Matemática Contemporânea, Vol 31, 29-52

QUARTIC DIFFERENTIAL FORMS ASSOCIATED TO COUPLES OF TRANSVERSAL NETS WITH SINGULARITIES

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Dedicated to Renato Tribuzy on the occasion of his 60th birthday

Abstract

For the first–order algebraic differential equation

$$a_4 dv^4 + 4a_3 du\, dv^3 + 6a_2 du^2 dv^2 + 4a_1 du^3 dv + a_0\, du^4 = 0$$

subject to certain constraints, where $a_i = a_i(u,v)$ with $a_i(0,0) = 0$ for $i = 0,1,2,3,4$, we give a complete local classification of generic singularities of the family of its phase curves up to topological orbital equivalence. This equation is related to geometric objects such as curvature and asymptotic lines of surfaces immersed in $\mathbb{R}^4$.

1 Introduction

Let M be a compact, connected, oriented two–manifold of class C^∞. We let $\mathcal{Q}(M)$ denote the set consisting of the smooth quartic differential forms ω defined on M which have the following property. If

$$(u,v)^*(\omega) \;=\; a_4 dv^4 \;+\; 4a_3 dv^3 du \;+\; 6a_2 dv^2 du^2 \;+\; 4a_1 dv du^3 \;+\; a_0 du^4 \qquad (1)$$

is the expression of ω in a local chart $(u,v) : U \subset M \longrightarrow \mathbb{R}^2$, then for all points $p \in (u,v)(U)$, we have that either $a_i(p) = 0$ for $i = 0,1,2,3,4$, or

$$J(p) \;=\; \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} (p) \;=\; 0 \qquad (2)$$

*The first author was supported in part by FONDECYT Grant #1050036.

†The second author was supported in part by by FAPESP Grant 03/03107-9 and by CNPq Grants #470957/2006–9 and #306328/2006–2, Brazil.

with

$$H(p) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}(p) < 0 \text{ and } I(p) = (a_4\, a_0 - 4\, a_1\, a_3 + 3\, a_2^2)(p) > 0\,. \quad (3)$$

For these quartic differential forms ω, the points p for which $\omega(p) \equiv 0$ are called **singular points** of ω. The remaining points, or in other words those that are not singular, are called **regular points** of ω. Conditions (2) and (3), which hold over the regular points p of ω, imply that $\omega(p)^{-1}(0)$ is the union of four distinct lines, say $L_1(\omega)(p)$, $L_2(\omega)(p)$, $L_3(\omega)(p)$, $L_4(\omega)(p)$ of the tangent space T_pM. (See [B-P].) In general, these line fields do not define foliations over the set of regular points of ω. However, as a consequence of condition (2), these lines may be grouped in a couple of pairs, say $\mathcal{N}_1(\omega) = \{L_1(\omega), L_2(\omega)\}$ and $\mathcal{N}_2(\omega) = \{L_3(\omega), L_4(\omega)\}$, so that each $\mathcal{N}_i(\omega)$, with $i = 1, 2$, defines a net.

Associated with each $\omega \in \mathcal{Q}(M)$ and each point p of M there are local coordinates (u, v) where ω has the simple form

$$(u,v)^*(\omega) \;=\; 4a\,(du^2 - dv^2)dudv \;+\; b\,(du^4 - 6du^2dv^2 + dv^4)\,. \qquad (4)$$

In these coordinates, we have

$$b \cdot (u,v)^*(\omega) \;=\; \omega^+ \cdot \omega^-$$

where

$$\omega^{\pm} \;=\; b\,(dv^2 - du^2) \;+\; 2\,(-a \pm \sqrt{a^2+b^2})\,du\,dv\,.$$

Observe that the quadratic forms ω^- and ω^+ are positive, or in other words at each point p, either the set $(\omega^{\pm}(p))^{-1}(0)$ consists of two transversal lines, or $\omega^{\pm}(p) \equiv 0$. (See for example [Gui1], [Gui2].) However, these quadratic forms are not differentiable at the singular points of the quartic. Further generically, each singular point p of the quartic ω is the center of a smooth curve γ :

$[-1,1] \to M$ such that $\gamma(0) = p$ and that $\gamma(t)$ is a singular point of the quadratic ω^+ (resp. ω^-), for all $t \geq 0$ (resp. $t \leq 0$). The nets $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$ correspond to the configuration of ω^- and the configuration of ω^+, respectively.

This type of quartic differential forms is related to principal curvature lines of surfaces immersed in $\mathbb{R}^4$. In fact, the principal directions at a point p are obtained by solving an equation $\omega(p) = 0$, with $\omega \in \mathcal{Q}(M)$. Conversely, given an $\omega \in \mathcal{Q}(M)$, with analytic coefficients a_i, and a point $p \in M$, there exists an immersion $f : V \to \mathbb{R}^4$, where V is some small open neighborhood of p so that the differential equation of the principal lines of curvature of f is given by $\omega/V = 0$. (See [GST], [GT1], [GT2], [GG], [Ga-S], [RS1], [RS2].) Further, these quartics are related to the first–order algebraic differential equation

$$a_4(u,v)(\frac{dv}{du})^4 + 4a_3(u,v)(\frac{dv}{du})^3 + 6a_2(u,v)(\frac{dv}{du})^2 + 4a_1(u,v)(\frac{dv}{du}) + a_0(u,v) = 0\,,$$

which is singular at $(u,v) = (0,0)$, subject to the constraints given by equations (2) and (3) . (See [BB], [Poi] [Mat].)

