Matemática Contemporânea, Vol 30, 109-123

AN ISOMETRIC IMMERSION THEOREM IN Sol^3

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Abstract

Using a recent result on the existence of G-structure preserving isometric immersions (see [1]), we prove an existence result for isometric immersion into the three dimensional Lie group Sol^3. This group is one of the eight simply connected homogeneous manifolds, as classified by Scott (see [6]).

1 Introduction

It is well known that an isometrically immersed hypersurface needs to satisfy the compatibility equations called the Codazzi, Gauss and Ricci equations. In the case where the ambient spaces has constant sectional curvature ($\mathbb{S}^n$, $\mathbb{H}^n$ and $\mathbb{R}^n$), these equations prove to be sufficient too. Recently, in [3], some isometric immersion theorems were proven in an ambient space given by a Riemannian product $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$; in [4] the author proves an immersion theorem in homogeneous 3-manifolds whose isometry group has dimension 4. As presented in [6], eight 3-dimensional Riemannian manifolds characterize all 3-dimensional geometries. The only simply connected, homogeneous Riemannian 3-manifold whose isometry group has dimension 3 is given by the Lie group Sol^3, endowed with a left-invariant metric. In this note we prove an isometric immersion theorem in Sol^3, and, more in general, into Lie groups endowed with a left-invariant Riemannian metric, using a recent result of existence of affine immersions, due to Piccione and Tausk (see [1]), that uses the notion of G-structure.

*The author is sponsored by Fapesp, Grant number 04/13586-4.

I wish to thank Daniel Victor Tausk and Paolo Piccione for the constant support throughout my work.

2 Preliminaries

Let V_0 and V be arbitrary vector spaces having the same dimension and the same field of scalars; a linear isomorphism $p : V_0 \to V$ will be called a V_0*-frame* of V. Let $\mathrm{GL}(V_0)$ denote the *general linear group* of V_0, i.e., the group of all linear isomorphisms of V_0. Then the set $\mathrm{FR}_{V_0}(V)$ of all V_0-frames of V is a $\mathrm{GL}(V_0)$-structure on the set V modeled upon V_0.

Let M be a differentiable manifold, E_0 be a real finite-dimensional vector space, E be a set and $\pi : E \to M$ be a map; for each $x \in M$ we denote by E_x the subset $\pi^{-1}(x)$ of E and we call it the *fiber* of E over x. Assume that for each $x \in M$ we are given a real vector space structure on the fiber E_x such that E_0 and E_x have the same dimension. The set $\mathrm{FR}_{E_0}(E_x)$ of all E_0-frames of E_x is thus a principal space with structural group $\mathrm{GL}(E_0)$.

Throughout this article we will use the concepts of *Christoffel tensor* and *inner-torsion* of an affine manifold with G-structure (see [1]).

To define the Christoffel tensor we must notice that, if $\pi : E \to M$ is a vector bundle with typical fiber E_0 and $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local E_0-frame of E, we can define a connection $\mathfrak{d}^s$ on $E|_U$ by:

$$(\mathfrak{d}^s_X V)(x) = s(x)[d\widetilde{V}_x(X)]$$

for all V smooth section of $E|_U$ ($V \in \Gamma(E|_U)$), X smooth section of TM ($V \in \Gamma(TM)$) and all $x \in U$, where $\widetilde{V}$ is such that $s \circ \widetilde{V} = V$.

If ∇ is a connection on E, we define the *Christoffel tensor* of ∇ with respect to s as the $C^\infty(M)$-bilinear map

$$\Gamma : \Gamma(TM|_U) \times \Gamma(E|_U) \to \Gamma(E|_U)$$

such that $\Gamma = \nabla - \mathfrak{d}^s$; recall that the difference of two connections is a tensor.

Let now $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇, let G be a Lie subgroup of $\mathrm{GL}(E_0)$ and $P \subset \mathrm{FR}_{E_0}(E)$ be a G-structure on E. For each $x \in M$, denote by $G_x \subset \mathrm{GL}(E_x)$ the Lie subgroup consisting of all G-structure preserving isomorphisms of E_x, i.e., $g \in G_x$ iff $g \circ p \in P_x$ for all $p \in P_x$; denote by $\mathfrak{g}_x \subset \mathfrak{gl}(E_x)$ its Lie algebra.

For $x \in M$ and $p \in P_x$, we can define $\sigma : \mathrm{GL}(E_0) \ni g \mapsto p \cdot g \in \mathrm{FR}_{E_0}(E)$ and then $I_p : \mathrm{GL}(E_0) \ni g \mapsto \sigma \circ L_g \circ \sigma^{-1} \in \mathrm{GL}(E_x)$. Define $\mathrm{Ad}_p : \mathfrak{gl}(E_0) \to \mathfrak{gl}(E_x)$ as the differential of the Lie group isomorphism I_p and $\overline{\mathrm{Ad}}_p : \frac{\mathfrak{gl}(E_0)}{\mathfrak{g}} \to \frac{\mathfrak{gl}(E_x)}{\mathfrak{g}_x}$ the map induced by Ad_p in the quotient. Observe that $\mathrm{Ad}(\mathfrak{g}) \subset \mathfrak{g}_x$.

