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OPTIMAL REGULARITY OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR SINGULAR EQUATIONS AND THEIR LIMITING FREE BOUNDARY PROBLEMS

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Abstract

In the present paper, we start the journey of investigation into fully nonlinear elliptic singular equations of the form $F(D^2u, x) = \beta_\varepsilon(u_\varepsilon)$, where $\beta_\varepsilon(u_\varepsilon)$ converges to the Dirac delta measure δ_0. We show optimal regularity, uniform in ε, as well as H^1 compactness for Bellman's singular equations. We also provide a complete picture of limiting one-dimensional profiles. The study of further geometric-measure properties as well as smoothness of the general limiting free boundary problem is currently in progress [16].

1 Introduction

Let $\partial\Omega$ be a smooth compact hypersurface in $\mathbb{R}^N$. Given a nonnegative function $f\colon \partial\Omega \to \mathbb{R}$ and a positive function $g\colon \overline{\Omega} \to \mathbb{R}$, an important question in geometry as well as in applied mathematics is whether one can find another compact hypersurface $\Gamma = \partial\Omega' \subset \Omega$ such that it is possible to solve the overdetermined elliptic boundary value problem

$$\left\{\begin{array}{rll} Lu = 0 & \text{in} & \Omega \setminus \Omega' \\ u = f & \text{on} & \partial\Omega \\ u = 0,\ u_\nu = g & \text{on} & \Gamma, \end{array}\right. \tag{1.1}$$

where ν is the unit outward normal vector on Γ and L is a linear elliptic operator, either in divergence or in non-divergence form.

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It turns out that a singular perturbation technique can be used as a successful systematic way to construct *possible* solutions of (1.1). The basic idea is the following: a *possible* solution to problem (1.1) will be obtained as a limit of the approximating regularized problem

$$\begin{cases} Lu_\varepsilon &= g^2(x)\beta_\varepsilon(u_\varepsilon) \text{ in } \Omega \\ u_\varepsilon &= f \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where β_ε is a suitable approximation of the Dirac Delta function. For each $\varepsilon > 0$ fixed, the regularizing problem (1.2) has its own importance by itself, as it appears, for instance, in the mathematical formulation of problems in flame propagation and combustion theory.

Uniform properties of problem (1.2) have attracted the attention of many mathematicians for more than three decades. The problem $\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$ has been fully studied in the late 70's and early 80's by Lawy-Stampacchia, Kinderlehrer-Nirenberg, Caffarelli, among others.

Berestycki, Caffarelli and Nirenberg, in [4], analyzed problem (1.2) for linear elliptic operators in non-divergence form

$$Lu := a_{ij}(x)D_{ij}u + b_iD_iu + cu,$$

with sufficient regular coefficients and $g \equiv 1$. Further regularity of the limiting free boundary could not be addressed, though.

In a very recent work, [15], D. Moreira and the author obtained a rather complete description of the limiting free boundary problem (1.2), for linear equations in divergence form:

$$Lu := \operatorname{div}\big(a_{ij}(x)\nabla u_\varepsilon\big) = g^2(x)\beta_\varepsilon(u_\varepsilon),$$

with merely Hölder continuous coefficients. We have shown that the limiting function u, obtained as the limit of minimizers of $E_\varepsilon(v) := \displaystyle\int_\Omega \frac{1}{2}\langle a_{ij}(x)Dv, Dv\rangle +$ $g^2(x)B_\varepsilon(v)dx$, where $B_\varepsilon(t) := \int_0^t \beta_\varepsilon(s)ds$, satisfies $Lu = 0$ in $\{u > 0\}$ and

$$u_\nu = g \cdot \sqrt{\frac{2A}{\langle a_{ij}\nu, \nu\rangle}}, \qquad \text{along } \partial\{u > 0\} \cap \Omega, \tag{1.3}$$

where A is the total mass of β_ε and ν is the outward normal vector on the free surface $\partial\{u>0\}$. Initially (1.3) is shown to hold in a weak sense; however after proving $C^{1,\alpha}$ regularity of the free boundary, we conclude the desired Neumann condition expressed in (1.3) does hold in the classical sense (for $N-1$ almost every point on $\partial\{u>0\}$). It is important to highlight that, in terms of free boundary problems like (1.1), operators in non-divergence form have advantages and disadvantages when compared with operators in divergence form. One advantage is that solutions of elliptic equations in non-divergence form tend to be regular up to the boundary. A disadvantage is that, for equations in non-divergence from one loses the variational characterization to select particular solutions to the approximating problem. The hardest situation then, yet in the linear setting, is when one has an operator of the form

$$Lu = \operatorname{div}\big(a_{ij}(x)Du\big) + b_i u_i,$$

with nonsmooth coefficients a_{ij}. This type of operator appears in the study of flame propagation. The author, in [17], suggests a geometric-measure variational approach to this problem. With this new machinery, one can establish important nondegeneracy geometric conditions of approximating level sets that, combined with a very recent and important technique developed by Ferrari and Salsa in [14], ultimately enable fine regularity results of the limiting free boundary. A similar free boundary condition as in (1.3) is obtained.

