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A SPLITTING THEOREM FOR KÄHLER SUBMANIFOLDS OF SPACE-FORMS

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Abstract

Isometric immersions from Kähler manifolds with parallel pluri-mean curvature (ppmc) generalize, in a natural way, the constant mean curvature (cmc) sufaces. The (2,0) part of the complexified second fundamental form is a holomorphic quadratic differential (Q) which plays a central role in the study of the cmc sufaces. Likewise, for ppmc immersions, Q is also a vector bundle valued holomorphic quadratic differencial, significant in the study of the geometry of the immersion. It is well known that those immersions with Q vanishing are extrinsically symmetric ([10] and [11]). In this work we study ppmc immersions with big nullity index of Q.

1 Introduction and statement of results

Let M^m be a Kähler manifold with complex dimension m and φ be an isometric immersion of M^m into a space form. We denote by α the second fundamental form of φ. The complexified α splits in a natural way, according to types, giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

Isometric immersions with $\alpha^{(1,1)} = 0$ are called pluriminimal immersions. Holomorphic immersions between Kähler manifolds are examples of pluriminimal immersions. Pluriminimal immersions have been extensively studied (see, by instance, [3] [4], [5], [6]). When M is a Riemann surface we have

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$\alpha^{(1,1)} =< .,. > H$, where $H = trace\ \alpha$ is the mean curvature of the immersion. In this case the pluriminimal immersions are precisely the minimal ones. In general, the immersion is pluriminimal if and only if its restriction to each holomorphic curve of M is a minimal immersion. The operator $\alpha^{(1,1)}$ is naturally called the plurimean curvature of the immersion. When the ambient space is R^n, it is well known that $\alpha^{(1,1)} = 0$ if and only if $H = 0$ ([4]), so that the class of pluriminimal immersions and the class of minimal immersions coincide.

We are mainly interested in isometric immersions from Kähler manifolds which have $\alpha^{(1,1)}$ parallel, called isometric immersions with *parallel plurimean curvature* operator (*ppmc* isometric immersions). They constitute a natural generalization to higher dimensions of the isometric immersions from Riemann surfaces with parallel mean curvature. In fact *ppmc* isometric immersions into space-forms display some special features of the parallel mean curvature surfaces, namely the existence of a $1-$parameter deformation through a smooth family of *ppmc* isometric immersions which, up to a parallel isomorphism, have the same normal bundle ([2]). Just as in the case of immersions with parallel mean curvature, *ppmc* isometric immersions can also be characterized by the pluriharmonicity of its Gauss map ([7]).

Studying immersed surfaces in R^3, H. Hopf discovered in 1955 that for surfaces with constant mean curvature (*cmc* surfaces) the complexification of the traceless part of α is a holomorphic quadratic differential Q. This holomorphic quadratic differential has been an important ingredient in the investigation of geometric properties of *cmc* surfaces ([13], [14]). The operator Q is nothing but the $(2,0)$ part of the complex bilinear extension of α. A straightforward computation shows that, for *ppmc* isometric immersions, Q is again a vector-bundle valued holomorphic quadratic differential. Isometric immersions with $\alpha^{(2,0)} = \alpha^{(0,2)} = 0$ are called $(2,0) - geodesic$ immersions. Curiously, that is a strong condition. Indeed, it can be deduced from Codazzi equation that $(2,0) - geodesic$ immersions into space forms have parallel second fundamental form. Ferus ([11], [10]), Takeuchi ([16]) and Strübing ([15]) classified the $(2,0) - geodesic$ immersed immersions into space forms. It turns out that they are extrinsically symmetric.