Globally, the quartic differential forms considered in this work are much more general than those forms associated to principal directions. Notwithstanding, we show that their generic singularities behave like the case of differential equations of principal directions. Thus all of the local results of [GT1], [GT2], [GG], [Ga-S] are applicable to the nets associated to an $\omega \in \mathcal{Q}(M)$.

The article is organized as follows:

Section 2 is devoted to showing that conditions (2) and (3) are preserved by changes of coordinates. We establish the existence of local coordinates (u,v) where $\omega \in \mathcal{Q}(M)$ has the simple form (4).

In Section 3, we introduce the **simple singular points**. We show that the set of those $\omega \in \mathcal{Q}(M)$ whose singular points are all simple is open and dense in $\mathcal{Q}(M)$, where the set $\mathcal{Q}(M)$ is considered endowed with the smooth Whitney topology. We give the local configuration of the nets $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$ around this type of points, and we characterize those singular points which are locally stable.

In Section 4, we show that a non–locally stable simple singular point is of codimension one or two, and we give its corresponding versal unfolding.

2 Preliminaries

We begin by showing that the coefficients $a_0, ..., a_4$ of the local expression of $\omega \in \mathcal{Q}(M)$ in local coordinates (u, v) satisfy relationships similar to those in the case of curvature lines of surfaces immersed in $\mathbb{R}^4$. (See [Ga-S, Lemma 2.1].)

Proposition 2.1. *Condition (2) holds if and only if there exist smooth functions* $E, F, G : (u, v)(U) \longrightarrow \mathbb{R}$ *such that* $(E, F, G)(p) \neq (0, 0, 0)$, *for all* $p \in (u, v)(U)$, *and furthermore that*

$$\begin{aligned} Ea_2 &= -Ga_0 + 2Fa_1 , \\ E^2a_3 &= -2FGa_0 + (4F^2 - EG)a_1 , \\ E^3a_4 &= G(EG - 4F^2)a_0 + 4F(2F^2 - EG)a_1 . \end{aligned} \tag{5}$$

Moreover, if condition (2) holds, then condition (3) holds if and only if $EG - F^2$ *is positive.*

Proof. Since $J = 0$ there exist smooth functions $E, F, G : (u, v)(U) \longrightarrow \mathbb{R}$, with $(E,F,G) \neq (0,0,0)$, so that

$$\begin{aligned} Ga_0 - 2Fa_1 + Ea_2 &= 0\,, \\ Ga_1 - 2Fa_2 + Ea_3 &= 0\,, \\ Ga_2 - 2Fa_3 + Ea_4 &= 0\,. \end{aligned} \tag{6}$$

Observe that the first relationship of (6) corresponds to the first relationship of (5), the second relationship of (6) multiplied by E corresponds to the second relationship of (5), and the last relationship of (6) multiplied by E^2 corresponds to the last relationship of (5).

Using relationships (5), we find

$$E^3\,(H,I) = G^2\,(Ea_1^2 - 2Fa_0a_1 + Ga_0^2)\,(-1, 4(EG - F^2))\,. \tag{7}$$

Therefore $EG - F^2 > 0$ imply $H < 0$ and $I > 0$. To obtain the converse is sufficient to prove that relationship (6), $H < 0$ and $I > 0$ imply $E \neq 0$. If $E = 0$ we have $G(a_0, a_1, a_2) = 2F(a_1, a_2, a_3)$. Then $F = 0$ (resp. $G = 0$) implies $G \neq 0$ (resp. $F \neq 0$) and $(a_0, a_1, a_2) = (0,0,0)$ (resp. $(a_1, a_2, a_3) = (0,0,0)$), which implies $I = 0$ (resp. $H = 0$). Hence $E = 0$ implies $FG \neq 0$ and $(a_1, a_2, a_3) = \frac{G}{2F}\,a_0 \left(1, \frac{G}{2F}, \left(\frac{G}{2F}\right)^2\right)$, which implies $(H, I) = a_0 \left(a_4 - \left(\frac{G}{2F}\right)^2\right) \left(\left(\frac{G}{2F}\right)^2, 1\right)$, that is impossible.

□

The following shows that the definition of the set $\mathcal{Q}(M)$ is independent of the coordinates chosen.

Proposition 2.2. *Conditions (2) and (3) are preserved by changes of coordinates.*

Proof. Assume that

$$(u,v)^*(\omega) = a_4dv^4 + 4a_3dv^3du + 6a_2dv^2du^2 + 4a_1dvdu^3 + a_0du^4$$

and that there exist smooth functions $E, F, G : (u,v)(U) \longrightarrow \mathbb{R}$ so that both $EG - F^2$ and E are positive, and relationships (5) hold. If

$$u = f(x,y)\,,\ v = g(x,y)$$

is a coordinate change and

$$(x,y)^*(\omega) \ = \ b_4 dy^4 \ + \ 4b_3 dy^3 dx \ + \ 6b_2 dy^2 dx^2 \ + \ 4b_1 dy dx^3 \ + \ b_0 dx^4\,,$$

then

$$\begin{aligned}
b_0 \ &= \ (f_x^4 - (6f_x^2 G g_x^2)/E - (8F f_x G g_x^3)/E^2 + (G((-4F^2)/E^2 + G/E)g_x^4)/E)a_0 \ + \\
&\quad (f_x^3 g_x + (3F f_x^2 g_x^2)/E + f_x((4F^2)/E^2 - G/E)g_x^3 + \\
&\quad (F((2F^2)/E^2 - G/E)g_x^4)/E)a_1
\end{aligned}$$

