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SYMPLECTIC FORMS ARE INTRINSICALLY HARMONIC

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Abstract

Using the symplectic Hodge theory, we prove that every nondegenerate closed two-form on a manifold M is intrinsically harmonic.

1 Introduction

Let M be a differentiable manifold and let $\omega \in \Omega^k(M)$. We say that ω is *intrinsically harmonic* if there exists a Riemannian metric g on M such that ω is harmonic relatively to g, i.e., if $\Delta\omega = 0$, where Δ is the Laplacian operator on forms of the metric g. It is well known that this is equivalent to ω being closed and co-closed. A celebrated result by E. Calabi (see [2]) gives a characterization of intrinsically harmonic 1-forms whose singularities are nondegenerate. In [1], the author proves that the symplectic form of a Kähler manifold is harmonic relatively to the Riemannian metric arising from a compatible complex structure; it is also observed that the result remains true for *almost Kähler* manifolds. No other result in this direction is available for forms of order greater than 1. In this notes we will discuss the proof of the following result:

Theorem 1 *Let ω be a symplectic form on a compact manifold M, i.e., ω is a closed and everywhere nondegenerate 2-form. Then, ω is intrinsically harmonic.*

We will use a suitable extension of the Hodge theory to symplectic manifolds, called *symplectic Hodge theory* (see [1]) for the proof of the result, that goes along

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the following lines. First, following [1], we recall a few basic facts from symplectic Hodge theory, and we show that ω is symplectically coclosed; We then choose a compatible quasi-complex structure J on M, and relate the symplectic Hodge theory with the Riemannian Hodge theory relative to the metric $g = \omega(\cdot, J\cdot)$ to conclude that ω is also coclosed relatively to g.

2 Riemannian Hodge Theory

Let (M, g) be a compact oriented n-dimensional Riemannian manifold. At a point $p \in M$, let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of the cotangent space T_p^*M, with respect to the induced inner product and orientation.

Definition 2.1 *The Hodge star operator with respect to g is the linear operator $*_g$ defined on the exterior algebra $\Lambda^k(T_pM)$ by*

$$\begin{aligned} *_g(1) &= e_1 \wedge \cdots \wedge e_n, \\ *_g(e_1 \wedge \cdots \wedge e_n) &= 1, \\ *_g(e_1 \wedge \cdots \wedge e_k) &= e_{k+1} \wedge \cdots \wedge e_n. \end{aligned}$$

We see that $*_g : \Lambda^k(T_pM) \to \Lambda^{n-k}(T_pM)$ and $*_g$ satisfies

$$*_g *_g = (-1)^{k(n-k)}. \tag{2.1}$$

Moreover, $*_g$ takes smooth forms to smooth forms, so we have a linear operator

$$*_g : \Omega^k(M) \to \Omega^{n-k}(M)$$

which satisfies (2.1) also.

The *codifferential* and the *Laplacian* are the linear operators

$$\begin{aligned} \delta &: \Omega^k(M) \to \Omega^{k-1}(M), \\ \Delta &: \Omega^k(M) \to \Omega^k(M), \end{aligned}$$

defined respectively by

$$\delta = (-1)^{n(k+1)+1} *_g d *_g, \tag{2.2}$$

$$\Delta = \delta d + d\delta. \tag{2.3}$$

The operator Δ is also called the *Laplace-Beltrami operator* and satisfies $\Delta *_g = *_g \Delta$. On $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, it is simply the usual Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Definition 2.2 *$\alpha \in \Omega^k(M)$ is called coclosed if $\delta\alpha = 0$ and is called harmonic if $\Delta\alpha = 0$.*

We define an inner product on $\Omega^k(M)$ by setting

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *_g \beta \quad \text{for all} \quad \alpha, \beta \in \Omega^k(M).$$

Such inner product satisfies

$$\begin{aligned} \langle d\alpha, \beta \rangle &= \langle \alpha, d\beta \rangle , \\ \langle \Delta\alpha, \beta \rangle &= \langle \alpha, \Delta\beta \rangle . \end{aligned}$$

We have the following

Lemma 2.3 *$\Delta\alpha = 0$ if and only if $d\alpha = \delta\alpha = 0$, that is, $\alpha \in \Omega^k(M)$ is harmonic if and only if α is closed an coclosed.*

Proof. See [4].

□

3 Intrinsically Harmonic Forms

Let M^n be a differentiable manifold and let $\omega \in \Omega^k(M)$.

Definition 3.1 *ω is called intrinsically harmonic if there exists a Riemannian metric g on M such that $\Delta\omega = 0$, where Δ denotes the Laplacian operator defined in (2.3).*

In general, there are no known criteria to establish whether a given k-form ω is intrinsically harmonic. A few partial results are known in the case of 0-forms, of n-forms, and in the case of 1-forms.