Recall that ∇ is associated to a unique principal connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ on the $\mathrm{GL}(E_0)$-principal bundle $\mathrm{FR}_{E_0}(E)$ (see [1], Proposition 2.5.4); denote by ω the connection form of this principal connection. Let $s : U \to P$ be a smooth local section of P. For $x \in U$ set $p = s(x)$ and $\overline{\omega} = s^*\omega$.

Define $\mathfrak{I}_x^P : T_xM \to \frac{\mathfrak{gl}(E_x)}{\mathfrak{g}_x}$ as the composition of maps illustrated in the following diagram:

$$\begin{tikzcd} T_xM \arrow[r, "\overline{\omega}_x"] \arrow[rrr, bend right, "\mathfrak{I}_x^P"'] & \mathfrak{gl}(E_0) \arrow[r, "\text{quotient}"] & \mathfrak{gl}(E_0)/\mathfrak{g} \arrow[r, "\overline{\mathrm{Ad}}_p"] & \mathfrak{gl}(E_x)/\mathfrak{g}_x \end{tikzcd} \tag{1}$$

It can be proved $\mathfrak{I}_x^P$ does not depend on the choice of the local section s (see [1], Lemma 2.10.3). We call $\mathfrak{I}_x^P$ the *inner torsion* of the G-structure P at the point x with respect to the connection ∇.

The result in [1, Lemma 2.11.1] gives a more convenient method for computing the inner torsion $\mathfrak{I}_x^P$ as the composition of the Christoffel tensor Γ_x of the connection ∇ with respect to s and the quotient map $\mathfrak{gl}(E_x) \to \frac{\mathfrak{gl}(E_x)}{\mathfrak{g}_x}$.

Now, let (M, ∇) be an n-dimensional affine manifold, let G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and let $P \subset \mathrm{FR}(TM)$ be a G-structure on M. Denote by T and R respectively the torsion and the curvature tensors of ∇. Given $x, y \in M$, a map $\sigma : T_xM \to T_yM$ is *G-structure preserving* if $\sigma \circ p \in P_y$ for some (and hence for all) $p \in P_x$. A smooth map $f : M \to M$ is said to be *G-structure preserving* if $\mathrm{d}f_x : T_xM \to T_{f(x)}M$ is G-structure preserving for all $x \in M$.

Denote by $\overline{\mathrm{Ad}}_\sigma$ is a linear isomorphism from $\frac{\mathfrak{gl}(T_xM)}{\mathfrak{g}_x}$ to $\frac{\mathfrak{gl}(T_yM)}{\mathfrak{g}_y}$ defined by passing to the quotient $\mathrm{Ad}_\sigma : \mathfrak{gl}(T_xM) \to \mathfrak{gl}(T_yM)$, the differential of the Lie group isomorphism $I_\sigma : GL(T_xM) \ni T \mapsto \sigma \circ T \circ \sigma^{-1} \in GL(T_yM)$ at identity (it's well defined since Ad_σ carries $\mathfrak{g}_x$ to $\mathfrak{g}_y$).

Definition 2.1 *We say that the triple* (M, ∇, P) *is an* infinitesimally homogeneous affine manifold with G-structure *if for all* $x, y \in M$, *every* G*-structure preserving map* $\sigma : T_xM \to T_yM$ *relates* T_x *with* T_y, R_x *with* R_y *and* $\mathfrak{I}^P_x$ *with* $\mathfrak{I}^P_y$, *i.e.,* $\mathrm{T}_y(\sigma\cdot, \sigma\cdot) = \sigma \circ \mathrm{T}_x$, $R_y(\sigma\cdot, \sigma\cdot) = \sigma \circ R(\cdot, \cdot) \circ \sigma^{-1}$ *and* $\mathfrak{I}^P_y \circ \sigma = \overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^P_x$.

With the notions of *Christoffel tensor* and *inner-torsion* presented let us now describe the canonical 1-structure associated to a Lie Group H. Let H be a Lie group and $\mathfrak{h}$ its Lie algebra; assume that $\mathfrak{h}$ is endowed with an inner product. The choice of an orthonormal basis of $\mathfrak{h}$ gives a 1-structure on the tangent bundle TH; more precisely, if $\{E_i\}$ denotes an orthonormal basis of $\mathfrak{h}$ and $\{e_i\}$ the canonical basis of $\mathbb{R}^n$, let $s : H \ni p \mapsto F_p \in \mathrm{FR}_{\mathbb{R}^n}(T_pH)$ with F_p such that $F_p(e_i) = E_i(p)$ be a global smooth section of the frame bundle associated to TG. Then $P = s(H)$ is a G-structure on TH with $G = \{\mathrm{Id}_{\mathbb{R}^n}\}$.

For each $p \in H$, we have $G_p = \{\mathrm{Id}_{T_pH}\}$ and $\mathfrak{g}_p = \{0\}$.