In ([3]) Dajczer and Rodrigues proved the following results:

Theorem 1 *Let M be a complete Kähler manifold with complex dimension m and $\varphi : M \to R^n$ be a pluriminimal immersion. Then if, for every $x \in M$, the index of relative nullity of α at x is greater or equal than $2m-2$, $M^m = R^{2m-2} \times M^1$ and $\varphi = id \times \varphi_2$.*

Theorem 2 *Let M be a complete Kähler manifold with complex dimension m such that, at a point $x_0 \in M$, the holomorphic sectional curvatures of M are all different from 0, and $\varphi : M \to R^n$ be a pluriminimal immersion. Then if, for every $x \in M$, the index of relative nullity of α at x is greater or equal than $2m-2k$, $M^m = R^{2m-2k} \times M^k$ and $\varphi = id \times \varphi_2$.*

Notice that that for pluriminimal immersions, the index of relative nullity of α and the index of relative nullity of $\alpha^{(2,0)}$ coincide.

From now on M^m will denote a connected complete Kähler manifold with complex dimension m, S^n_c $(c > 0)$ the n-dimensional euclidean sphere with sectional curvature c and H^n_c the n-dimensional hyperbolic space with constant seccional curvature c $(c < 0)$.

For *ppmc* immersions we have proved [8] that:

Theorem 3 *Let $\varphi : M^m \to R^n$ be a ppmc immersion. If the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2m-2$, one of the following conditions hold:*

1. *φ is extrinsically symmetric*

2. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to R^{n_1} \times R^{n_2}$, where φ_2 has parallel mean curvature and $\varphi_1 : M^{m-1} \to R^{n_1}$ is extrinsically symmetric.*

Corollary 4 *Let $\varphi : M^m \to S^n$ be a ppmc immersion such that, for every $x \in M^m$, the index of nullity of $\alpha^{(2,0)}$ at x is greater or equal than $2m-2$. Then one of the following conditions hold:*

1. *f is extrinsically symmetric*

2. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to S_a^{n_1} \times S_b^{n_2}$ $(a^{-1/2} + b^{-1/2} = 1)$, where φ_2 $M^1 \to S_b^{n_2}$ has parallel mean curvature and $\varphi_1 : M^{m-1} \to S_a^{n_1}$ is extrinsically symmetric.*

Corollary 5 *Let $\varphi : M^m \to H^n$ be a ppmc immersion such that, for every $x \in M^m$, the index of nullity of $\alpha^{(2,0)}$ at x is greater or equal than $2m-2$. Then one of the following conditions hold:*

1. *φ is extrinsically symmetric*

2. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to S_a^{n_1} \times H_b^{n_2}$ $(a^{-1} + b^{-1} = -1)$, where φ_2 $M^1 \to H_b^{n_2}$ has parallel mean curvature and $\varphi_1 : M^{m-1} \to S_a^{n_1}$ is extrinsically symmetric.*

3. $M^m = M^{m-1} \times M^1$ *and* $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to H_a^{n_1} \times S_b^{n_2}$ $(a^{-1}+b^{-1}=-1)$*, where* φ_2*:* $M^1 \to S_b^{n_2}$ *has parallel mean curvature and* $\varphi_1 : M^{m-1} \to H_a^{n_1}$ *is extrinsically symmetric.*

In the present work we assume that M has non zero holomorphic sectional curvatures and prove the following results:

Theorem 6 *Let* M *be a Kähler manifold with non zero holomorphic sectional curvatures and* $\varphi : M^m \to R^n$ *be a ppmc immersion. If the index of nullity of* $\alpha^{(2,0)}$ *is everywhere greater or equal than* $2m-2k$ $(k \geq 1)$*, one of the following conditions hold:*

1. M *is extrinsically symmetric;*
2. $M^m = M^r \times M^{m-r}$ $(1 \leq r \leq k)$ *and* $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to R^{n_1} \times R^{n_2}$*, where* $\varphi_2 : M^{m-r} \to R^{n_2}$ *is extrinsically symmetric and* φ_1 *is ppmc.*

