definition: A submanifold $\Sigma^n \subset M^{2n}$ is *Lagrangian* if $\omega\Big|_\Sigma \equiv 0$.

This is equivalent to the condition $J(T_x\Sigma) = (T_x\Sigma)^\perp$ for all $x \in \Sigma$, so that the tangent space is canonically isometric to the normal space.

Local Description: Assume that Σ is given as a graph $y_j = F_j(x)$. It then follows that Σ is Lagrangian if and only if $\sum_j F_j dx_j$ is a closed 1-form. This is equivalent to saying that locally there exists a potential function $u(x)$ such that Σ is described by $y_j = \frac{\partial u}{\partial x_j}$.

Variational Problem: Seek lagrangian submanifolds which minimize volume among "all" lagrangian competitors which have the same boundary, or which agree outside a compact set. The Euler-Lagrange equation for this problem can be described as follows. If Σ is a graph $y = \frac{\partial u}{\partial x}$ over $\Omega \subset \mathbb{R}^n$, and we look at the variational problem with respect to a Dirichlet boundary condition, then we have

$$|\Sigma_u| \leq |\Sigma_v| \text{ if } \nabla u = \nabla v \text{ on } \partial\Omega,$$

where $\Sigma_v = graph(\nabla v)$. Since $|\Sigma_u| = \int_\Omega \sqrt{\det(g_{ij})}dx$, and $g_{ij} = \delta_{ij} + \sum_k u_{ik}u_{kj}$, the relation

$$\frac{d}{dt}|\Sigma_{u+th}|\Big|_{t=0} = 0 \ \ \forall h \in C_c^\infty(\Omega)$$

is equivalent to the fourth order equation of bi-harmonic type

$$\sum_{j=1}^{n} \Big(\Delta_g(\frac{\partial u}{\partial x^j})\Big)_{x^j} = 0. \tag{1}$$

The boundary condition corresponds to specifying ∇u on $\partial\Omega$.

Hamiltonian stationary equation: Let $h(x, y)$ be of compact support, and $X_h = J(\nabla h)$ be its Hamiltonian gradient. Note that in the Euclidean space $X_h = \sum_i h_{x^i}\frac{\partial}{\partial y_i} - h_{y^i}\frac{\partial}{\partial x_i}$. The vector field X_h generates a flow F_t through symplectic maps ($F_t^*\omega = \omega$), so that $F_t(\Sigma)$ is Lagrangian if Σ is Lagrangian. In particular, this implies the Lagrangian condition has a huge amount of flexibility (infinite dimensional space of deformations). The equation (1) holds if and only if Σ is stationary for all hamiltonian deformations. We may use this formulation to remove the graphical restriction in the Euler-Lagrange equation.

Important fact: If M^{2n} is Kähler-Einstein, the Euler-Lagrange equation assumes a nice global form. In this case the mean curvature vector $\vec{H}$ is itself Hamiltonian. More precisely, $\vec{H} = J(\nabla\beta)$ for a multi-valued hamiltonian function β. If $X_h = J\nabla h$, then

$$\begin{aligned}\frac{d}{dt}|\Sigma_t|\Big|_{t=0} &= -\int_\Sigma \langle \vec{H}, X_h\rangle \, dv \\ &= -\int_\Sigma \langle d\beta, dh\rangle \, dv.\end{aligned}$$

Therefore the equation (1) reduces to $\Delta\beta = 0$, which is equivalent to the 3rd order system

$$\begin{aligned} d\sigma_H &= 0, \\ \delta\sigma_H &= 0, \end{aligned}$$

where $\sigma_H = \vec{H}\lfloor\omega$ is the 1-form on Σ associated with the mean curvature vector field.

A Lagrangian submanifold $\Sigma^n \subset \mathbb{R}^{2n}$ is Hamiltonian stationary if and only if $\Delta\beta = 0$, where $dz\Big|_\Sigma = e^{-i\beta} dvol_\Sigma$ and $dz = dz_1 \wedge \cdots \wedge dz_n$. In the particular case $n = 1$, we see that the angle function β satisfies $\frac{d^2\beta}{ds^2} = 0$, where s denotes the arclength. Therefore the only Hamiltonian stationary curves are the lines and circles. This example shows that Hamiltonian stationary submanifolds are not necessarily minimal. In the two dimensional case the hamiltonian stationary equations have been shown to be an integrable system and there has been lots of interesting work done from that point of view. We will not treat that direction in this paper.