Using the notions of *Christoffel tensor* and *inner-torsion* presented above we see that, if ∇ is the Levi-Civita connection in TH, then the inner-torsion $\mathfrak{I}^P_p : T_pH \to \mathfrak{gl}(T_pH)$ is equal to the Christoffel tensor $\Gamma_p : T_pH \to \mathfrak{gl}(T_pH)$ corresponding to s.

3 Isometric immersions into Lie groups

In this section we discuss an immersion theorem into Lie groups (Corollary 3.3) as an application of Theorem 3.2 whose proof can be found at [1] (Theorem 3.5.2).

Let $\pi : E \to M$ be a vector bundle on M with typical fiber $\mathbb{R}^k$ endowed with a semi-Riemannian structure g^E of index s. Denote by $\langle\cdot,\cdot\rangle_s$ the standard Minkowski metric in $\mathbb{R}^k$ having index s, and let $\widetilde{G} = \mathrm{O}(k, k-s)$ the be Lie subgroup of $\mathrm{GL}(\mathbb{R}^k)$ consisting of all linear isomorphisms of $\mathbb{R}^k$ preserving $\langle\cdot,\cdot\rangle_s$. Define $\mathrm{FR}^o(E)$ as the $\widetilde{G}$-structure consisting of all linear isometries $p : \mathbb{R}^k \to E_x$, $x \in M$; these will be called the *orthonormal frames* of E.

Definition 3.1 *Suppose we are given the following data:*

- *an* $\overline{n}$*-dimensional semi-Riemannian manifold* $(\overline{M}, \overline{g})$*, where the semi-Riemannian metric* $\overline{g}$ *has index* $\overline{r}$*;*
- *an* n*-dimensional semi-Riemannian manifold* (M, g)*, where the semi-Riemannian metric* g *has index* r*;*
- *a vector bundle* $\pi : E \to M$ *with typical fiber* $\mathbb{R}^k$ *and a semi-Riemannian structure* g^E *of index* s*, where* $\overline{n} = n + k$ *and* $\overline{r} = r + s$*;*
- *a connection* ∇^E *on* E *compatible with* g^E*;*
- *a smooth section* α^0 *of* $\mathrm{Lin}_2^{\mathrm{s}}(TM, E)$*.*

By a local solution of the semi-Riemannian isometric immersion problem *corresponding to the data above we mean a pair* (f, S)*, where* $f : U \to \overline{M}$ *is an isometric immersion defined in an open subset* U *of* M *and* $S : E|_U \to f^\perp$ *is an isomorphism of vector bundles such that:*

- $\overline{g}_{f(x)}\big(S_x(e), S_x(e')\big) = g_x^E(e, e')$*, for all* $x \in U$ *and all* $e, e' \in E_x$*;*
- S *is connection preserving if* E *is endowed with* ∇^E *and* $f^\perp$ *is endowed with the normal connection* $\nabla^\perp$*;*
- S *carries* α^0 *to the second fundamental form* α *of the isometric immersion* f*, i.e.,* $S_x \circ \alpha_x^0 = \alpha_x$*, for all* $x \in U$*.*

We call U *the* domain *of the local solution* (f, S)*. By a* solution of the semi-Riemannian isometric immersion problem *we mean a local solution* (f, S) *whose domain is* M*.*

Consider the vector bundle $\widehat{E} = TM \oplus E$ endowed with the semi-Riemannian structure $\widehat{g}$ whose restrictions to TM and E are g and g^E respectively and such that TM and E are orthogonal. Let G be a Lie subgroup of $\mathrm{O}_{\overline{r}}(\mathbb{R}^{\overline{n}})$, $\widehat{P}$ be a G-structure on $\widehat{E}$ and $\overline{P}$ be a G-structure on $\overline{M}$ such that $\widehat{P} \subset \mathrm{FR}^{\mathrm{o}}(\widehat{E})$

and $\overline{P} \subset \mathrm{FR}^{\mathrm{o}}(T\overline{M})$. A local solution (f, S) of the semi-Riemannian isometric immersion problem with domain $U \subset M$ is said to be *G-structure preserving* if for all $x \in U$, the linear isomorphism:

$$\mathrm{d}f_x \oplus S_x : \widehat{E}_x = T_xM \oplus E_x \longrightarrow \mathrm{d}f_x(T_xM) \oplus f_x^{\perp} = T_{f(x)}\overline{M}$$

is G-structure preserving.