and

$$\begin{aligned}
b_1 \ &= \ (4f_x^3 f_y - (12 f_x f_y G g_x^2)/E - (8F f_y G g_x^3)/E^2 - (12 f_x^2 G g_x g_y)/E \\
&\quad -(24F f_x G g_x^2 g_y)/E^2 + (4G((-4F^2)/E^2 + G/E)g_x^3 g_y)/E)a_0 \ + \\
&\quad (3f_x^2 f_y g_x + (6F f_x f_y g_x^2)/E + f_y((4F^2)/E^2 - G/E)g_x^3 + f_x^3 g_y + \\
&\quad (6F f_x^2 g_x g_y)/E + 3f_x((4F^2)/E^2 - G/E)g_x^2 g_y + \\
&\quad (4F((2F^2)/E^2 - G/E)g_x^3 g_y)/E)a_1\,.
\end{aligned}$$

Setting

$$\begin{aligned}
\tilde{E} \ &= \ E(f_x)^2 + 2F f_x g_x + G(g_x)^2\,, \\
\tilde{F} \ &= \ E f_x f_y + F(f_x g_y + f_y g_x) + G g_x g_y\,, \\
\tilde{G} \ &= \ E(f_y)^2 + 2F f_y g_y + G(g_y)^2
\end{aligned}$$

we have

$$\tilde{E}\tilde{G} - \tilde{F}^2 \ = \ (EG - F^2)(f_x g_y - f_y g_x)^2,$$

and thus we again obtain the relationships

$$\begin{aligned}
\tilde{E}b_2 &= -\tilde{G}b_0 + 2\tilde{F}b_1 \\
\tilde{E}^2 b_3 &= -2\tilde{F}\tilde{G}b_0 + (4\tilde{F}^2 - \tilde{E}\tilde{G})b_1\,, \\
\tilde{E}^3 b_4 &= \tilde{G}(\tilde{E}\tilde{G} - 4\tilde{F}^2)b_0 + 4\tilde{F}(2\tilde{F}^2 - \tilde{E}\tilde{G})b_1\,.
\end{aligned}$$

The Proposition results from these facts.

□

We next show the existence of local coordinates (u, v) where $\omega \in \mathcal{Q}(M)$ takes a very simple form. In the case of curvature lines, these coordinates correspond to the isothermal coordinates.

Proposition 2.3. *Given $\omega \in \mathcal{Q}(M)$ and $p \in M$, there exist an open neighborhood U of p in M and a local chart $(u, v) : U \subset M \longrightarrow \mathbb{R}^2$ such that*

$$(u, v) * (\omega) = 4a(du^2 - dv^2)dudv + b(du^4 - 6du^2dv^2 + dv^4)\,. \qquad (8)$$

Proof. First observe that taking $F = 0$ and $E = G$ in (5), the local expression (1) has the form (8). Hence, given a local chart (u, v) at p and associated (smooth) maps E, F, G to the quartic differential form ω, it suffices to find a (smooth) coordinate change

$$u = f(x, y)\,,\ v = g(x, y)$$

so that, in a neighborhood of the origin, we have

$$\begin{aligned}
Ef_xf_y + F(f_xg_y + f_yg_x) + Gg_xg_y &= 0 \quad \text{and} \\
E(f_x)^2 + 2Ff_xg_x + G(g_x)^2 &= E(f_y)^2 + 2Ff_yg_y + G(g_y)^2\,.
\end{aligned}$$

Therefore, the problem is equivalent to finding isothermal coordinates in a neighborhood of a point of a surface. (See [Spi, Vol. IV, Addendum 1 of Chapter 9].) The conclusion follows.

□

3 Simple singular points

Let ω be a quartic differential form in $\mathcal{Q}(M)$, and let p be a singular point of ω. Assume that (1) is the local expression of ω in a chart $(u,v):(M,p)\to(\mathbb{R}^2,0)$ with coefficients a_0,a_1,a_2,a_3,a_4 satisfying relationships (5). The point p is called a **simple singular point** of ω if $\{a_0=0\}$ and $\{a_1=0\}$ are regular curves meeting each other transversally at the origin.

The next two results, Lemma 3.1 and Proposition 3.2, which we state without proof correspond to Lemma 3.2 and Proposition 3.1 of [GT1].

Lemma 3.1. *Let $\omega\in\mathcal{Q}(M)$, and let $p\in M$ be a simple singular point of ω. There are coordinates $(u,v):(M,p)\to(\mathbb{R}^2,0)$ such that the quartic differential form ω in these coordinates is of the form*

$$(u,v)*(\omega)=4(Au+Bv+S)(du^2-dv^2)dudv+(v+R)(du^4-6du^2dv^2+dv^4) \quad (9)$$

where $A\neq 0$ and B are real numbers, and $S=S(u,v)$ and $R=R(u,v)$ are real–valued functions which satisfy

$$S(0,0)=R(0,0)=\frac{\partial S}{\partial u}(0,0)=\frac{\partial S}{\partial v}(0,0)=\frac{\partial R}{\partial u}(0,0)=\frac{\partial R}{\partial v}(0,0)=0\,.$$

For the rest of the article, we henceforth endow the set $\mathcal{Q}(M)$ with the smooth Whitney topology.