In special ambient manifolds there are special classes of hamiltonian stationary submanifolds which are of particular interest. The first of these is the class of minimal lagrangian submanifolds. These satisfy the equations $\vec{H} = 0$ and Σ is lagrangian. These submanifolds exist generally in Kähler-Einstein manifolds.

In a Calabi-Yau manifold this problem is particularly natural. Recall that a Calabi-Yau manifold has trivial canonical bundle and hence carries a parallel $(n, 0)$-form α_0. The minimal lagrangian condition is then equivalent to the equation $\alpha_0\Big|_\Sigma = e^{i\theta} dv$, where θ is a constant called the phase. This is also equivalent to the fact that $Re(e^{-i\theta}\alpha_0)$ "calibrates" Σ or that Σ is *special Lagrangian* for a suitable choice of the canonical form α.

Main Goal: Develop existence theory for minimal and special lagrangian submanifolds.

There are several reasons why this would be an important thing to do including the following:

1) To better understand the SYZ proposal (mirror symmetry, construction of a geometric mirror manifold to a Calabi-Yau).

2) To construct very rigid canonical class of representatives for a "hopefully" large part of the homology of Kähler-Einstein manifolds.

3) The methods should be important; for example, we hope that variational methods can be used to construct other classes of calibrated submanifolds (e.g. holomorphic).

We have the following result which asserts that a sufficiently smooth minimizer among lagrangian competitors will in fact be minimal.

Proposition 4.1. *If Σ is a smooth lagrangian submanifold of Kähler-Einstein M which is stationary for lagrangian variations, then Σ is minimal.*

Proof: The mean curvature vector $H = J\nabla\beta$ is locally hamiltonian so mean curvature deformation preserves the lagrangian condition and reduces volume unless $\vec{H} = 0$.

□

There is a nice global picture behind the above result. Let $\mathcal{L}$ denote the subspace of lagrangian submanifolds in the space $\mathcal{S}$ of submanifolds in a given homology class and of volume bounded above by V_0. The mean curvature flow provides a vector field $\mathcal{H}$ tangent to $\mathcal{L}$. The idea is to take completions (e.g. integral currents) $\overline{\mathcal{L}} \subset \overline{\mathcal{S}}$, and observe that the elements $\Sigma \in \overline{\mathcal{L}}$ are weakly lagrangian. We can carry out this completion process and prove the following.

Proposition 4.2. *There exists a volume minimizer in $\overline{\mathcal{L}}$. This minimizer is an integral current which is lagrangian in the sense that its tangent planes are almost surely lagrangian planes.*

A key question which appears to be very difficult is the following. Can $\mathcal{H}$ be defined on $\overline{\mathcal{L}}$ such that $\overline{\mathcal{L}}$ is preserved under the flow? Making such an extension would show that a minimizing lagrangian current would necessarily be stationary for volume among all deformations.

There are some specific results for $n = 2$ ([26]); that is, the ambient manifold is a Kähler 4-manifold (M^4, g, J). Even in this case we do not have a complete understanding of when our lagrangian minimizers are actually minimal. We do have an existence and regularity theory for the problem of least area lagrangian surfaces. We describe the results of [26] beginning with two general results.

Proposition 4.3. *A homology class $\sigma \in H_2(M, \mathbb{Z})$ has a piecewise smooth Lagrangian representative Σ if and only if $\omega(\sigma) = \int_\sigma \omega = 0$.*

Proposition 4.4. *A homology class* $\sigma \in H_2(M, \mathbb{Z})$ *has a smoothly immersed Lagrangian representative if and only if* $\omega(\sigma) = 0$ *and* $c_1(\sigma) = 0$*, where* c_1 *denotes the first Chern class.*

Geometrically, $c_1(\sigma) = \int_\sigma \rho$, where ρ is the Ricci 2-form ($\rho\Big|_\Sigma = d\sigma_H$). If M is Kähler-Einstein, then $\rho = c\omega$, so $\omega(\sigma) = 0$ and $c_1(\sigma) = 0$ are the same condition.