Theorem 3.2 *Suppose we are given data as in Definition 3.1; denote by ∇ the Levi-Civita connection of (M, g) and by $\overline{\nabla}$ the Levi-Civita connection of $(\overline{M}, \overline{g})$. Consider the vector bundle $\widehat{E} = TM \oplus E$ endowed with the semi-Riemannian structure $\widehat{g}$ whose restrictions to TM and E are g and g^E respectively and such that TM and E are orthogonal. Let $\widehat{\nabla}$ be the connection on $\widehat{E}$ that is compatible with $\widehat{g}$ and whose components are ∇, ∇^E and α^0 (see [1, Subsection 2.8.1]). Let G be a Lie subgroup of $\mathrm{O}_{\overline{r}}(\mathbb{R}^{\overline{n}})$, $\widehat{P}$ be a G-structure on $\widehat{E}$ and $\overline{P}$ be a G-structure on $\overline{M}$ such that $\widehat{P} \subset \mathrm{FR}^{\mathrm{o}}(\widehat{E})$ and $\overline{P} \subset \mathrm{FR}^{\mathrm{o}}(T\overline{M})$. Assume that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous and that for all $x \in M$, $y \in \overline{M}$ and every G-structure preserving map $\sigma : \widehat{E}_x \to T_y\overline{M}$, the following conditions hold:*

(a) σ relates the inner torsion of $\widehat{P}$ with the inner torsion of $\overline{P}$, i.e.:

$$\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}_x^{\widehat{P}} = \mathfrak{I}_y^{\overline{P}} \circ \sigma;$$

(b) the Gauss equation holds:

$$\overline{g}_y\big[\overline{R}_y\big(\sigma(v), \sigma(w)\big)\sigma(u), \sigma(z)\big] = g_x\big(R_x(v, w)u, z\big) - g_x^E\big(\alpha_x^0(w, u), \alpha_x^0(v, z)\big) + g_x^E\big(\alpha_x^0(v, u), \alpha_x^0(w, z)\big),$$

for all $u, v, w, z \in T_xM$;

(c) the Codazzi equation holds:

$$\overline{g}_y\big[\overline{R}_y\big(\sigma(v), \sigma(w)\big)\sigma(u), \sigma(e)\big] = g_x^E\big((\nabla^{\otimes}\alpha^0)_x(v, w, u), e\big) - g_x^E\big((\nabla^{\otimes}\alpha^0)_x(w, v, u), e\big),$$

for all $u, v, w \in T_xM$ and all $e \in E_x$, where $\nabla^{\otimes}$ denotes the connection induced by ∇ and ∇^E on $\mathrm{Lin}_2(TM, E)$;

(d) the Ricci equation holds:

$$\overline{g}_y\big[\overline{R}_y\big(\sigma(v),\sigma(w)\big)\sigma(e),\sigma(e')\big] = g_x^E\big(R_x^E(v,w)e,e'\big) \\ + g_x\big(\alpha_x^0(v)^*\cdot e,\alpha_x^0(w)^*\cdot e'\big) - g_x\big(\alpha_x^0(w)^*\cdot e,\alpha_x^0(v)^*\cdot e'\big),$$

for all $v,w\in T_xM$ and all $e,e'\in E_x$, where R^E denotes the curvature tensor of ∇^E.

Then, for all $x_0\in M$, all $y_0\in\overline{M}$ and for every G-structure preserving map $\sigma_0:\widehat{E}_{x_0}\to T_{y_0}\overline{M}$ there exists a G-structure preserving local solution (f,S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0)=y_0$,

$$\sigma_0 = \mathrm{d}f_{x_0}\oplus S_{x_0}:\widehat{E}_{x_0}=T_{x_0}M\oplus E_{x_0}\longrightarrow \mathrm{d}f_{x_0}(T_{x_0}M)\oplus f_{x_0}^{\perp}=T_{y_0}\overline{M}. \qquad (2)$$

If M is connected and simply-connected and if $(\overline{M},\overline{\nabla})$ is geodesically complete then one can find a unique G-structure preserving global solution (f,S) of the semi-Riemannian isometric immersion problem satisfying the initial condition above.

Applied to the case of Lie groups endowed with a left invariant metric and endowed with the 1-structure defined by the choice of a left invariant referential, the theorem above leads us to the following Corollary:

Corollary 3.3 *Let H be a $\overline{n}$-dimensional Lie group with Lie algebra $\mathfrak{h}$, and let $\overline{g}$ be a left-invariant Riemannian metric tensor on H, with Levi-Civita connection $\overline{\nabla}$. Let (M,g) a n-dimensional Riemannian manifold ($n=\overline{n}-1$) with Levi-Civita connection ∇. Let A be a smooth field of symmetric operators*

$$A_p:T_pM\to T_pM,$$

let T_i ($1\le i\le n$) be smooth vector fields on M and let f_i be smooth real functions on M such that $||T_i||^2+f_i^2=1$. Let $\{E_i;1\le i\le n\}$ be an orthonormal basis of $\mathfrak{h}$, and set:

$$\overline{\nabla}_{E_i}E_j=\sum_k\Gamma_{ij}^k\,E_k, \quad \forall\, i,j=1,\dots,\overline{n}.$$

and

$$\overline{R}_{ijkl} = \overline{g}(\overline{R}(E_i, E_j)E_k, E_l) \quad \forall\, i, j, k, l = 1, \ldots, \overline{n}.$$

Assume that the following relations hold:

- *for* $1 \le i, j \le \overline{n}$*:*

$$\begin{cases} \nabla_{T_i} T_j - f_j A(T_i) = \sum_{k,l} g(T_i, T_k) \Gamma^l_{kj} T_l \\ g(A(T_i), T_j) + T_i(f_j) = \sum_{k,l} g(T_i, T_k) \Gamma^l_{kj} f_l \end{cases}$$

Assume that, for all $x \in M$*:*

- *the Gauss equation holds:*

$$\sum_{i,j,k,l} v^i w^j u^k z^l \overline{R}_{ijkl} = g_x(R_x(v, w)u, z) \\ - g_x(A(w), v) g_x(A(v), z) + g_x(A(u), v) g_x(A(w), z)$$

  for all $u, v, w, z \in T_x M$*;*

  Where, for $1 \le i \le \overline{n}$*:*

$$v^i = g_x(v, T_i(x)) \qquad w^i = g_x(w, T_i(x))$$
$$u^i = g_x(u, T_i(x)) \qquad z^i = g_x(z, T_i(x))$$

- *the Codazzi equation holds:*

$$\sum_{i,j,k,l} v^i w^j u^k f_l(x) \overline{R}_{ijkl} = v(g(A(W), U)) - w(g(A(V), U)) \\ + g_x(A(u), [W, V]) + g_x(A(v), \nabla_w U) - g_x(A(w), \nabla_v U)$$

  for all $u, v, w \in T_x M$ *(*v^i*,* w^i *and* u^i *as defined above) and* U, V, W *local extensions of* u, v, w *respectively;*

Then, for all $x_0 \in M$ *and* $y_0 \in H$*, there exists* $f : U \to H$ *(where* U *is an open neighborhood of* x_0 *in* M*) isometric immersion into* H *such that* $f(x_0) = y_0$ *and* $S : U \times \mathbb{R} \to f^{\perp}$ *isomorphism of vector bundles such that:*

- $\overline{g}_{f(x)}(S_x(e), S_x(e')) = ee'$, *for all* $x \in U$ *and for all* $e, e' \in x \times \mathbb{R}$*;*
- S *is connection preserving if* $U \times \mathbb{R}$ *is endowed with* ∇^E *such that* $\nabla^E_V e = V(e)$ *(for* V *vector field in* U *and* e *real function in* U*) and* $f^{\perp}$ *is endowed with the normal connection* $\nabla^{\perp}$*;*
- S *carries* $\alpha^0 := g(A(\cdot), \cdot)\frac{\partial}{\partial t}$ *to the second fundamental form* α *of the isometric immersion* f*, i.e.,* $S_x \circ \alpha^0_x = \alpha_x$*, for all* $x \in U$*.*

Moreover, if M *is complete, connected and simply-connected there exists a unique global isometric immersion* (f, S) *of* M *into* H *with the properties above.*

Proof. As described in the preliminaries, we have $\overline{s} : H \ni p \mapsto F_p \in \mathrm{FR}_{\mathbb{R}^{\overline{n}}}(T_pH)$ with F_p such that $F_p(e_i) = E_i(p)$ $(1 \leq i \leq \overline{n})$ and $\overline{P} = \overline{s}(H)$ the G-structure of H $(G = \{\mathrm{Id}_{\mathbb{R}^{\overline{n}}}\})$. In the following, if $x \in H$, we'll denote by $\overline{G}_x$ the Lie group $\{\mathrm{Id}_{T_xH}\}$ and by $\overline{\mathfrak{g}}_x$ its Lie algebra.

Endow the vector bundle $\widehat{E} = TM \oplus \mathbb{R}$ with the Riemannian structure $\widehat{h}$ whose restrictions to TM and $\mathbb{R}$ are g and g^E $(g^E_x(a, b) := a(x)b(x),\ a, b$ real functions in $M)$ respectively and such that TM and $\mathbb{R}$ are orthogonal.

Let $\widehat{\nabla}$ be the connection on $\widehat{E}$ given by $\widehat{\nabla}_V(W, a) = (\nabla_V W - aA(V), g(A(V), W) + V(a))$ for V and W vector fields on M and a a real function on M (connection which is compatible with $\widehat{g}$).

Consider $\widehat{s} : M \ni p \mapsto F_p \in \mathrm{FR}_{\mathbb{R}^{\overline{n}}}(T_pM \oplus \mathbb{R})$ with F_p such that $F_p(e_i) = (T_i(p), f_i(p))$ $(1 \leq i \leq \overline{n})$ and $\widehat{P} = \widehat{s}(M)$ a G-structure on $\widehat{E}$. Denote by $\widehat{G}_x$ the Lie group $\{Id_{\widehat{E}_x}\}$ and by $\widehat{\mathfrak{g}}_x$ its Lie algebra (where $x \in M$).

For $x \in M$ and $y \in H$, let $\sigma : \widehat{E}_x \to T_yH$ be the linear map such that $\sigma(T_i(x), f_i(x)) = E_i(y)$ (G-structure preserving map).

Since $(H, \overline{\nabla}, \overline{P})$ is homogeneous, we can easily conclude from Proposition 6.4 of [2] that it's indeed infinitesimally homogeneous.