Proposition 3.2. *The set of quartic differential forms $\omega\in\mathcal{Q}(M)$ whose singular points are simple is dense in $\mathcal{Q}(M)$.*

Each simple singular point of a quartic differential form $\omega\in\mathcal{Q}(M)$ has a smooth continuation in a neighborhood of ω. We explain this fact in the next Proposition.

Proposition 3.3. *Let p_0 be a simple singular point of a quartic differential form $\omega_0 \in \mathcal{Q}(M)$. Then there exist a neighborhood U of p in M, a neighborhood $\mathcal{V}$ of ω_0 in $\mathcal{Q}(M)$, and a smooth map $P : \mathcal{V} \to U$ which associates each $\omega \in \mathcal{V}$ with the unique singular point of ω in U. Furthermore, the singular point $P(\omega)$ is simple.*

Proof. The local expression of a quartic differential form ω in $\mathcal{Q}(M)$ associated to an arbitrary chart (u, v) is given by

$$\begin{array}{rl} (u,v)*(\omega) = & (A_{40}a_0 + A_{41}a_1)dv^4 + (A_{30}a_0 + A_{31}a_1)dv^3du \\ & +(A_{20}a_0 + A_{21}a_1)dv^2du^2 + \ a_1dvdu^3 + a_0du^4 \end{array} \tag{10}$$

where $a_0 = a_0(u, v)$, $a_1 = a_1(u, v)$, and $A_{ij} = A_{ij}(u, v)$, for $i = 2, 3, 4$ and $j = 0, 1$, are smooth functions. Moreover, the singular points of ω are given by the equations $a_0 = a_1 = 0$.

Consider a local chart $(u, v) : (M, p) \longrightarrow (\mathbb{R}^2, 0)$ such that the local expression ω_0 is of the form (9). Therefore, $A \neq 0$. For ω in a neighborhood $\tilde{\mathcal{V}}$ of ω_0 in $\mathcal{N}(M)$, the local expression in the same coordinates is of the form (10), where $a_0(\omega_0)(u, v) = v + R(u, v)$ and $a_1(\omega_0)(u, v) = 4(Au + Bv + S(u, v))$.

Consider next the smooth map $F : \tilde{\mathcal{V}} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$F(\omega, (u, v)) = (a_0(\omega)(u, v),\, a_1(\omega)(u, v)).$$

Since $F(\omega_0,\, (0, 0)) = (0, 0)$, and since the matrix

$$D_2\, F(\omega_0, (0, 0)) = \begin{pmatrix} 4A & 4B \\ 0 & 1 \end{pmatrix}$$

is non–singular, there exist a neighborhood $\tilde{U}$ of $(0, 0)$ in $\mathbb{R}^2$, a neighborhood $\mathcal{V} \subset \tilde{\mathcal{V}}$ of ω_0 in $\mathcal{Q}(M)$, and a smooth map $Q : \mathcal{V} \to \tilde{U}$ such that $Q(\omega_0) = (0, 0)$ and $F(\omega, Q(\omega)) = (0, 0)$, for all $\omega \in \mathcal{V}$. The proof now follows.

$\square$

The next two results are contained in [GG, Theorem 1.1]. We do not give their proofs.

Theorem 3.4. *Let* $\omega \in \mathcal{Q}(M)$, *and let* $p \in M$ *be a simple singular point of* ω. *Let* $(u,v) : (M,p) \longrightarrow (\mathbb{R}^2, 0)$ *be a local chart such that*

$$(u,v)*(\omega) = 4(Au+Bv+S(u,v))(du^2-dv^2)dudv+(v+R(u,v))(du^4-6du^2dv^2+dv^4)$$

where $A \neq 0$ *and* B *are real numbers, and* $S(u,v)$ *and* $R(u,v)$ *are real–valued functions which satisfy*

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0.$$

Then, under each of the conditions (a) through (e), the corresponding phase portrait is obtained by making into one, through a rigid translation, the pair of pictures (that is, nets) of the indicated figure.

(a) *Condition* H_3 : $\Delta < 0$. *(Figure 1)*

(b) *Condition* H_4 : $\Delta > 0$, $A < 0$ *and* $A \neq -1/4$. *(Figure 2)*

(c) *Condition* H_5 : $\Delta > 0$, $A > 0$. *(Figure 3)*

(d) *Condition* H_{34}: $\Delta > 0$ *and* $A = -1/4$ *and* $B \neq 0$. *(Figure 4)*

(e) *Condition* $\tilde{H}_3$: $A = -1/4$ *and* $B = 0$. *(Figure 5)*

Here

$$\begin{aligned} \Delta &= 16[4(1+B^2)^3 + 24(1+B^2)^2 A + 8(5-B^2)(1+B^2)A^2 + \\ &\quad 4(9+B^2)A^3 + (17+4B^2)A^4 + 4A^5]\,. \end{aligned} \tag{11}$$

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Definition 3.5. *Let p be a singular point of a quartic differential form $\omega \in \mathcal{Q}(M)$. We will say that ω is* **locally topologically stable** *at p when both nets $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$ are locally topologically stable at p.*

Theorem 3.6. *Let $\omega \in \mathcal{Q}(M)$, and let $p \in M$ be a simple singular point of ω. Consider a local chart $(u, v) : (M, p) \longrightarrow (\mathbb{R}^2, 0)$ as in Theorem 3.4. Then ω is locally topologically stable at p if and only if either condition H_3, or condition H_4, or condition H_5 holds.*

4 Non–locally stable simple singular points

In this section, we obtain versal unfoldings of the singular points H_{34} and $\tilde{H}_3$, thus showing that the former is of codimension one, and the latter is of codimension two. For this, we need the following characterization of simple singular points.