- 0-forms: If $f \in \Omega^0(M)$ is closed, i.e., $df = 0$, then f is locally constant and therefore f is harmonic relatively to any metric.

- n-forms: Assume that M oriented and connnected. If $\omega \in \Omega^n(M)$ then ω is obviously closed. Moreover, ω is harmonic relatively to a metric g if and only if $*_g\omega$ is closed, and therefore constant. Assume that $\omega_p \neq 0$ for all $p \in M$. We define a metric by stating a orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM if $\omega_p(e_1, \ldots, e_n) = 1$. Relatively to such metric, we have $*_g\omega_p = 1$ for all $p \in M$ and therefore ω is harmonic.

- For 1-forms, the following beautiful criterion is due to Calabi (see [2]). Given $\omega \in \Omega^1(M)$ and $p \in M$, we define the *future* and the *past* of the point p relatively to ω to be respectively the sets

$$F_\omega(p) = \{q \in M : \omega_{\gamma(t)}(\gamma^{'}(t)) > 0,\ t \in [0,1]\}$$

and

$$P_\omega(p) = \{q \in M : \omega_{\gamma(t)}(\gamma^{'}(t)) < 0,\ t \in [0,1]\},$$

where $\gamma : [0,1] \to M$ is a differentiable curve such that $\gamma(0) = p$ and $\gamma(1) = q$.

Definition 3.2 *If the sets $F_\omega(p)$ and $P_\omega(p)$ are dense on M for all $p \in M$, ω is called a transitive form on M.*

Theorem 3.3 (Calabi) Let $\omega \in \Omega^1(M)$ whose singularities are nondegenerate. Then ω is intrinsically harmonic if and only if ω is transitive.

- If M is a Kähler manifold, i.e., M is a complex manifold endowed with an Hermitian metric h, then the imaginary part of h is a symplectic form which is harmonic with respect to the Riemannian metric g on M given by the real part of h. This follows directly from Brylinski's original paper (see [1, Proposition 2.2.9, Theorem 2.4.1]. It is also observed (see [1, Remark 2.4.4]) that the same conclusion holds if M is almost Kähler, i.e., when the symplectic form has a compatible almost complex structure.

No other result is known for differential forms of order greater than or equal to 2.

We will discuss next the case where ω is a general symplectic form, i.e., ω is a nondegenerate closed 2-form.

4 Symplectic Hodge Theory

Let V be a finite-dimensional real vector space and let $\omega \in \Lambda^2(V)$.

Definition 4.1 *If ω is nondegenerate, the pair (V, ω) is called a symplectic space.*

Let (V, ω) be a symplectic space. Then the following conditions are satisfied:

- There is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V, called *symplectic basis* of V, for which:

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij},$$

  for all $1 \leq i, j \leq n$. Thus, the dimension of V must be even, $\dim(V) = 2n$.

- There is a complex structure J on V such that J is *ω-compatible*, i.e., the bilinear map $g : V \times V \to \mathbb{R}$ defined by

$$g(u, v) = \omega(u, Jv),$$

  for all $u, v \in V$, is a inner product on V.

For a symplectic space (V, ω), we denote by ω^* the induced symplectic structure on V^*. The symplectic structure ω^* induces a bilinear map

$$\omega^r : \Lambda^r(V) \times \Lambda^r(V) \to \mathbb{R}$$

defined by

$$\omega^r(f_{i_1} \wedge \cdots \wedge f_{i_r}, f_{j_1} \wedge \cdots \wedge f_{j_r}) = \det(\omega^*(f_{i_k}, f_{j_l})),$$

where $\{f_1, \cdots, f_{2n}\}$ is a basis of V, and we extend it by linearity.

Definition 4.2 *The symplectic Hodge operator is the linear operator*

$$*_\omega : \Lambda^r(V) \to \Lambda^{2n-r}(V)$$

defined by relation

$$\alpha \wedge *_\omega \beta = \omega^r(\alpha, \beta)\mathrm{vol}_\omega(V),$$

for all $\alpha, \beta \in \Lambda^r(V)$, *where* $\mathrm{vol}_\omega(V) = \frac{\omega^n}{n!}$.

We have the following

Lemma 4.3 *The following properties of* $*_\omega$ *are satisfied:*

$$*_\omega *_\omega = Id, \tag{4.1}$$

$$*_\omega(\omega) = \frac{1}{(n-1)!}\omega^{n-1}. \tag{4.2}$$

Proof. See [1].

□

Let (V, ω) be a symplectic space and let J be a ω-compatible complex structure on V. Denoting by g the associated inner product, we have the volume form $\mathrm{vol}_g(V)$ defined on V relatively to g. The following lemmas relates the volume form and symplectic and Riemannian Hodge operators.