In general, the strategy to study the limiting free boundary problem obtained as the limit, as $\varepsilon \to 0$ of problem (1.2) is as follows:

1. Obtain optimal regularity, uniform in ε, of solutions to problem (1.2). This regularity depends upon the a priori estimates available for the operator L. It can never be higher than Lipschitz, though, since any L-harmonic function reaches the free boundary $\partial\{u>0\}$ with a positive angle. This is an important and crucial step towards the study of further geometric properties of the limiting free boundary, because it enables compactness of the family of solutions of (1.2).

2. Establish, uniform in ε, nondegeneracy. This is done in terms of the maximum admissible growth rate the optimal regularity obtained in the 1st step allows. This is an important geometric information that prevents blow-up sequences to vanish at the limit.

3. Prove uniform positive density of particular approximating level sets $\{u_\varepsilon = c_\varepsilon\}$, with $c_\varepsilon \to 0$ as $\varepsilon \to 0$. In the limit, this is used to obtain finiteness of the $n-1$ Hausdorff measure of the free boundary $\mathcal{F} = \partial\{u > 0\}$. Furthermore, one should be able to verify that $H^{n-1}\big(\mathcal{F} \cap B_r(x_0)\big) \sim r^{n-1}$, for any ball centered at a free boundary point.

4. Find the desired free boundary condition. Since the function u is, at most, Lipschitz continuous, one cannot use inverse function theorem to conclude any regularity of the free boundary. The free boundary condition is the additional geometric information we need to go forward in our regularity investigation of the free boundary $\mathcal{F}$.

5. Study further regularity properties of the free boundary. This final step, in general, involves a rather delicate analysis. One first shows the surface $\mathcal{F}(u) = \partial\{u > 0\}$ is Flat. Roughly speaking, this means that $\mathcal{F}(u)$ is (locally) close enough to the graph of a Lipschitz function. A good free boundary condition would do it for us. Now, we have to establish a result that says that Flat surfaces are Lipschitz. Afterwards, we show that Lipschitz free boundaries are $C^{1,\alpha}$. This corresponds, in the theory of minimal surfaces, to the seminal work of De Giorgi. Higher regularity of the free boundary, can, in general, be obtained by a recursive process.

For singular perturbation problem with gradient dependence upon the penalization: $\Delta u = \beta_\varepsilon(u) H(Du)$, Caffarelli, Jerison and Kenig established, in [9], with the aid of a new and rather powerful monotonicity formula, the first step of the journey: uniform Lipschitz regularity.

In this present work, we bring the attention to singular approximating problem (1.2) for fully nonlinear elliptic operators. We shall establish the optimal regularity of solutions to the limiting free boundary problem obtained as a limit as $\varepsilon \to 0$ of (1.2), where $Lu = F(D^2u, x)$. We will also establish H^1 compactness of $\{u_\varepsilon\}$ for approximating equations governed by Bellman's operator. A complete picture of one-dimensional profiles are derived in Section 4, which suggests that, for higher dimensional equation, under reasonable orthornormal invariance conditions on F, a similar free boundary condition should hold in the normal direction.

2 Optimal regularity uniform in ε

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $F\colon \mathcal{S}(N) \times \Omega \to \mathbb{R}$ a fully nonlinear operator. The symbol $\mathcal{S}(N)$ stands for the space of all symmetric $N \times N$ matrices. We shall assume standard and mild assumptions on F. Namely

- (Uniform ellipticity): There exist two constants $0 < \lambda \le \Lambda$ such that

$$F(M+N,x) \le F(M,x) + \Lambda \|N^+\| - \lambda \|N^-\|, \ \forall M, N \in \mathcal{S}(n), \ \forall x \in \Omega.$$

- (Hölder continuity in x): There exist $\mu \in (0,1)$ and $K > 0$, such that

$$\|F(M,\cdot)\|_{C^\mu(\Omega)} \le K, \ \forall M \in \mathcal{S}(n).$$

- (Normalization) $F(0,x) \equiv 0$.

Given a nonnegative function $f\colon \partial\Omega \to \mathbb{R}$ and a bounded and positive function $g\colon \overline{\Omega} \to \mathbb{R}$, we shall be concerned with the optimal regularity, uniform in ε, of solutions to

$$\left\{ \begin{array}{rcl} F(D^2u_\varepsilon, x) & = & g^2(x)\beta_\varepsilon(u_\varepsilon) \text{ in } \Omega \\ u_\varepsilon & = & f \text{ on } \partial\Omega. \end{array} \right. \tag{2.1}$$

The perturbation family is constructed as follows: let β_1 be a smooth function supported in $[0,1]$ and positive in $(0,1)$. We then define

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta_1\left(\frac{s}{\varepsilon}\right). \tag{2.2}$$

The family β_ε approximates $A\delta_0$, where δ_0 stands for the delta Dirac measure supported at 0 and A is the total mass of β_1: $A := \int_{\mathbb{R}} \beta_1(\tau)d\tau$.

Notice that we have not assumed convexity of F; consequently, it is not available any a priori $C^{2,\alpha}$ estimate. Hence, solutions to (2.1) may not be classical and therefore these shall be understood in the viscosity sense.