Proposition 4.1. *Let p be a simple singular point of $\omega \in \mathcal{Q}(M)$. Consider a local chart $(u,v) : (M,p) \to (\mathbb{R}^2, 0)$ such that the local expression of ω at p is of the form*

$$\begin{aligned} (u,v)*(\omega) \;&=\; 4(Au+Bv+S(u,v))(du^2-dv^2)dudv + \\ &\quad (v+R(u,v))(du^4-6du^2dv^2+dv^4)\,. \end{aligned} \tag{12}$$

Consider also the separatrix polynomial

$$g(s) = -s\,Q(s) \tag{13}$$

where

$$Q(s) \;=\; s^4 - 4Bs^3 - 2(3+2A)s^2 + 4Bs + 1 + 4A\,.$$

Then the point p is:

a) a locally stable singular point if the separatrix polynomial (13) has only simple roots;

b) an $H_{34}-$singular point if the separatrix polynomial (13) has a root of multiplicity two;

c) an $\tilde{H}_3-$singular point if the separatrix polynomial (13) has a root of multiplicity three.

Proof. This is a direct consequence of Theorems 3.4 and 3.6 , and the result [GG, Theorem 5.3].

□

The notion of equivalence of families of quartic differential forms in $\mathcal{Q}(M)$ used in this article is the following.

Definition 4.2. *Consider two smooth families (ω_μ) and (v_μ) in $\mathcal{Q}(M)$ with (the same) parameter $\mu \in \mathbb{R}^k$. Let $\mathcal{N}_1(\omega_\mu)$ and $\mathcal{N}_2(\omega_\mu)$ (resp. $\mathcal{N}_1(v_\mu)$ and $\mathcal{N}_2(v_\mu)$) be the nets associated to ω_μ (resp. v_μ). The families (ω_μ) and (v_μ) are called* ***equivalent*** *(over the identity) if for every $\mu \in \mathbb{R}^k$, the nets $\mathcal{N}_i(\omega_\mu)$ and $\mathcal{N}_i(v_\mu)$ are topologically equivalent, with $i = 1, 2$.*

The problems in this section are local problems around a simple singular point. Thus we will work with quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$. Our next result gives a normal form for these quartics at a simple singular point.

Proposition 4.3. *If $(0,0)$ is a simple singular point of $\omega \in \mathcal{Q}(\mathbb{R}^2)$, then there exists a local chart (u, v) such that*

$$\begin{gathered}(u,v)*(\omega) = 4(Au+Bv)(du^2-dv^2)dudv + v(du^4-6du^2dv^2+dv^4)+\\ A_4(u,v)dv^4 + A_3(u,v)dv^3du + A_2(u,v)dv^2du^2+\\ A_1(u,v)dvdu^3 + A_0(u,v)du^4\end{gathered} \tag{14}$$

where $A \neq 0$ and $A_i(0,0) = \dfrac{\partial A_i}{\partial u}(0,0) = \dfrac{\partial A_i}{\partial v}(0,0) = 0$, for $i = 0,1,2,3,4$.

Proof. We refer the reader to [Ga-S, Proposition 3.1] for a proof.

□

A consequence of Proposition above, is that the phase portrait of the nets $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$ at a small neighborhood of a simple singular point p is determined by the linear part of ω at p.

Our next result asserts that for a smooth family $\omega(\mu)$ in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a simple singular point at the origin, without loss of generality we may assume that the origin is a singular point of $\omega(\mu)$, for small $\mid \mu \mid$. Here is a precise statement.

Lemma 4.4. *Let $\omega(\mu)$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family of quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a simple singular point at the origin. Let $(u,v) : (U,(0,0)) \longrightarrow (\mathbb{R}^2,(0,0))$ be a local chart. Then there exists a change of coordinates of the form $(x,y,\mu) = (h(u,v,\mu),\mu)$ such that, for each μ with small $\mid \mu \mid$, the origin is a singular point of the quartic*

$$(x,y)^*(\omega(\mu))\,.$$

Proof. Assume that

$$\omega(\mu) \;=\; a_4\,dv^4 \;+\; a_3\,dv^3du \;+\; a_2\,dv^2du^2 \;+\; a_1\,dvdu^3 \;+\; a_0\,du^4 \qquad (15)$$

with $a_i = a_i(u,v,\mu)$, for $i = 0,1,2,3,4$.

By hypothesis we have that $(a_0,a_1)((0,0),\bar{0})) = (0,0)$, and that

$$D_1(a_0,a_1)\,((0,0),\bar{0}) = \begin{pmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{pmatrix}$$

is non–singular.

Since the map $(a_0,a_1) : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^2$ is smooth, it follows from the implicit function theorem that there exists a smooth map S defined on a small neighborhood of $\bar{0} \in \mathbb{R}^k$ so that $S(\bar{0}) = (0,0)$ and

$$(a_0,a_1)(S(\mu),\mu) = (0,0)$$

for all μ in such a neighborhood.

Using the change of coordinates

$$(x,y,\mu) = (u,v,\mu) - (S(\mu),\bar{0})$$

we obtain the Lemma.

□

Our next result shows that the normal form (14) also holds for families in $\mathcal{Q}(\mathbb{R}^2)$ which pass through a quartic having a simple singular point.