Lemma 4.4 *Since in the discussion above, we have:*

$$\mathrm{vol}_\omega(V) = \mathrm{vol}_g(V). \tag{4.3}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a subset of V, orthonormal relatively to g, such that $\omega(e_i, e_j) = 0$, for all $1 \le i, j \le n$. Then, $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ is a symplectic basis of V, orthonormal relatively to g. Now consider the orthonormal basis of V

$$\mathcal{B} = \{e_1, f_1, \ldots, e_n, f_n\},$$

where $f_i = Je_i$, for all $1 \le i \le n$. By induction over n, we have:

$$\mathrm{vol}_\omega(V)(e_1, f_1, \ldots, e_n, f_n) = 1. \tag{4.4}$$

The equality (4.4) says that $\mathcal{B}$ is a positive basis of V. Thus, if $\{\theta_1, \ldots, \theta_{2n}\}$ denotes the dual basis of $\mathcal{B}$, we have $\mathrm{vol}_g(V) = \theta_1 \wedge \cdots \wedge \theta_{2n}$ and thus

$$\mathrm{vol}_g(V)(e_1, f_1, \ldots, e_n, f_n) = 1. \tag{4.5}$$

The equality (4.3) follows from (4.4) and (4.5).

□

Lemma 4.5 *For any $\alpha \in \Lambda^k(V)$, we have:*

$$*_\omega \alpha = *_g J^* \alpha, \tag{4.6}$$

where $J^*\alpha$ denotes the pull-back of α by J.

Proof. By definition of ω^k, we have

$$\omega^k(\alpha, \beta) = g^k(\alpha, J^*\beta), \tag{4.7}$$

for all $\alpha, \beta \in \Lambda^k(V)$. Thus, from (4.3) and (4.7) we have:

$$\begin{aligned} \alpha \wedge *_\omega \beta &= \omega^k(\alpha, \beta)\mathrm{vol}_\omega(V) \\ &= g^k(\alpha, J^*\beta)\mathrm{vol}_g(V) \\ &= \alpha \wedge *_g J^*\beta \end{aligned}$$

for all $\alpha, \beta \in \Lambda^k(V)$. This concludes the proof.

□

5 Symplectic Manifolds

Let M be a differentiable manifold and $\omega \in \Omega^2(M)$. The 2-form ω is called *nondegenerate* if the pair (T_pM, ω_p) is a symplectic space for all $p \in M$.

Definition 5.1 *A symplectic manifold is a pair (M, ω), where M is a differentiable manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate.*

It can be proved (see [3]) that a nondegenerate 2-form ω is closed if and only if there exists a *symmetric* connection ∇ on M which is compatible with ω, i.e., with $\nabla\omega = 0$, in which case there are infinitely many symmetric connections compatible with ω.

For any symplectic manifold (M, ω), the following conditions are satisfied:

- $\dim(M) = 2n$, for some positive entire number n.
- $\omega^n = \omega \wedge \cdots \wedge \omega$ is a on M. Thus, any symplectic manifold is orientable.
- There exists an almost complex structure J on M which is ω-compatible. This means that, for each point $p \in M$, the bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$ defined by

$$g_p(u, v) = \omega_p(u, J_p v), \tag{5.1}$$

for all $u, v \in T_pM$, is a Riemannian metric on M.

Let (M, ω) be a compact symplectic manifold. Applying the above constructions to each tangent space we obtain a symplectic Hodge operator

$$*_\omega : \Omega^k(M) \to \Omega^{2n-k}(M),$$

relatively to the volume form $\mathrm{vol}_\omega(M)$, satisfying the properties (4.1) and (4.2). Then, we have the following

Theorem 5.2 *For any symplectic manifold (M, ω), ω is intrinsically harmonic.*

Proof. We will show that there exists a Riemannian metric g on M such that $\delta\omega = 0$. Let J be a almost complex structure which is ω-compatible. We denote by g the Riemannian metric defined in (5.1). By Lemma 4.5, we have in each tangent space T_pM:

$$*_{\omega_p}\alpha = *_{g_p} J_p^* \alpha,$$

for all $\alpha \in \Lambda^k(T_pM)$. Thus, globally, we have:

$$*_\omega \alpha = *_g J^* \alpha, \tag{5.2}$$

for all $\alpha \in \Omega^k(M)$. On the other hand, it follows from (4.2) that $d *_\omega (\omega) = 0$. Since ω is invariant by J, i.e., $J^*\omega = \omega$, it follows from (5.2) that $d *_g \omega = 0$. Thus,

$$\delta\omega = *_g d *_g \omega = 0.$$

This concludes the proof.

□

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