Lemma 2.1. *Let u_ε be any viscosity solution of equation (2.1). Then $0 \le u_\varepsilon \le \|f\|_\infty$ in Ω.*

Proof. Recall $\beta_\varepsilon \ge 0$. Define $\tilde{u}_\varepsilon := u_\varepsilon - \|f\|_\infty$. Notice that $\tilde{u}_\varepsilon \le 0$ on $\partial\Omega$ and

$$F(D^2\tilde{u}_\varepsilon, x) = F(D^2 u_\varepsilon, x) \ge 0.$$

The estimate by above now follows as an immediate application of Alexandroff-Bakelman-Pucci estimate adapted for viscosity solutions (see, [7] or [8]). Let us prove the nonnegativity of u_ε. Suppose, by contradiction, the region $A_\varepsilon := \{x \in \Omega : u_\varepsilon(x) < 0\}$ is nonempty. Since β_ε is supported on $[0, \varepsilon]$,

$$F(D^2 u_\varepsilon, x) = 0 \text{ in } A_\varepsilon,$$

in the viscosity sense. However, because $u_\varepsilon \ge 0$ on $\partial\Omega$, we have $u_\varepsilon \ge 0$ on ∂A_ε, thus, applying, once more, the Alexandroff-Bakelman-Pucci estimate adapted for viscosity solutions, we conclude $u_\varepsilon \ge 0$ in A_ε, leading us to a contradiction. □

Next Theorem gives the optimal regularity, uniform in ε, one can hope for: Lipschitz continuity.

Theorem 2.2 (Optimal Regularity). *Let $\tilde{\Omega} \subset\subset \Omega$. There exists a constant C depending on $\|f\|_\infty$, $\|g\|_\infty$, $\|\beta_1\|_\infty$, $\tilde{\Omega}$, dimension, ellipticity and C^μ norm of $F(M, \cdot)$, but independent of ε, such that, any family of viscosity solutions $\{u_\varepsilon\}_{\varepsilon>0}$ of (2.1) is C-Lipschitz continuous in $\tilde{\Omega}$.*

Proof. The proof concerns to analyze the gradient of u_ε in two regions:

Region 1: $\Omega_1 = \Omega_1(\varepsilon) := \tilde{\Omega} \cap \{0 \le u_\varepsilon(x) \le \varepsilon\}$.

We can assume $\varepsilon < \operatorname{dist}(\partial\tilde{\Omega}, \partial\Omega)$. Let x_0 be a generic point in Ω_1 and define $v\colon B_1 \to \mathbb{R}$ to be

$$v(y) := \frac{1}{\varepsilon} u_\varepsilon(x_0 + \varepsilon y).$$

Let $P(y)$ be a paraboloid touching v, at some point z_0, by below. Define

$$\tilde{P}(x) := \varepsilon P\left(\frac{x - x_0}{\varepsilon}\right).$$

Then, $\tilde{P}$ touches u_ε by below at $x_1 := x_0 + \varepsilon z_0$. Since u_ε is a viscosity solution of (1.2), we have

$$F\left(D^2\tilde{P}(x_1), x_1\right) \leq g^2(x_1)\beta_\varepsilon(u_\varepsilon(x_1)). \tag{2.3}$$

Furthermore,

$$D^2\tilde{P}(x_1) = \frac{1}{\varepsilon} D^2 P(z_0) \qquad \text{and} \qquad \beta_\varepsilon(u_\varepsilon(x_1)) = \frac{1}{\varepsilon}\beta_1(v(z_0)). \tag{2.4}$$

Combining (2.3) and (2.4), we end up with

$$\varepsilon F\left(\varepsilon D^2 P(z_0), x_0 + \varepsilon z_0\right) \leq g^2(x_0 + \varepsilon z_0)\beta_1(v(z_0)).$$

Taking a paraboloid touching v by above and arguing similarly, we conclude v is a viscosity solution of

$$\varepsilon F\left(\varepsilon D^2 v, x_0 + \varepsilon x\right) = g^2(x_0 + \varepsilon x)\beta_1(v) \text{ in } B_1. \tag{2.5}$$

If we define

$$F_\varepsilon\left(M, x\right) := \varepsilon F\left(\varepsilon D^2 v, x_0 + \varepsilon x\right),$$

it is standard to verify that F_ε is uniformly elliptic with the same ellipticity constant as F. It now follows from Caffarelli's $C^{1,\alpha}$ estimate for viscosity solution of fully nonlinear elliptic equations, see [7], there exists a constant C, depending only on λ, Λ and K, such that

$$|\nabla v(0)| \leq C\left\{\|v\|_{L^\infty(B_{1/2})} + \|g^2\beta_1\|_{L^n(B_{1/2})}\right\}. \tag{2.6}$$

Notice moreover that

$$v(0) = \frac{1}{\varepsilon} u_\varepsilon(x_0) \leq 1,$$

since $x_0 \in \Omega_1$. Hence, from Harnack inequality (see, for instance, [8]) there exists a universal constant C such that

$$\|v\|_{L^\infty(B_{1/2})} \leq C. \tag{2.7}$$

Combining (2.6) and (2.7) we finally end up with

$$|\nabla u_\varepsilon(x_0)| = |\nabla v(0)| \leq C,$$

where C is independent of ε.