Corollary 7 *Let* M *be a Kähler manifold with non zero holomorphic sectional curvatures and* $\varphi : M^m \to S^n$ *be a ppmc immersion. If the index of nullity of* $\alpha^{(2,0)}$ *is everywhere greater or equal than* $2m-2k$ $(k \geq 1)$*, then one of the following conditions hold:*

1. M *is extrinsically symmetric;*
2. $M^m = M^r \times M^{m-r}$ $(1 \leq r \leq k)$ *and* $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to S_a^{n_1} \times S_b^{n_2}$ $(a^{-1}+b^{-1}=1)$*, where* $\varphi_2 : M^{m-r} \to S_b^{n_2}$ *is extrinsically symmetric and* φ_1 *is ppmc.*

Corollary 8 *Let Let* M *be a Kähler manifold with non zero holomorphic sectional curvatures and* $\varphi : M^m \to H^n$ *be a ppmc immersion. If the index of* *nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2m-2k$ ($k \geq 1$), one of the follwing conditions hold:*

1. *M is extrinsically symmetric;*
2. *$M^m = M^r \times M^{m-r}$ ($1 \leq r \leq k$) and $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to S_a^{n_1} \times H_b^{n_2}$, where $(a^{-1} + b^{-1} = -1)$, $\varphi_2 : M^{m-r} \to H^{n_2}$ is extrinsically symmetric and φ_1 is ppmc;*
3. *$M^m = M^r \times M^{m-r}$ ($1 \leq r \leq k$) and $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to H_a^{n_1} \times S_b^{n_2}$, where $(a^{-1} + b^{-1} = -1)$, $\varphi_2 : M^{m-r} \to S^{n_2}$ is extrinsically symmetric and φ_1 is ppmc.*

Remark 9 *Theorem 6, Corolaries 7 and 8 remain true if we repalce the assumption on the holomorphic sectional curvatires of M by the non negativity of all sectional curvatures.*

2 Preliminaries

Let (M^m, J) be a Kähler manifold with complex dimension m and $\varphi : M^m \to F_c$ be an isometric immersion into a space form with sectional curvature c. We let $C(TM)$ denote the space of smooth sections of TM. We use the notation TM and $T^{\perp}M$ for the the tangent and normal bundles of φ. The complexification of TM, denoted by $T^C M$, decomposes as

$$T^C M = T'M + T''M$$

where $T'M$ and $T''M$ are the eigenbundles of J corresponding respectively to the eigenvalues i and $-i$ of J. The orthogonal projections of $T^C M$ onto $T'M$ and $T''M$ will be represented respectively by π' and π''. Of course, for any section X of TM, we have $X = \pi'(X) + \pi''(X)$.

The complex bilinear extension of the second fundamental form α splits in a natural way, according to types, giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

We have

$$\alpha^{(1,1)}(X,Y) = \alpha(X', Y'') + \alpha(X'', Y')$$

where $Z' = \pi'(Z)$ and $Z'' = \pi''(Z)$ for every $Z \in C(T^C M)$. We can also write $\alpha^{(1,1)}(X,Y) = C(X,Y)$, where $C(X,Y) = \frac{1}{2}\{\alpha(X,Y) + \alpha(JX,JY)\}$

Similarly

$$\alpha^{(2,0)}(X,Y) = \alpha(X',Y') = \frac{1}{2}Q(X,Y) - i\frac{1}{2}Q(X,JY)$$

where $Q(X,Y) = \frac{1}{2}\{\alpha(X,Y) - \alpha(JX,JY)\}$

We will use the same symbol ∇ to represent, either the Levi-Civita connection of TM, or the induced connections on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$. The symbol $\overset{\perp}{\nabla}$ will be used to represent either the induced connection on $\overset{\perp}{T}M$ or on $T^*M \otimes \overset{\perp}{T}M$.

Let $\Delta_x = \{X \in T_xM : Q(X,Y) = 0 \quad \forall\ Y \in T_xM\}$ and $\Delta_x^\perp$ be its orthogonal complement in T_xM.