Let's prove now that our assumptions imply that $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma|_{T_xM}$, where $\overline{\mathrm{Ad}}_\sigma$ is a linear isomorphism from $\frac{\mathfrak{gl}(\widehat{E}_x)}{\widehat{\mathfrak{g}}_x}$ to $\frac{\mathfrak{gl}(T_yH)}{\overline{\mathfrak{g}}_y}$ defined by passing to the quotient $\mathrm{Ad}_\sigma : \mathfrak{gl}(\widehat{E}_x) \to \mathfrak{gl}(T_yH)$, the differential of the Lie group isomorphism

$I_\sigma : GL(\widehat{E}_x) \ni T \mapsto \sigma \circ T \circ \sigma^{-1} \in GL(T_yH)$ at identity (it's well defined since Ad_σ carries $\widehat{\mathfrak{g}}_x$ to $\overline{\mathfrak{g}}_y$).

Since for the 1-structures already mentioned $\widehat{\mathfrak{g}}_x$ and $\overline{\mathfrak{g}}_y$ are null Lie algebras then, for all $x \in M$ and all $y \in H$, we have $\overline{\mathrm{Ad}_\sigma} = \mathrm{Ad}_\sigma$.

Remember now that $\overline{\Gamma} = \overline{\nabla} - \mathfrak{d}^{\overline{s}}$, where

$$(\mathfrak{d}^{\overline{s}}_X V)(x) = \overline{s}(x)[d\widetilde{V}_x(X(x))]$$

(X and V vector fields on H, and $\overline{s} \circ \widetilde{V} = V$). And since $\mathfrak{d}^{\overline{s}}_X V = 0$ when V is left-invaiant ($d\widetilde{V}_x$ is constant), $\overline{\Gamma} = \overline{\nabla}$ on left-invariant vector fields.

It's also true that $\widehat{\Gamma}(T_i, (T_j, f_j)) = \widehat{\nabla}_{T_i}(T_j, f_j)$ because if $\widetilde{Y}_i$ is a section of $\widehat{E}$ such that $\widehat{s} \circ \widetilde{Y}_i = (T_i, f_i)$ then $\widetilde{Y}_i(p) = e_i$ (which implies $(d\widetilde{Y}_i)_p = 0$). Since

$$(\mathfrak{d}^{\widehat{s}}_X (T_i, f_i))(p) = \widehat{s}(p)[(d\widetilde{Y}_i)_p(X(p))] = 0$$

we have $\widehat{\Gamma}(T_i, (T_j, f_j)) = \widehat{\nabla}_{T_i}(T_j, f_j)$ as previously mentioned.

Then $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma|_{T_xM}$ if and only if $\widehat{\Gamma}_x(T_i, (T_j, f_j)) = (\mathrm{Ad}_{\sigma^{-1}} \circ \overline{\Gamma}_y \circ \sigma)(T_i, (T_j, f_j))$ (observe that $\widehat{\Gamma}$ and $\overline{\Gamma}$ are tensors, $\mathrm{Ad}_\sigma^{-1} = \mathrm{Ad}_{\sigma^{-1}}$ and the following diagram commutes).

$$\begin{tikzcd}
T_xM \arrow[r, "\sigma|_{T_xM}"] \arrow[d, "\widehat{\Gamma}_x"] & T_yH \arrow[d, "\overline{\Gamma}_y"] \\
\mathfrak{gl}(\widehat{E}_x) \arrow[r, "\mathrm{Ad}_\sigma"'] & \mathfrak{gl}(T_yH)
\end{tikzcd}$$

Since

$$(T_i, 0) = \sum_j \widehat{g}((T_i, 0), (T_j, f_j))(T_j, f_j) = \sum_j g(T_i, T_j)(T_j, f_j)$$

it's true that

$$\widehat{\nabla}_{T_i}(T_j, f_j) = \widehat{\Gamma}_x(T_i, (T_j, f_j)) = (\mathrm{Ad}_{\sigma^{-1}} \circ \overline{\Gamma}_y \circ \sigma)(T_i, (T_j, f_j)) =$$
$$= \sum_k g(T_i, T_k)\sigma^{-1} \circ \overline{\Gamma}_y(E_k, E_j) = \sum_k g(T_i, T_k)\sigma^{-1} \circ \overline{\nabla}_{E_k} E_j$$

And, since $\widehat{\nabla}_{T_i}(T_j, f_j) = (\nabla_{T_i} T_j - f_j A(T_i), g(A(T_i), T_j) + T_i(f_j))$ and $\overline{\nabla}_{E_i} E_j = \sum_k \Gamma^k_{ij} E_k$, for all $i, j = 1, \ldots, \overline{n}$, we have:

$$\begin{cases} \nabla_{T_i}T_j - f_jA(T_i) = \sum_{k,l} g(T_i,T_k)\Gamma^l_{kj}T_l \\ g(A(T_i),T_j) + T_i(f_j) = \sum_{k,l} g(T_i,T_k)\Gamma^l_{kj}f_l \end{cases}$$

This proves these equations are necessary and sufficient for $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma|_{T_xM}$ to be true.