Region 2: $\Omega_2 = \Omega_2(\varepsilon) := \tilde{\Omega} \cap \{u_\varepsilon(x) > \varepsilon\}$.

Let us denote by $\Gamma_\varepsilon := \big\{x \in \Omega : u_\varepsilon(x) = \varepsilon\big\}$. Select any point $x_0 \in \Omega_2$ and denote by $r = \operatorname{dist}(x_0, \Gamma_\varepsilon)$. Since β_ε is supported in $[0, \varepsilon]$, we have

$$F(D^2 u_\varepsilon, x) = 0 \text{ in } \Omega_2,$$

in the viscosity sense. We now rescale u_ε with respect to r, i.e., we consider the auxiliary function $v_r \colon B_1 \to \mathbb{R}$, defined by

$$v_r(y) := \frac{1}{r} u_\varepsilon(x_0 + ry).$$

As before, v_r is a viscosity solution of

$$rF\left(rD^2 v_r(y), x_0 + ry\right) = 0 \text{ in } B_1,$$

and $v_r(0) \leq \dfrac{\|f\|_\infty}{r}$. Applying once more Caffarelli's $C^{1,\alpha}$ estimate and afterwards Harnack inequality, we obtain

$$|\nabla u_\varepsilon(x_0)| = |\nabla v_r(0)| \leq \frac{C}{\operatorname{dist}(x_0, \Gamma_\varepsilon)}. \tag{2.8}$$

We now have to care about a universal bound of the gradient of u_ε for points $x \in \Omega_2$ that are close to Γ_ε, since the above estimate, in principle blows up when x approaches Γ_ε. It is worthwhile to comment that, for each $\varepsilon > 0$, we can find a $1 \leq c_\varepsilon \leq 2$, so that $\Gamma_{c_\varepsilon \varepsilon}$ is $C^{1,\alpha}$ (Sard's Theorem) and thus we can control the Lipschitz norm of u_ε up to the boundary of $\Omega_{c_\varepsilon \varepsilon}$ ($C^{1,\alpha}$ regularity up to the boundary). However, since we do not know yet any smoothness of the limiting free boundary, such a control could deteriorate as $\varepsilon \to 0$.

This step requires a more delicate analysis. The idea will be to obtain an estimate of $u_\varepsilon(x)$ in terms of the distance of x to Γ_ε, $\operatorname{dist}(x, \Gamma_\varepsilon)$. For that, let $\chi \in \Gamma_\varepsilon$ and choose ρ small enough so that $B(\chi, \rho) \subset \Omega$. Fix $x_0 \in B(\chi, \rho)$ and let us, for convenience, keep calling $\operatorname{dist}(x_0, \Gamma_\varepsilon) = r$. Set λ so that $u_\varepsilon(x_0) = \lambda \cdot r$. We now consider the Lipschitz re-normalization

$$w(y) := \frac{1}{r} u_\varepsilon(x_0 + ry).$$

The important features of w to be noticed are

1. $w(0) = \lambda$,
2. $w \geq \varepsilon > 0$ in B_1,
3. There exists a $Y_1 \in \partial B_1$ such that $w(Y_1) = \varepsilon$,
4. $F(D^2 w(y), x_0 + ry) = 0$ in B_1, in the viscosity sense,
5. $\nabla w(y) = \nabla u_\varepsilon(x_0 + ry)$.

We also know, from our analysis on the first region that $|\nabla w(Y_1)| \leq C$ and C does not depend on ε. By Harnack inequality, there exists a constant c such that

$$w(y) \geq c\lambda \quad \forall y \in B_{1/2}.$$

Using Perron's method, we can construct the following barrier

$$\begin{cases} F(D^2 z, x_0 + ry) &= \quad 0 \text{ in } B_1 \setminus \overline{B_{1/2}} \\ z\Big|_{\partial B_1} = 0, & \quad z\Big|_{\partial B_{1/2}} = 1. \end{cases} \tag{2.9}$$

The function z is $C^{1,\alpha}$ up to the boundary and furthermore, there exists a $\delta > 0$, depending only on ellipticity, such that

$$z_\nu \geq \delta \text{ on } \partial B_1,$$

where ν denotes the inward normal vector $-x$. From the comparison principle and the homogeneity of F, we conclude

$$w \geq c\lambda z \text{ in } B_1 \setminus B_{1/2};$$

therefore

$$C \geq |\nabla w(Y_1)| \geq w_\nu(Y_1) \geq c\lambda z_\nu(Y_1) \geq \delta c\lambda.$$

We have conclude that

$$u_\varepsilon(x_0) \leq C \cdot \operatorname{dist}(x, \Gamma_\varepsilon)$$

for a universal constant $C > 0$, independent of ε. Finally, applying $C^{1,\alpha}$ estimate followed by Harnack inequality and estimate (2.8), we end up with

$$|\nabla u_\varepsilon(x_0)| \leq C \frac{u_\varepsilon(x_0)}{\operatorname{dist}(x, \Gamma_\varepsilon)} \leq C,$$

and the theorem is proven.