Δ_x and $\Delta_x^\perp$ are J_x invariant.since $Q(X,JY) = Q(JX,Y)$, for any $X, Y \in T_xM$.

Proposition 10 *[8] On an open set where the dimension of Δ is constant, Δ is a smooth integrable distribution whose leaves are totally geodesic in M.*

We let U denote the open subset of M where $\dim \Delta$ is minimal and let $r = \dim \Delta_x$, for $x \in U$. From now on Δ will be considered defined on U.

For the study of this nullity foliation it is useful to consider the tensor $C_T : \Delta^\perp \to \Delta^\perp$ defined by

$$C_T(X) = -(\nabla_X T)^{\Delta^\perp},$$

where $T \in \Delta$ and $(\ \ )^{\Delta^\perp}$ denotes the orthogonal projection onto $\Delta^\perp$.

Proposition 11 *[8] The following conditions hold:*

1. *C_T commutes with J, for all $T \in \Delta$.*
2. *$Q(C_T(Y), Z) = Q(Y, C_T(Z))$, for $T \in \Delta$ and $Y, Z \in \Delta^\perp$.*

When $\xi \in \overset{\perp}{T_x}M$ let A_ξ denote the Weingarten operator at x associated to α. We represent by $N_x(M)$ the first normal space of the immersion at x.

Lemma 12 *[8] $A_{\alpha(S,T)}Y$, $A_{\alpha(S,Y)}T \in \Delta^\perp$ whenever $S, T \in \Delta$ and $Y \in \Delta^\perp$*

Lemma 13 *The following equalities hold:*

$$R(T,Y)S = 0 \tag{1}$$

$$A_{\alpha(T,S)}Y = A_{\alpha(T,Y)}S + A_{\alpha(S,Y)}T, \tag{2}$$

whenever $T, S \in \Delta$ and $Y \in \Delta^\perp$

Proof. Equality (1) follows easily if we prove that

$$R^M(T,Y)S = A_{\alpha(T,Y)}S. \tag{3}$$

Observe that the left hand side of equality (3) anti-commutes in the variables Y and T, while in the right hand side the same variables commute.

We first prove equality (3) repeating a proof presented in [8].

We conclude, from lemma 12 and Gauss equation, that $< R(T,Y)S, X >= 0$, whenever $X \in \Delta$, hence $R(T,Y)S \in \Delta^{\perp}$.

Using Codazzi equation and fact that φ is *ppmc* we get that

$$\nabla_{Y''} X' \in \Delta' \qquad \forall\ X \in C(\Delta),\ \forall\ Y, W \in C(TM), \tag{4}$$

since $\alpha(\nabla_{Y''} X', W') = 0$.

Condition (4) implies that

$$R(S', Y'')T' \in \Delta'. \tag{5}$$

We also remark that, for every $Z \in \Delta^{\perp}$,

$$< \alpha(T', Y''), \alpha(S', Z'') >=< \alpha(T'', Y'), \alpha(S'', Z') >= 0 \tag{6}$$

Indeed, using Gauss equation and the fact that S and T are sections of Δ, we have

$$0 =< R(T', Z'')S', Y'' >= - < \alpha(T', Y''), \alpha(S', Z'' > .$$

From (5), (6) and Gauss equation we get that, for whenever $Z \in \Delta^{\perp}$,

$$< R^M(T,Y)S, Z >=< R^M(T', Y'')S'', Z' > + < R^M(T'', Y')S', Z'' >=$$

$$=< \alpha(T', S''), \alpha(Y'', Z') > + < \alpha(T'', S'), \alpha(Y', Z'') >=$$

$$=< \alpha(T', Y''), \alpha(S'', Z') > + < \alpha(T'', Y'), \alpha(S', Z'') >=< \alpha(T,Y), \alpha(S,Z) >,$$

thus $R^M(T,Y)S = A_{\alpha(T,Y)}S$.