Lemma 4.5. *Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$, such that $\omega(0)$ has a simple singular point at the origin. Then there exits a local chart $\phi : (U \times V, ((0,0), \bar{0})) \to (\mathbb{R}^2 \times \mathbb{R}^k, ((0,0), \bar{0}))$ of the form $\phi(p, \mu) = (u(p, \mu), v(p, \mu), \mu)$, with $\phi(p, \bar{0}) = (u(p), v(p), \bar{0})$ for all $p \in U_0$, such that in the chart $\phi_\mu : (U_0, p(\mu)) \to (\mathbb{R}^2, (0,0))$ defined by $\phi_\mu(p) = \phi(p, \mu)$ for all $\mu \in V$, the local expression of $\omega(\mu)$ is*

$$\begin{aligned}\phi_\mu^*(\omega(\mu)) \;=\; & 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4) + \\ & A_4(\mu)(u,v)dv^4 + A_3(\mu)(u,v)dv^3du + A_2(\mu)(u,v)dv^2du^2 + \\ & A_1(\mu)(u,v)dvdu^3 + A_0(\mu)(u,v)du^4\end{aligned} \tag{16}$$

with $A_i(\mu)(0,0) = \dfrac{\partial A_i}{\partial u}(\mu)(0,0) = \dfrac{\partial A_i}{\partial v}(\mu)(0,0) = 0,$ *for* $i = 0,1,2,3,4.$

Proof. For μ in a neighborhood V of the origin in $\mathbb{R}^k$, there exists local chart (s, t, μ) such that the local expression of $\omega(\mu)$ is

$$\begin{aligned}& 4(\tilde{A}(\mu)s + \tilde{B}(\mu)t)(ds^2 - dt^2)dsdt + (\tilde{C}(\mu)s + \tilde{D}(\mu)t)(ds^4 - 6ds^2dt^2 + dt^4) + \\ & \tilde{A}_4(\mu)(s,t)dt^4 + \tilde{A}_3(\mu)(s,t)dt^3ds + \tilde{A}_2(\mu)(s,t)dt^2ds^2 + \\ & \tilde{A}_1(\mu)(s,t)dtds^3 + \tilde{A}_0(\mu)(s,t)ds^4\end{aligned}$$

where $A = \tilde{A}(0) \neq 0$, $\tilde{C}(0) = 0$, $\tilde{D}(0) = 1$, and $\tilde{A}_i(\mu)(0,0) = \dfrac{\partial \tilde{A}_i}{\partial s}(\mu)(0,0) = \dfrac{\partial \tilde{A}_i}{\partial t}(\mu)(0,0) = 0$, for $i = 0,1,2,3,4$.

Let $L_{(\alpha,\beta)} : \mathbb{R}^3 \to \mathbb{R}^3$, with parameter $(\alpha, \beta) \in \mathbb{R}^2$, be the family of linear isomorphisms such that the inverse of $L = L_{(\alpha,\beta)}$ is given by

$$L^{-1}(s, t, \mu) = ((1+\alpha)s - \beta t, \beta s + (1+\alpha)t, \mu)\,.$$

Observe that for all $(\alpha,\beta)\in\mathbb{R}^2$, the map $L_{(\alpha,\beta)}$ is a linear rotation at the first two coordinates. Therefore, in the chart

$$(s,t,\mu)=((1+\alpha)u-\beta v,\beta u+(1+\alpha)v,\mu)$$

the local expression of $\omega(\mu)$ is given by

$$\begin{aligned}(u,v)^*(\omega(\mu)) &= 4(A(\mu)u+B(\mu)v)(du^2-dv^2)dudv\\ &+(C(\mu)u+D(\mu)v)(du^4-6du^2dv^2+dv^4)+\\ &A_4(\mu)dv^4+A_3(\mu)dv^3du+A_2(\mu)dv^2du^2+A_1(\mu)dvdu^3+A_0(\mu)du^4\end{aligned}$$

with $A_i(\mu)(0,0)=\dfrac{\partial A_i}{\partial u}(\mu)(0,0)=\dfrac{\partial A_i}{\partial v}(\mu)(0,0)=0$, for $i=0,1,2,3,4.$

To complete the proof, it suffices to show that there exists $(\alpha,\beta)=(\alpha(\mu),\beta(\mu))$ so that $(C(\mu),D(\mu))\equiv(0,1)$, for $\mu\in V$ sufficiently close to $\bar{0}$. In fact,

$$\begin{aligned}C(\mu) &= 4(1+\alpha)^4\tilde{A}(\mu)\beta+4(1+\alpha)^3\tilde{B}(\mu)\beta^2-4(1+\alpha)^2\tilde{A}(\mu)\beta^3-\\ &\quad 4(1+\alpha)\tilde{B}(\mu)\beta^4+(1+\alpha)^4\beta\tilde{D}(\mu)-6(1+\alpha)^2\beta^3\tilde{D}(\mu)+\beta^5\tilde{D}(\mu)+\\ &\quad (1+\alpha)^5\tilde{C}(\mu)-6(1+\alpha)^3\beta^2\tilde{C}(\mu)+(1+\alpha)\beta^4\tilde{C}(\mu)\end{aligned}$$

and

$$\begin{aligned}D(\mu) &= 4(1+\alpha)^4\tilde{B}(\mu)\beta-4(1+\alpha)^3\tilde{A}(\mu)\beta^2-4(1+\alpha)^2\tilde{B}(\mu)\beta^3+4(1+\\ &\quad \alpha)\tilde{A}(\mu)\beta^4+(1+\alpha)^5\tilde{D}(\mu)-6(1+\alpha)^3\beta^2\tilde{D}(\mu)+(1+\alpha)\beta^4\tilde{D}(\mu)-\\ &\quad (1+\alpha)^4\beta\tilde{C}(\mu)+6(1+\alpha)^2\beta^3\tilde{C}(\mu)-\beta^5\tilde{C}(\mu)\,.\end{aligned}$$

If $\tilde{C}(\mu)=0$, then $\tilde{D}(\mu)\neq 0$. We may set $\beta=0$ and $1+\alpha=\frac{1}{\tilde{D}(\mu)^{\frac{1}{5}}}\cdot$ Then $C(\mu)=0$ and $D(\mu)=1$.