The Gauss equation as presented bellow follows immediately from Gauss equation as presented in the Theorem 3.2.

In our case $\alpha_x(\cdot,\cdot) = g_x(A(\cdot),\cdot)$.

$$\overline{g}_y[\overline{R}_y(\sigma(v),\sigma(w))\sigma(u),\sigma(z)] = g_x(R_x(v,w)u,z) -$$

$$-g_x(A(w),v)g_x(A(v),z) + g_x(A(u),v)g_x(A(w),z)$$

for all $u,v,w,z \in T_xM$;

Notice then that:

$$v = \sum v^i(T_i,f_i)_x \qquad w = \sum w^i(T_i,f_i)_x$$
$$u = \sum u^i(T_i,f_i)_x \qquad z = \sum z^i(T_i,f_i)_x$$

The Codazzi equation on the other hand follows easily when noticed that

$$(\nabla^{\otimes}\alpha^0)_x(v,w,u) = v(g(A(W),U)) - g_x(\nabla_vW,u) - g_x(w,\nabla_vU)$$

for all $u,v,w \in T_xM$ and U,V,W local extensions of u,v,w respectively.

$$\overline{g}_y[\overline{R}_y(\sigma(v),\sigma(w))\sigma(u),\sigma(1)] = v(g(A(W),U)) - w(g(A(V),U)) +$$

$$+g_x(A(u),[W,V]) + g_x(A(v),\nabla_wU) - g_x(A(w),\nabla_vU)$$

for all $u,v,w \in T_xM$ and U,V,W local extensions of u,v,w respectively;

As $v = \sum v^i(T_i,f_i)_x$, $w = \sum w^i(T_i,f_i)_x$, $u = \sum u^i(T_i,f_i)_x$, $1 = \sum f_i(x)(T_i,f_i)_x$ Codazzi equation, as presented in the theorem, follows.

Ricci equation naturally holds:

Notice that $\overline{g}_y[\overline{R}_y(\sigma(v),\sigma(w))\sigma(e'),\sigma(e)] = 0$ and $g_x[R^E_x(v,w)e',e] = 0$ for all $v, w \in T_xM$ and all $e, e' \in E_x$ since:

$$\overline{g}_y[\overline{R}_y(\cdot,\cdot)\cdot,\cdot] \qquad \text{and} \qquad g_x[R^E_x(\cdot,\cdot)\cdot,\cdot] = 0$$

are anti-symmetric in the last 2 variables and e' and e are linearly dependents ($\dim E_x = 1$).

Moreover, we have

$$g_x(\alpha^0_x(v)^* \cdot e, \alpha^0_x(w)^* \cdot e') - g_x(\alpha^0_x(w)^* \cdot e, \alpha^0_x(v)^* \cdot e')$$
$$= ee'[g_x(A(v),A(w)) - g_x(A(v),A(w))] = 0$$

($v, w \in T_xM$, $e, e' \in \mathbb{R}$).

Then, Ricci equation holds immediately.

From the facts presented above, Theorem 3.2 concludes the proof.

□

4 Isometric immersions into Sol^3

By Sol^3, or simply Sol, we mean the Lie group whose base manifold is $\mathbb{R}^3$, endowed with group operation:

$$(x,y,z)\cdot(x',y',z') = (x + e^{-z}x', y + e^{z}y', z + z')$$

and the left-invariant metric $\mathrm{d}s^2 = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$.

We consider the $\{\mathrm{Id}_{\mathbb{R}^3}\}$-structure on Sol given by the orthonormal left-invariant frame X_1, X_2 and X_3 defined by:

$X_1 : \mathrm{Sol} \ni (x,y,z) \mapsto (e^z, 0, 0) \in T_{(x,y,z)}\mathrm{Sol}$

$X_2 : \mathrm{Sol} \ni (x,y,z) \mapsto (0, e^{-z}, 0) \in T_{(x,y,z)}\mathrm{Sol}$

$X_3 : \mathrm{Sol} \ni (x,y,z) \mapsto (0, 0, 1) \in T_{(x,y,z)}\mathrm{Sol}$

Corollary 4.1 *Let (M, g) be a Riemannian surface with Levi-Civita connection ∇. Let A be a smooth field of symmetric operators*

$$A_p : T_pM \to T_pM,$$

let T_i $(1 \leq i \leq 3)$ be smooth vector fields on M and let f_i be smooth real functions on M such that $||T_i||^2 + f_i^2 = 1$.