□

An immediate, yet important, consequence of Theorem 2.2 is the fact that *any* family, $\{u_\varepsilon\}_{\varepsilon>0}$, of viscosity solutions of the singular perturbation problem (2.1) is compact in the C^α_{loc} topology, for any $\alpha < 1$. We therefore obtain

Theorem 2.3. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of viscosity solutions of equation (2.1). Then a subsequence of u_ε converges, uniformly over compact sets, to a nonnegative, locally Lipschitz function u_0. Furthermore, u_0 satisfies*

$$F(D^2 u_0, x) = 0 \text{ in } \Omega_0 := \{x \in \Omega \,:\, u_0(x) > 0\}, \tag{2.10}$$

in the viscosity sense.

3 Bellman's operator and H^1 compactness

In this section we shall derive strong H^1 convergence for solutions of (2.1) when F is a Bellman operator:

$$F(D^2 u, x) := \inf_{\alpha \in \mathcal{A}} \left(\operatorname{Tr}\left[A_\alpha(x) D^2 u\right] \right), \tag{3.1}$$

where $\{A_\alpha(x)\}$ is a family of uniform elliptic matrices. Equations of the form

$$\inf_{\alpha\in\mathcal{A}} \left(\mathrm{Tr}\left[A_\alpha(x)D^2u\right]\right) = 0,$$

appear, for instance, when one seeks for optimal costs in stochastic control problems. Notice that Bellman's operators are concave, therefore, Evans-Krylov Theorem (adapted to viscosity solutions, see [8]) gives $C^{2,\alpha}$ estimate for viscosity solutions of Bellman's equation, as long as the family $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ is uniform-in-α Hölder continuous. For simplicity, we will assume $\mathcal{A}$ is finite. We will often write

$$\mathrm{Tr}\left[A_\alpha(x)D^2u\right] = a^\alpha_{ij}(x)\partial_{ij}u.$$

Theorem 3.1. *Let $A_\alpha \in C^\mu(\Omega)\cap H^1(\Omega)$, for some $\mu > 0$. Then, any family of solutions, $\{u_\varepsilon\}$ of*

$$\left\{\begin{array}{rcl} \inf\limits_{\alpha\in\mathcal{A}} \left(Tr\left[A_\alpha(x)D^2u_\varepsilon\right]\right) &=& g^2(x)\beta_\varepsilon(u_\varepsilon) \ in\ \Omega \\ u_\varepsilon &=& f \ on\ \partial\Omega. \end{array}\right. \tag{3.2}$$

is pre-compact in $H^1(\Omega')$, for any $\Omega' \subset\subset \Omega$. In particular, up to a subsequence,

$$\nabla u_\varepsilon \longrightarrow \nabla u_0 \ in\ L^2(\Omega'),$$

as $\varepsilon \to 0$.

Proof. For each $\beta\in\mathcal{A}$ let us consider the set

$$G_\beta := \left\{x\in\Omega' \ : \ \inf_{\alpha\in\mathcal{A}}\left(\mathrm{Tr}\left[A_\alpha(x)D^2u_0\right]\right) = \mathrm{Tr}\left[A_\beta(x)D^2u_0\right]\right\}.$$

By taking a nested sequence of smooth open subsets of G_β, we may assume G_β has a smooth boundary. In G_β, we have

$$\mathrm{Tr}\left[A_\beta(x)D^2u_\varepsilon\right] \geq F(D^2u_\varepsilon, x) = g^2(x)\beta_\varepsilon(u_\varepsilon).$$

Taking into account Lemma 2.1, if we multiply above inequality by u_ε and integrate the result over G_β we obtain

$$\begin{aligned} 0 &\leq \int_{G_\beta} u_\varepsilon\cdot a^\beta_{ij}\partial_{ij}u_\varepsilon dx \\ &= \int_{\partial G_\beta} u_\varepsilon\cdot a^\beta_{ij}\cdot\partial_i u_\varepsilon \nu_j dS - \int_{G_\beta} a^\beta_{ij}\cdot\partial_i u_\varepsilon\cdot\partial_j u_\varepsilon + u_\varepsilon\cdot\partial_j\left(a^\beta_{ij}\right)\cdot\partial_i u_\varepsilon dx, \end{aligned}$$

where ν_j stands for the outward unit normal vector on ∂G_β. The above inequality can be re-written as

$$\int_{G_\beta} a_{ij}^\beta \cdot \partial_i u_\varepsilon \cdot \partial_j u_\varepsilon dx \leq \int_{\partial G_\beta} u_\varepsilon \cdot a_{ij}^\beta \cdot \partial_i u_\varepsilon \nu_j dS - \int_{G_\beta} u_\varepsilon \cdot \partial_j \left(a_{ij}^\beta\right) \cdot \partial_i u_\varepsilon dx. \quad (3.3)$$