If $\tilde{C}(\mu)\neq 0$, we set $1+\alpha=m\,\beta$, with m a real root of the equation

$$\begin{aligned}&\tilde{C}(\mu)x^5+2\,(2\tilde{B}(\mu)-3\tilde{C}(\mu))x^4+2\,(2\tilde{B}(\mu)-3\tilde{C}(\mu))x^3-\\ &2\,(2\tilde{A}(\mu)+3\tilde{D}(\mu))x^2+(\tilde{C}(\mu)-4\tilde{B}(\mu))x+\tilde{D}(\mu)=0\,.\end{aligned}$$

Then $C(\mu)=0$, and we are under the condition of the first case. The proof now follows. □

To obtain a versal unfolding for a simple singular point, we will need the following.

Lemma 4.6. *Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a simple singular point at the origin. Consider a local chart (u, v, μ) as in Lemma 4.5 such that, in this chart, our family has the local expression*

$$\begin{array}{rcl}\omega(\mu) & = & 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4) + \\ & & A_4(\mu)(u,v)dv^4 + A_3(\mu)(u,v)dv^3du + A_2(\mu)(u,v)dv^2du^2 + \\ & & A_1(\mu)(u,v)dvdu^3 + A_0(\mu)(u,v)du^4 \end{array} \tag{17}$$

where $A_i(\mu)(0,0) = \dfrac{\partial A_i}{\partial u}(\mu)(0,0) = \dfrac{\partial A_i}{\partial v}(\mu)(0,0) = 0,$ for $i = 0,1,2,3,4.$
Then, for small $\mid \mu \mid$, the family $(\omega(\mu))$ is equivalent to the family

$$\tilde{\omega}(\mu) \;=\; 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4)\,. \tag{18}$$

Proof. The Lemma is clear from the fact that both families have the same linear part at the origin.

□

We next give a versal unfolding for singular points of type H_{34}.

Lemma 4.7. *Let $\omega \in \mathcal{Q}(\mathbb{R}^2)$ be a quartic with the origin an $H_{34}-$singular point. Then there exist coordinates (u, v) such that the local expression of ω is of the form (12), with $A \neq -\dfrac{1}{4}$ ·*

Proof. Consider a local chart (u, v) such that the local expression of ω at the origin is of the form (12). Then $A = -\dfrac{1}{4}$ if and only if the root of multiplicity two of the polynomial $g(s)$ is $s = 0$. Now if $A = -\dfrac{1}{4}$ and s_0 is a simple root of $g(s)$, then we make a rotation which sends s_0 over $s = 0$. In the resulting chart, the local expression of ω is also of the form (12). Hence, the corresponding coefficient $A \neq -\dfrac{1}{4}$, which completes the proof.

$\square$

Theorem 4.8. *A versal unfolding of an $H_{34}-$singular point is the family of quartic $\upsilon(\lambda)$, with $\lambda \in \mathbb{R}$, given by*

$$\upsilon(\lambda) \;=\; 4\left(\left(\lambda-\frac{125}{32}\right)u+\frac{51}{32}\,v\right)\,(du^2-dv^2)dudv+v\,(du^4-6du^2dv^2+dv^4)\,.$$

Proof. Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$ so that $\omega(0)$ has an $H_{34}-$singular point at the origin. By Lemmas 4.6 and 4.7 we may suppose that

$$\omega(\mu) \;=\; 4(A(\mu)u+B(\mu)v)(du^2-dv^2)dudv+v(du^4-6du^2dv^2+dv^4)$$

with $A(0) \neq -\frac{1}{4}\cdot$

Consider the real–valued function ψ defined on a neighborhood of the origin of $\mathbb{R}^k$ by

$$\begin{aligned}\psi(\mu) \;&=\; 16\,[4(1+B(\mu)^2)^3+24(1+B(\mu)^2)^2A(\mu)+\\ &\quad 8(5-B(\mu)^2)(1+B(\mu)^2)A(\mu)^2+4(9+B(\mu)^2)A(\mu)^3+\\ &\quad (17+4B(\mu)^2)A(\mu)^4+4A(\mu)^5]\,.\end{aligned}$$

Then the unfolding induced by ψ from the family $(\upsilon(\lambda))_{\lambda\in\mathbb{R}}$ is

$$\tilde{\upsilon}(\mu) \;=\; 4\left(\left(\psi(\mu)-\frac{125}{32}\right)u+\frac{51}{32}\,v\right)\,(du^2-dv^2)dudv+v\,(du^4-6du^2dv^2+dv^4)\,.$$

Since the discriminant (11) associated to the family $(\omega(\mu))$ is $\Delta(\mu) = \psi(\mu)$, and since the discriminant associated to the family $(\tilde{\upsilon}(\mu))$ is of the form

$$\tilde{\Delta}(\mu) \;=\; \psi(\mu)\,h(\psi(\mu))$$

where $h(x)$ is a degree 4 polynomial with $h(0) > 0$, both families are equivalent for small $|\mu|$. The proof is now complete.