Corollary 4.5. *If* $\omega(\sigma) = 0$ *and* $c_1(\sigma) \neq 0$*, then a least area Lagrangian* $\Sigma \in \sigma$ *must be singular (worse than branch points).*

Before presenting the general existence theory we describe the singularities which arise in this problem. Let $\gamma \subset \mathbb{S}^3$ be a curve and $C(\gamma) \subset \mathbb{R}^4$ be the cone over γ. When is $C(\gamma)$ hamiltonian stationary?

To answer this question, let $\pi : S^3 \to S^2$ be the Hopf map, $\pi(z_1, z_2) = \frac{z_1}{z_2} \in \mathbb{P}^1 \approx S^2$. Then $\pi^{-1}(x)$ is a great circle $(e^{i\theta} z_1, e^{i\theta} z_2)$, $0 \leq \theta \leq 2\pi$.

1) The cone $C(\gamma)$ is lagrangian if and only if $\gamma \subset \mathbb{S}^3$ is legendrian, i.e., $\gamma' \perp$ Hopf circle $e^{i\theta}\gamma$;

2) For a closed curve γ, the cone $C(\gamma)$ is hamiltonian stationary if $\pi(\gamma)$ is a round circle in S^2 dividing S^2 into rationally related areas $\frac{A_1}{A_2} \in \mathbb{Q}$. Conversely, any such circle is $\pi(\gamma)$ for some closed curve γ. The interesting case is when $\pi(\gamma)$ is not a great circle.

Formula: Given $p, q \in \mathbb{Z}_+$ relatively prime ($(p, q) = 1$), define

$$\gamma(\theta) = \frac{1}{\sqrt{p+q}}\Big(\sqrt{q}e^{ip\theta}, i\sqrt{p}e^{-iq\theta}\Big),$$

$0 \leq \theta \leq 2\pi$.

Then $C(\gamma)$ is hamiltonian stationary, thus $dz_1 \wedge dz_2\Big|_C = e^{i\beta} dv$, where $\Delta\beta = 0$.

Remark: The 1-form $d\beta$ represents the Maslov class and it is a hamiltonian isotopy invariant.

In our case we have

$$\beta = (p - q)\theta.$$

Thus any curve on $C(\gamma)$ which winds once around the origin in the positive direction has Maslov index given by $p - q$. Since we are only interested in volume minimizing cones, we should ask whether $C(\gamma)$ is minimizing. This question can be posed as follows. Is it true that

$$|C(\gamma) \cap B_1(0)| \leq |\Sigma|$$

for any Lagrangian disk Σ with $\partial\Sigma = \gamma$?

A partial answer to this question is given by the following result from [26].

Proposition 4.6. *If $|p - q| > 1$, then there exists $h \in C_c^\infty(\mathbb{R}^4 \setminus \{0\})$ which generates a hamiltonian deformation F_t such that $|F_t(\Sigma)| < |\Sigma| - ct^2$ for small t, $c > 0$.*

This is an instability result whose proof relies on a careful analysis of the Jacobi operator. The idea is to look for hamiltonians of the form $h(r, \theta) = f(r)\sin(m\theta)$ and study the second variation of area (fourth order Jacobi operator). Unfortunately that same analysis also shows the following.

Corollary 4.7. *If $|p - q| = 1$ (primitive case), then $C(\gamma)$ is stable for hamiltonian deformations with compact support in $\mathbb{R}^4 \setminus \{0\}$.*

This result shows that the mean curvature deformation does not work to reduce area for $C(\gamma)$. We now state the general existence and regularity result of [26].

Theorem 4.8. *Let $\Sigma \in \sigma \in H_2(M, \mathbb{Z})$ be a lagrangian area minimizing surface (image of a map). Then Σ is a smooth immersion away from a finite set of points $\{P_1, \ldots, P_k, Q_1, \ldots, Q_l\}$ (the map is globally Lipschitz) such that:*

1. *P_j are branch points (mean curvature goes to zero at P_j);*
2. *Q_j are nonflat singular points with tangent cone (unitarily equivalent to one of those described above) of nonzero Maslov index $Ind(Q_j) \in \mathbb{Z}$.*

In fact $Ind(Q_j) = \pm 1$*. Moreover,*

$$\sum_{j=1}^{l} Ind(Q_j) = \frac{1}{2} c_1(\sigma).$$

Corollary 4.9. *At least one of the cones is minimizing.*

Proof: If we choose a homology class σ so that $\omega(\sigma) = 0$ and $c_1(\sigma) \neq 0$, then any lagrangian minimizer in the class would have to have at least one singular point with a nonflat tangent cone. The tangent cone at such a point Q_j would then have to be minimizing.