Notice that, for any $\delta > 0$, $\nabla u_\varepsilon \to \nabla u_0$ uniformly in $\Omega_\delta := \{x \in \Omega' \ : \ u_0 \geq \delta\}$. Indeed, since u_ε converges uniformly to u_0 (Theorem 2.3) for ε small enough, $u_\varepsilon > \frac{\delta}{2}$. Because support $\beta_\varepsilon \subset [0, \varepsilon]$, if ε is small enough,

$$F(D^2 u_\varepsilon, x) = F(D^2 u_0, x) = 0.$$

The claim now follows by $C^{1,\alpha}$ a priori estimate. Thus, if we let $\varepsilon \to 0$ in (3.3), we obtain:

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{G_\beta} a_{ij}^\beta \cdot \partial_i u_\varepsilon \cdot \partial_j u_\varepsilon dx \ \leq \ & \int_{\partial G_\beta \cap \{u_0>0\}} u_0 \cdot a_{ij}^\beta \cdot \partial_i u_0 \nu_j dS \\ - & \int_{G_\beta \cap \{u_0>0\}} u_0 \cdot \partial_j \left(a_{ij}^\beta\right) \cdot \partial_i u_0 dx. \end{aligned} \quad (3.4)$$

Now, taking into account that

$$a_{ij}^\beta(x) \partial_i \partial_j u_0 = 0 \text{ in } G_\beta \cap \{u_0 > 0\},$$

using a truncation argument and Green's Theorem, one verifies that

$$\begin{aligned} \int_{G_\beta} a_{ij}^\beta \cdot \partial_i u_0 \cdot \partial_j u_0 dx \ = \ & \int_{\partial G_\beta \cap \{u_0>0\}} u_0 \cdot a_{ij}^\beta \cdot \partial_i u_0 \nu_j dS \\ - & \int_{G_\beta \cap \{u_0>0\}} u_0 \cdot \partial_j \left(a_{ij}^\beta\right) \cdot \partial_i u_0 dx. \end{aligned}$$

Combining the above with (3.4), we conclude

$$\limsup_{\varepsilon\to 0} \int_{G_\beta} a_{ij}^\beta \cdot \partial_i u_\varepsilon \cdot \partial_j u_\varepsilon dx \leq \int_{G_\beta} a_{ij}^\beta \cdot \partial_i u_0 \cdot \partial_j u_0 dx.$$

Finally, because of the uniform ellipticity of a_{ij}^β and the fact that, up to a subsequence, u_ε converges weakly in $H^1(\Omega')$ to u_0, (for instance because of Theorem 2.2), we conclude u_ε converges H^1-strong to u_0 in G_β. Now it is simple to show that indeed u_ε converges strongly to u_0 in $H^1(\Omega')$, and the Theorem is proven. □

Notice that Theorem 2.2 says that ∇u_ε is bounded in Ω', but no limiting behavior is given. Theorem 3.1 provides L^2 convergence of ∇u_ε. We believe this information will help to prove weak versions of a possible limiting free boundary condition.

4 One-dimensional profile

In order to understand the limiting free boundary condition, it is important to analyze the one-dimensional profile. In other words, we should analyze the limiting configuration for

$$f(u_\varepsilon'') = \beta_\varepsilon(u_\varepsilon), \tag{4.1}$$

where $\beta_\varepsilon(s) := \frac{1}{s}\beta\left(\frac{s}{\varepsilon}\right)$, for some non-negative continuous function β supported on $[0,1]$ and positive on $(0,1)$. The nonlinearity f is uniformly elliptic, i.e., for positive constants $\lambda,\ \Lambda$,

$$\lambda b \le f(a+b) - f(a) \le \Lambda b, \tag{4.2}$$

for any non-negative real number b. We further assume $f(0) = 0$. Notice that (4.2) implies f is Lipschitz continuous and strictly increasing. We define

$$f_\varepsilon(s) := \varepsilon f\left(\frac{s}{\varepsilon}\right).$$

It is simple to verify that f_ε satisfies (4.2) for the same ellipticity constants $\lambda,\ \Lambda$ and to prove the following result.

Proposition 4.1. *There exists a function $f^\star$ satisfying (4.2), so that*

$$f^\star(s) := \lim_{\varepsilon\to 0} f_\varepsilon(s), \tag{4.3}$$

uniformly over compact sets.

Next proposition will play an important role in the future free boundary condition one may establish to the limiting free boundary problem obtained as $\varepsilon \to 0$ in (2.1). It says that the limiting function u_0 in (4.1) reaches the zero set with a prescribed slope. For higher dimensional equations, under a natural orthonormal invariance condition on F, we hope to obtain a similar free boundary condition in the viscosity sense. Basically, for higher dimensional equation, $F(D^2u) = \beta_\varepsilon(u)$, under the assumption that F depends only upon the eigenvalues of D^2u, a similar result as in Proposition 4.2 should hold near a point of the reduced free boundary in the direction of the theoretic-measure unit outer vector.