We now prove the second equality.

Notice that

$$< A_{\alpha(T,T)}Y, Z >= 2 < \alpha(T', T''), \alpha(Y', Z'') > +2 < \alpha(T', T''), \alpha(Y'', Z') > .$$

Then, again by Gauss equation and equation (6), we conclude that

$$< A_{\alpha(T,T)}Y, Z >= 2 < \alpha(T', Z''), \alpha(T'', Y') > +2 < \alpha(T', Y''), \alpha(T'', Z') >=$$

$$2 < \alpha(T, Y), \alpha(T, Z) >= 2 < A_{\alpha(T,Y)}T, Z >,$$

hence $2A_{\alpha(T,Y)}T = A_{\alpha(T,T)}Y$.

A polarization argument leads to $A_{\alpha(T,Y)}S + A_{\alpha(S,Y)}T = A_{\alpha(T,S)}Y$.

□

Proposition 14 *The following equality holds:*

$$(\nabla_S C_T)(Y) = C_T C_S Y + C_{\nabla_S T} Y \tag{7}$$

where $S, T \in \Delta$ *and* $Y \in \Delta^\perp$

Proof. We have proved in [8] the equality

$$(\nabla_S C_T)(Y) = C_T C_S Y + C_{\nabla_S T} Y + R(S, Y)T. \tag{8}$$

Now the result is a consequence of equation (1) in lemma 13.

□

3 The splitting theorem

We will prove first that $\Delta^\perp$ is integrable. Assume that, on U, $\dim \Delta = 2m - k$ $(k > 1)$. The case $k = 1$ was proved in [8] without the assumption on the holomorphic sectional curvatures of M.

Lemma 15 *Let* $T \in \Delta$*. Then the eigenvalues of* C_T *vanish identically.*

Proof. Consider a geodesic γ, definided on the real line, starting at T and let T denote also the velocity field of γ. Along γ take any vector field $Y \in \Delta^\perp$.

We know, from equation 7, that

$$(\nabla_T C_T)(Y) - C_T C_T Y = 0$$

for all $Y \in \Delta^{\perp}$.

Now, following the proof presented in[3], assume that λ is a real eigenfunction of C_T along γ. We get then that λ is a solution of the equation $\lambda' = \lambda^2$ defined on the real line, hence λ vanishes identically, that is, C_T has no non-zero real eigenvalues.

To conclude that zero is the only eigenvalue of C_T, assume that $a + ib$ were an eigenvalue of C_T. Then $a^2 + b^2$ would be an eigenvalue of C_{aT-bJT} so that $a = 0$ and $b = 0$.

We have thus proved that C_T is nilpotent.

□

Proposition 16 $C_T \equiv 0$

Proof. From lemma 15 we know that $C_T^k = 0$, since C_T is complex.

From lemma 13 and Gauss equation we know that $< \alpha(T, Y), \alpha(T, Y) >= 0$ for every $T \in \Delta$ and $Y \in \Delta^{\perp}$, hence

$$\alpha(T, Y) = 0. \tag{9}$$

Assume that, for some T, C_T were not identically zero. Let Γ represent, the kernel of C_T.

Take $Z \in \Delta^{\perp}$ such that $Y = C_T(Z) \neq 0$. We will prove first that $\alpha(Y, V) = 0$, for any V in Γ.

$$\alpha(Y, V) = \alpha(C_T(Z), V) = C(C_T(Z), V) + Q(C_T(Z), V) =$$

$$C(C_T(Z), V) + Q(Z, C_T(V)) = C(C_T(Z), V)$$

Now, using equation 4, we know that $C_{T'}Z'' = C_{T''}Z' = 0$, so that

$$C(C_T(Z), V) = C(C_{T'}(Z'), V'') + C(C_{T''}(Z''), V') =$$

$$-\alpha((\nabla_{Z'}T')^{\Delta^\perp}, V'') - \alpha((\nabla_{Z''}T'')^{\Delta^\perp}, V').$$