Assume that the following relations hold:

- *for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$:*

$$\begin{cases} \nabla_{T_i}T_j - f_jA(T_i) = (-1)^j g(T_i, T_j)T_3 \\ g(A(T_i), T_j) + T_i(f_j) = (-1)^j g(T_i, T_j)f_3 \end{cases}$$

- *for $i \in \{1, 2, 3\}$:*

$$\begin{cases} \nabla_{T_i}T_3 - f_3A(T_i) = \sum_{j=1}^{2}(-1)^{(j+1)}g(T_i, T_j)T_j \\ g(A(T_i), T_3) + T_i(f_3) = \sum_{j=1}^{2}(-1)^{(j+1)}g(T_i, T_j)f_j \end{cases}$$

Assume that:

- *the Gauss and Codazzi equations as presented in the corollary above hold.*

Then, for all $x_0 \in M$ and $y_0 \in$ Sol, there exists $f : U \to$ Sol (where U is an open neighborhood of x_0 in M) isometric immersion into Sol *such that $f(x_0) = y_0$ and $S : U \times \mathbb{R} \to f^{\perp}$ isomorphism of vector bundles such that:*

- *$\overline{g}_{f(x)}(S_x(e), S_x(e')) = ee'$, for all $x \in U$ and for all $e, e' \in x \times \mathbb{R}$;*

- *S is connection preserving if $U \times \mathbb{R}$ is endowed with ∇^E such that $\nabla^E_V e = V(e)$ (for V vector field in U and e real function in U) and $f^{\perp}$ is endowed with the normal connection $\nabla^{\perp}$;*

- *S carries $\alpha^0 := g(A(\cdot), \cdot)\frac{\partial}{\partial t}$ to the second fundamental form α of the isometric immersion f, i.e., $S_x \circ \alpha^0_x = \alpha_x$, for all $x \in U$.*

Moreover, if M is complete, connected and simply-connected there exists a unique global isometric immersion (f, S) of M into Sol *with the properties above.*

Proof. First of all let's calculate the value of $\overline{\Gamma}$ on the left-invariant vector fields X_i of Sol^3.

The Lie brackets $[X_i, X_j]$ can be deduced by its values on real functions. We see that:

$$[X_1, X_2] = 0 \qquad [X_1, X_3] = X_1 \qquad [X_2, X_3] = -X_2$$

Koszul formula for the Levi-Civita connection enable us to calculate $\overline{\nabla}_{X_i} X_j$. We obtain:

$$\begin{gathered} \overline{\nabla}_{X_1} X_1 = -X_3 \qquad \overline{\nabla}_{X_1} X_3 = X_1 \\ \overline{\nabla}_{X_2} X_2 = X_3 \qquad \overline{\nabla}_{X_2} X_3 = -X_2 \end{gathered} \tag{3}$$

and $\overline{\nabla}_{X_i} X_j = 0$ otherwise.

The first equations of the preceding corollary are, now, equivalent to:

$$\begin{cases} \nabla_{T_i} T_j - f_j A(T_i) = (-1)^j g(T_i, T_j) T_3 & \text{for } i \in \{1,2,3\} \\ g(A(T_i), T_j) + T_i(f_j) = (-1)^j g(T_i, T_j) f_3 & \text{and } j \in \{1,2\} \end{cases}$$

$$\begin{cases} \nabla_{T_i} T_3 - f_3 A(T_i) = \sum_{j=1}^{2} (-1)^{(j+1)} g(T_i, T_j) T_j & \text{for } i \in \{1,2,3\} \\ g(A(T_i), T_3) + T_i(f_3) = \sum_{j=1}^{2} (-1)^{(j+1)} g(T_i, T_j) f_j & \end{cases}$$

These equations, together with the equations of Gauss and Codazzi, imply our thesis, by the Corollary 3.3.

□

Remark 4.2 Sol*'s curvature can be easily computed since:*

$$2\langle Z, \overline{\nabla}_X Y\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle,$$

where X, Y, Z are left-invariants vector fields of Sol *(see [5] for more details).*

We have:

$$\overline{R}_{1212} = -1, \quad \overline{R}_{1313} = 1, \quad \overline{R}_{2323} = 1.$$

and the others $\overline{R}_{ijkl} = 0$ or given by curvature symmetries.

References

[1] Piccione, P., Tausk, D. V., *The theory of connections and* **G***-structures. Applications to affine and isometric immersions*, XIV Escola de Geometria, UFBA, Salvador, ISBN 85-244-0248-2, (2006).

[2] Piccione, P., Tausk, D. V. *An existence theorem for* **G***-structure preserving affine immersions*, preprint, 2006(`arXiv:math/0610703`).

[3] Daniel, B. *Isometric immersions into* $S^n \times \mathbb{R}$ *and* $H^n \times \mathbb{R}$ *and applications to minimal surfaces*, preprint (`perso-math.univ-mlv.fr/users/daniel.benoit/codazzi.pdf`).

[4] Daniel, B. *Isometric Immersions into* 3*-Dimensional Homogeneous Manifolds*, (2005), to appear in Commentarii Mathematici Helvetici.

[5] Milnor, J.*Curvature of Left Invariant Metrics on Lie Groups*, Advances in Mathematics, 21, (1976), 293-329.

[6] Scott, P. *The Geometries of 3-Manifolds*, The Bulletin of the London Mathematical Society, 15, (1983), 401-487.

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