□

Open Question: There are infinitely many cones with $|p - q| = 1$. Which of them are minimizing?

A general unresolved question in this subject is the question of when a lagrangian minimizer is actually minimal. Some observations on this follow:

1) If M is Kähler-Einstein, then $\sum_{j=1}^{l} Ind(Q_j) = 0$.

2) The condition $\vec{H} = 0$ is equivalent to $l = 0$, i.e, there are no Q_j's.

3) In general for M^4 Kähler and Ricci flat there exists a Lagrangian class σ for which any Lagrangian minimizer is not minimal (see Micallef-Wolfson [21]).

The following theorem holds in the case $n = 2$ and is a deformation result for singular special lagrangian surfaces.

Theorem 4.10. *Consider a family* ω_t *of Calabi-Yau Kähler forms on* M^4*. Assume* $\omega_t(\sigma) = 0$ *and for* $t = 0$ *there is a connected minimal lagrangian representative of* σ*. Then a least area lagrangian representative of* σ *is minimal for small* t*.*

The following conjecture would be a breakthrough on the problem in case $n \geq 3$.

Conjecture: The theorem above holds for M^{2n} with $n \geq 3$.

Finally we mention other methods which have been successful in constructing special lagrangian submanifolds, or which have potential to attack the existence question.

Gluing Theorems:

1) A. Butscher [6]: Given Σ_1, Σ_2 intersecting transversally, find Σ_ε special Lagrangian $\Sigma_\varepsilon \to \Sigma_1 + \Sigma_2$. The idea is to introduce a small neck (local model constructed by Lawlor) of size ε, and to construct a special lagrangian submanifold Σ_ε which is topologically $\Sigma_1 \# \Sigma_2$ and geometrically resembles Σ_1 and Σ_2 joined by a neck of size ε.

2) Y. Lee [16]: For $n = 2, 3$, given a connected smooth immersed special lagrangian $\Sigma^n \subset M^{2n}$ with transverse self-intersections, there is an embedded $\Sigma_\varepsilon \subset M$ special lagrangian which approximates Σ away from the self intersection points and in which each self intersection is replaced by a neck of approximate size ε.

3) D. Lee [15]: Given $\Sigma_1, \Sigma_2 \subset (M, \omega)$ such that $\Sigma_1 \cap \Sigma_2$ consists of a finite set of points at which the intersection is transverse, there exist $\omega_\varepsilon, \Sigma_\varepsilon$ such that $\omega_\varepsilon \approx \omega$ and $\Sigma_\varepsilon \approx \Sigma_1 \cup \Sigma_2$ in the sense described above, where Σ_ε is special lagrangian with respect ω_ε.

Corollary 4.11. *([15]) Any flat complex torus can be perturbed by an arbitrarily small amount to a new flat complex flat torus which contains nonflat special lagrangian submanifolds.*

Mean Curvature Flow:

We close with a very brief mention of the mean curvature flow which also provides a plausible approach to construction of special lagrangian submanifolds. The flow is given by:

$$\begin{aligned} \frac{\partial X}{\partial t} &= \vec{H} \\ X(0) &= \Sigma_0 \end{aligned}$$

where X_t is an embedding whose image is Σ_t. The fact that the mean curvature vector is hamiltonian implies that if Σ_0 is a smooth lagrangian submanifold, then Σ_t is smooth and lagrangian for small t

Various conjectures on the long time existence and convergence of mean curvature flow are given in the paper of Thomas-Yau [29]. Several types of counterexamples to natural conjectures are given by A. Neves [23]. For mean curvature flow the basic problems are to understand what type of singularities can occur and what happens for large values of t and as $t \to \infty$?

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