Proposition 4.2. *Let $\{u_\varepsilon\}$ be a family of solutions to the regularizing problem (4.1), with a fixed positive boundary data at $\{-1, 1\}$. Then a limiting function $u_0 = \lim_{\varepsilon_j \to 0} u_\varepsilon$ satisfies*

$$\left(u_0'(\xi)\right)^2 = 2\alpha \quad in \ \{u_0 > 0\},$$

where

$$\alpha := \int_0^1 (f^\star)^{-1} \circ \beta(\tau) d\tau.$$

Proof. From (4.1) we verify that

$$u_\varepsilon''(x) = f^{-1}\left(\beta_\varepsilon\left(u_\varepsilon(x)\right)\right) =: \varrho_\varepsilon\left(u_\varepsilon(x)\right). \tag{4.4}$$

If we multiply equation (4.4) by $u_\varepsilon'(x)$, we obtain

$$\frac{1}{2}\frac{d}{dx}\left(u_\varepsilon'(x)\right)^2 = \frac{d}{dx} G_\varepsilon(u_\varepsilon(x)), \tag{4.5}$$

where

$$G_\varepsilon(s) := \int_0^s \varrho_\varepsilon(\sigma) d\sigma.$$

It is simple to see that, for any $s > 0$,

$$\begin{aligned} G_\varepsilon(s) &= \int_0^{\frac{s}{\varepsilon}} \varepsilon f^{-1}\left(\frac{1}{\varepsilon}\beta(\tau)\right) d\tau \\ &= \int_0^{\frac{s}{\varepsilon}} f_\varepsilon^{-1} \circ \beta(\tau) d\tau \\ &\longrightarrow \int_0^1 (f^\star)^{-1} \circ \beta(\tau) d\tau, \end{aligned} \tag{4.6}$$

as $\varepsilon \to 0$. From weak compactness, up to a subsequence, $\mathcal{G}_\varepsilon := G_\varepsilon \circ u_\varepsilon$ converges to a function $\mathcal{G}$ in $L^2(-1, 1)$. Computation performed in (4.6) implies $\mathcal{G} \equiv \alpha$ in $\{u_0 > 0\}$. We claim that $\mathcal{G} = \alpha \chi_{\{u_0 > 0\}}$.

Indeed, by arguing as in the proof of Theorem 3.1, we obtain u'_ε converges weakly in $L^2(-1,1)$ to u'_0. In particular, from (4.5),

$$\int_{-1}^{1} \frac{1}{2}(u'_0(\xi))^2\phi'(\xi)d\xi = \int_{-1}^{1} \mathcal{G}(\xi)\phi'(\xi)d\xi,$$

for any $\phi \in C^1_0(-1,1)$. If we select ϕ supported in the interior of $F := \{u_0 = 0\}$, we conclude $\mathcal{G}$ is constant in F. To show that $\mathcal{G} \equiv 0$ in F, we argue as follows: let y be an interior point to F. We must show $u_\varepsilon(y) = o(\varepsilon)$. Assume by contradiction,

$$\frac{1}{\varepsilon}u_\varepsilon(y) > \lambda > 0.$$

By continuity and uniform convergence, $u_\varepsilon(x) > \lambda\varepsilon$, in $I := (y-h, y+h)$. Consequently,

$$\beta_\varepsilon\left(u_\varepsilon(x)\right) \geq \frac{c}{\varepsilon},$$

in I, for a constant $c > 0$ that depends only on β. From (4.4),

$$\varepsilon u''_\varepsilon(x) \geq \kappa > 0, \tag{4.7}$$

in I, for some $\kappa = \kappa(c, f)$. Integrating (4.7) drives us to a contradiction on the Lipschitz estimate previously obtained in Theorem 2.2.

Finally, if we return to (4.5), we conclude

$$\left(u'_\varepsilon(\xi)\right)^2 = G\left(u_\varepsilon(\xi)\right) + b_\varepsilon,$$

for some constant b_ε. By Lipschitz continuity $\{b_\varepsilon\}$ is a bounded sequence, thus, up to a subsequence, $b_\varepsilon \to b$. Because $\mathcal{G} \equiv 0$ in F, $b = 0$. This finishes the proof of the proposition.

□

5 Some general follow up ideas

To carry the whole journey (listed in the introduction of this paper) of studying geometric and regular properties of the limiting free boundary problem, obtained as $\varepsilon \to 0$, in

$$\left\{\begin{array}{rcl} F(D^2u_\varepsilon, x) & = & g^2(x)\beta_\varepsilon(u_\varepsilon) \text{ in } \Omega \\ u_\varepsilon & = & f \text{ on } \partial\Omega. \end{array}\right. \tag{5.1}$$

is certainly a very challenging and important mathematical problem. It is still too soon to try to predict what kind of free boundary condition will (in general) appear by letting ε go to zero in (5.1). Nevertheless, as pointed out in Section 4, under reasonable orthornormal invariance assumption on F, i.e., if F depends only upon the eigenvalues of D^2u, Proposition 4.2 indicates that a local free boundary condition in the viscosity sense should hold. If so, this will enable the employment of the results established by Wang in [18, 19] to study regularity of the free boundary near flat points.