□

We now consider the singular points of type $\tilde{H}_3$.

Theorem 4.9. *A versal unfolding of an $\tilde{H}_3-$singular point is the family of quartic $\upsilon(\lambda)$, with $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, given by*

$$\upsilon(\lambda) \;=\; 4\left(\left(\lambda_1 - \frac{1}{4}\right)u + \lambda_2\, v\right)\,(du^2 - dv^2)dudv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Proof. Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$ so that $\omega(0)$ has an $\tilde{H}_3-$singular point at the origin. By Lemma 4.6 we may suppose

$$\omega(\mu) \;=\; 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4)$$

with $A(0) = -\frac{1}{4}$ and $B(0) = 0$. Let us consider the real bi–valued function ψ defined on a neighborhood of the origin of $\mathbb{R}^k$ by

$$\psi(\mu) \;=\; (A(\mu), B(\mu))\,.$$

Then the unfolding induced by ψ from the family $(\upsilon(\lambda))_{\lambda\in\mathbb{R}^2}$ is

$$\tilde{\upsilon}(\mu) \;=\; 4\left(\left(A(\mu) - \frac{1}{4}\right)u + B(\mu)\, v\right)\,(du^2 - dv^2)dudv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Since the discriminant (11) associated to the family $(\omega(\mu))$ is equal to that associated to the family $(\tilde{\upsilon}(\mu))$, for every μ, we conclude that both families are equivalent. The proof is now complete.

□

The next two theorems give the bifurcation diagrams of these types of singular points.

Theorem 4.10. *Consider the one–parameter family of quartic $\omega(\lambda)$ given by*

$$\omega(\lambda) \;=\; 4\left(\left(\lambda-\frac{125}{32}\right)u+\frac{51}{32}\,v\right)\,(du^2-dv^2)dudv+v\,(du^4-6du^2dv^2+dv^4)\,.$$

Then, for all values of λ, the origin is a singular point of $\omega(\lambda)$. Moreover, for small $\mid\lambda\mid$, the origin is of type H_3 for $\lambda<0$, of type H_{34} for $\lambda=0$, and of type H_4 for $\lambda>0$.

Proof. Since the associated discriminant is

$$\Lambda=\frac{\lambda}{8192}(6640625-48348750\lambda+30426304\lambda^2-6680064\lambda^3+524288\lambda^4)\,,$$

the proof follows.

□

Theorem 4.11. *Consider the two–parameter family of quartic $\omega(\lambda)$, with $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2$, given by*

$$\omega(\lambda) \;=\; 4\left(\left(\lambda_1-\frac{1}{4}\right)u+\lambda_2\,v\right)\,(du^2-dv^2)dudv+v\,(du^4-6du^2dv^2+dv^4)\,.$$

Then the origin is a singular point for all values of $\lambda=(\lambda_1,\lambda_2)$. Moreover, for small $\mid\lambda\mid$, we have that:

i) The origin is of type H_3 if $\Lambda<0$.

ii) The origin is of type H_{34} if $\Lambda=0$ and $\lambda_1\neq 0$, or if $\lambda_1=0$ and $\lambda_2\neq 0$.

iii) The origin is of type H_4 for $\Lambda>0$ and $\lambda_1\neq 0$.

iv) The origin is of type $\tilde{H}_3$ for $\lambda=(\lambda_1,\lambda_2)=(0,0)$.

Here

$$\begin{aligned}\Lambda \;=\; &\frac{1}{4}\,(625\,\lambda_1\,+\,1200\,\lambda_2\,+\,1376\,\lambda_1^3\,+\,768\,\lambda_1^4\,+\,256\,\lambda_1^5\,+\,125\,\lambda_2^2\,+\\ &2080\,\lambda_1\,\lambda_2^2\,+\,1952\,\lambda_1^2\,\lambda_2^2\,+\,256\,\lambda_1^4\,\lambda_2^2\,+\,352\,\lambda_2^4\,+\,1792\,\lambda_1\,\lambda_2^4\\ &-512\,\lambda_1^2\,\lambda_2^4\,+\,256\,\lambda_2^6)\,.\end{aligned}$$

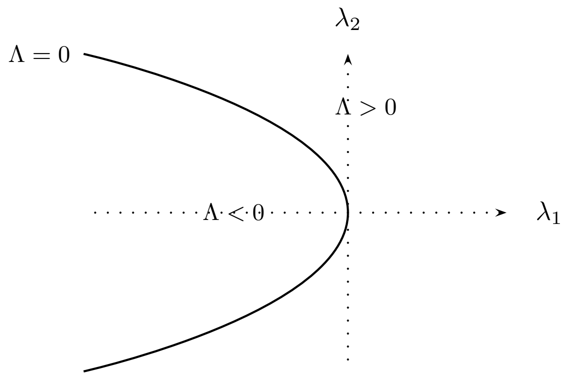

Figure 6

Proof. For the proof, it suffices to observe that the corresponding values of A and B are

$$A = \lambda_1 - \frac{1}{4} \quad \text{and} \quad B = \lambda_2 .$$

$\square$

Acknowledgement. The authors want to thank the referee for his/her comments which helped us improve the paper.

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