Using Codazzi equation and equation 9,

$$C(C_T(Z), V) = -\alpha(\nabla_{Z'}T', V'') - \alpha(\nabla_{Z''}T'', V') =$$

$$\alpha(T', \nabla_{Z'}V'') + \alpha(T'', \nabla_{Z''}V') = \alpha(T', (\nabla_{Z'}V'')^\Delta) + \alpha(T'', (\nabla_{Z''}V')^\Delta) =$$

$$= \alpha(T, (\nabla_{Z'}V'')^\Delta) + \alpha(T, (\nabla_{Z''}V')^\Delta)$$

Notice now that, for every $S \in \Delta$, $\langle \nabla_{Z''}V'', S' \rangle = \langle V'', C_{S'}(Z'') \rangle = 0$ since $C_{S'}(Z'') = 0$, as we know from equation 4. Therefore,

$$\alpha(T, (\nabla_{Z'}V'')^\Delta) + \alpha(T, (\nabla_{Z''}V')^\Delta) = \alpha(T, (\nabla_Z V'')^\Delta) + \alpha(T, (\nabla_Z V')^\Delta) =$$

$$\alpha(T, (\nabla_Z V)^\Delta)$$

We conclude then that

$$\langle \alpha(Y, V), \alpha(Y, V) \rangle = \left\langle \alpha(T, (\nabla_Z V)^\Delta), \alpha(Y, V) \right\rangle =$$

$$\left\langle \alpha(T, Y), \alpha((\nabla_Z Y)^\Delta, V) \right\rangle + \left\langle \alpha(T, V), \alpha((\nabla_Z Y)^\Delta, Y) \right\rangle,$$

where we have applied lemma 13 to get the last equality. Thus, from equation 9, $\alpha(Y, V) = 0$.

Now we end up taking $Z \in \Delta^\perp$ such that $Y = C_T(Z) \in \Gamma$ (if C_T^{k-1} is not identically zero choose W such that $C^{k-1}(W) \neq 0$ and consider $Z = C^{k-1}(W)$). Clearly $JY \in \Gamma$. Therefore, using Gauss equation we conclude that, $\langle R(Y,JY)Y,JY\rangle = \langle(\alpha(Y,Y),\alpha(JY,JY)\rangle - \langle(\alpha(Y,JY),\alpha(Y,JY)\rangle = 0$, which cannot happen. Thus $\Gamma = \emptyset$.

According with remark 9, allowing the hypothesis that M has non negative sectional curvatures we argue in the following way: taking Y as above, that is $Y = C_T(Z) \in \Gamma$, we obtain, through Gauss equation, that, for every $Z \in \Delta^\perp$,

$$\langle R(Y,Z)Y,Z\rangle = -\langle \alpha(Y,Z),\alpha(Y,Z)\rangle\,, \tag{10}$$

since $\alpha(Y,Y) = 0$. The assumption on the sectional curvatures implies that $\alpha(Y,Z) = 0$ for every $Z \in TM$ which entails that $Y \in \ker\alpha$, so that $Y \in \ker\alpha^{(2,0)}$, which cannot happen. Thus C_T vanishes identically.

$\square$

Δ and $\Delta^\perp$ are now two parallel distributions. Since Δ and $\Delta^\perp$ are invariant under the action of the holonomy group of M, we infer from the De Rham decomposition theorem that U is a product of two Kähler manifolds and $\varphi_{|U}$ is a product of two immersions, since $\alpha(T,Y) = 0$ whenever $T \in \Delta$, $Y \in \Delta^\perp$ [12]. An analiticity argument allows the conclusion that M^m is a product of two Kähler manifolds and φ is a product immersion.

The proof of corollaries 4 and 5 is analogous to that of [8] for $r = 2$.

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