We would like to point out that the Theorems proven in the present paper, namely, Theorem 2.2, Theorem 2.3 and Theorem 3.1 refer to *any* family of viscosity solution, regardless the way one chooses to select particular solutions of problem (2.1). However, we should not expect, for instance, nondegeneracy, i.e., linear growth away from the free boundary, to hold in such generality. We do believe, in order to establish an appropriate nondegeneracy condition, one must find a stable way to select special viscosity solutions of problem (2.1). A natural guess would be the solutions generated by Perron's method, i.e.,

$$u_\varepsilon(x) = \inf \left\{ \begin{array}{c} h_\varepsilon(x) \ : \ h_\varepsilon \geq f \text{ on } \partial\Omega, \text{ and} \\ F(D^2 h_\varepsilon, x) \leq g^2(x)\beta_\varepsilon(h_\varepsilon), \text{ in the viscosity sense.} \end{array} \right\}.$$

The study of uniform weak geometric properties of equation (2.1) as well as the regularity of the limiting free boundary problem is currently in progress, [16].

Another important and interesting perspective one could explore is the study of the approximation problem (5.1) without sign constraint on the boundary data. The two phase singular perturbation problem

$$\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$$

was fully described in [10, 11] and [13]. Under some geometric assumptions upon the limiting function, the authors show the free boundary condition

$$(u_\nu^+)^2 - (u_\nu^+)^2 = 2A$$

holds in the viscosity sense, where u_ε^- and u_ε^+ stand for the negative and the positive part of u_ε, respectively. Further regularity of the free boundary near flat points could then be derived by Caffarelli's theory, [5] and [6].

It is noteworthy to mention that, for the two phase fully nonlinear singular perturbation problem (equation (5.1) without sign constraint on f), the strategy of the proof of Theorem 2.2 gives Uniform Lipschitz Regularity of $\{u_\varepsilon\}$, provided one knows an a priori uniform bound for either $\nabla u_\varepsilon^- \cdot \nu$ or $\nabla u_\varepsilon^+ \cdot \nu$. In order to establish a complete corresponding result of Theorem 2.2, when u_ε changes sign, one would need a sort of Monotonicity Formula for Fully Nonlinear Equation, as in [1] or in [9]. So far, this task remains unaccomplished. It would certainly be a major achievement to the theory of non-linear elliptic equations and its applications to free boundary problems.

Another important line of research would be the study of non-isotropic fully nonlinear singular equations, that is, when the singular term also depends upon direction:

$$F(D^2u_\varepsilon, x) = g^2(x)\beta_\varepsilon(u_\varepsilon)H(\nabla u_\varepsilon).$$

As we reported in the introduction, uniform Lipschitz regularity of a family of solutions to

$$\Delta u = \beta_\varepsilon(u)H(Du), \tag{5.2}$$

has been established by Caffarelli, Jerison and Kenig in [9]. D. Moreira has recently studied problem (5.2) by a least supersolution method, obtaining a nice geometric description of the limiting free boundary. The complete study of non-isotropic singular perturbation problems for more general (non-linear) elliptic equations is currently under investigation, [16].

The study of parabolic versions of equation (5.1) is also an important line of investigation. Evolutionary equation of the form

$$\Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon), \tag{5.3}$$

has been considered by Caffarelli and Vázquez in [12], Caffarelli, Lederman and Wolanski in [10, 11], among others. Danielli has recently studied this free boundary problem for a rather general parabolic operator in divergence form. One of the main contributions of this work is to show the "expected" free boundary condition holds in the viscosity sense, leading to a weak solution to certain problems in combustion theory.

The similarity of elliptic free boundary problems and their parabolic versions lies on the philosophical strategy to approach them. However, due to different homogeneities corresponding to the equation (parabolic dilatation invariant) and to the free boundary (scale of degree one), regularity of parabolic free boundary problems are, in general, extremely more delicate than its stationary version. For instance, it is not true, in general, that Lipschitz parabolic free boundaries are $C^{1,\alpha}$ (see, for instance, [3]) contradicting what happens in the elliptic environment (Caffarelli's theorem). On the other hand, Lipschitz regularity of the free boundary does imply Lipschitz regularity of the viscosity solution, see [2].

We believe some of the advances needed to understand the stationary fully nonlinear singular perturbation problem (5.1) will play an important role in the study of its parabolic version

$$F(D^2u^\varepsilon) - u_t^\varepsilon = \beta_\varepsilon(u_\varepsilon). \tag{5.4}$$

It is definitely a rewarding project (if accomplished) to try to establish similar results obtained for the elliptic case, respecting of course, the bounds already pointed above, imposed by the parabolic nature of the problem.

It seems that if a reasonable weak type of nondegeneracy condition for viscosity solutions of (5.4) can be established, one might be able to obtain interior improvement of the Lipschitz norm in both time and space, leading us to a C^1 regularity result for the free boundary. For the evolutionary equation (5.4), again, if a uniform control in x is known upon either $u_\varepsilon^- \cdot \nu$ or $u_\varepsilon^+ \cdot \nu$, the strategy of Theorem 2.2 can be used to obtain full uniform in ε Lipschitz regularity for the family $\{u_\varepsilon\}$. In accordance with the stationary situation, for the two phase case of equation (5.4) a generalization of the Parabolic Monotonicity Formula for harmonic functions would be needed in order to obtain a full uniform in ε Lipschitz continuity of solutions.

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