REMARKS ON THE ALEXANDER-WERMER THEOREM FOR CURVES

F. Reese Harvey  H. Blaine Lawson, Jr.*

To Renato Tribuzy on occasion of his 60th birthday

Abstract

We give a new proof of the Alexander-Wermer Theorem that characterizes the oriented curves in $\mathbb{C}^n$ which bound positive holomorphic chains, in terms of the linking numbers of the curve with algebraic cycles in the complement. In fact we establish a slightly stronger version which applies to a wider class of boundary 1-cycles. Arguments here are based on the Hahn-Banach Theorem and some geometric measure theory. Several ingredients in the original proof have been eliminated.

1. The Alexander-Wermer Theorem

We present a different proof of the Alexander-Wermer Theorem [AW], [W2] for curves which uses the Hahn-Banach Theorem and techniques of geometric measure theory. Several ingredients of the original proof are eliminated, such as the reliance on the result in [HL1] that if a curve satisfies the moment condition, then it bounds a holomorphic 1-chain. The arguments given here have been adapted to study the analogous problem in general projective manifolds (cf. [HL3,4,5]).

Our arguments will also apply to a more general class of curves which we now introduce.

*Partially supported by the N.S.F.
Definition 1.1. Let $X$ be a complex manifold and suppose there exists a closed subset $\Sigma(\Gamma)$ of Hausdorff 1-measure zero and an oriented, properly embedded $C^1$-submanifold of $X - \Sigma(\Gamma)$ with connected components $\Gamma_1, \Gamma_2, \ldots$. If, for given integers $n_1, n_2, \ldots$, the sum $\Gamma = \sum_{k=1}^{\infty} n_k \Gamma_k$ defines a current of locally finite mass in $X$ which is $d$-closed (i.e., without boundary), and if $\text{spt} \Gamma$ has only a finite number of connected components$^1$, then $\Gamma$ will be called a **scarred 1-cycle** (of class $C^1$) in $X$. By a unique choice of orientation on each $\Gamma_k$ we may assume each $n_k > 0$.

Exemple 1.2. Any real analytic 1-cycle is automatically a scarred 1-cycle (of class $C^\infty$) – see [F, p. 433].

Definition 1.3. By a **positive holomorphic 1-chain with boundary** $\Gamma$ we mean a sum $V = \sum_{k=1}^{\infty} m_k[V_k]$ with $m_k \in \mathbb{Z}^+$ and $V_k$ an irreducible 1-dimensional complex analytic subvariety of $X - \text{spt} \Gamma$ such that $V$ has locally finite mass in $X$ and, as currents in $X$,

$$dV = \Gamma$$

Remark 1.4. Standard projection techniques (cf. [Sh], [H]) show that any 1-dimensional complex subvariety $W$ of $X - \text{spt} \Gamma$ automatically has locally finite 2-measure at points of $\Gamma$, and furthermore, its current boundary is of the form $dW = \sum \epsilon_k \Gamma_k$ where $\epsilon_k = -1, 0$ or 1 for all $k$. See [H] and the “added in proof” for the more general case where $T$ is a positive $d$-closed current on $C^2 - \text{spt} \Gamma$.

Definition 1.5. A **scarred 1-cycle** $\Gamma$ in $\mathbb{C}^n$ satisfies the **(positive) winding**

---

$^1$More generally we need only assume that $\text{spt} \Gamma$ is contained in a compact connected set of finite linear measure.
condition if
\[ \frac{1}{2\pi i} \int_{\Gamma} \frac{dP}{P} > 0 \]
for all polynomials \( P \in \mathbb{C}[z] \) with \( P \neq 0 \) on \( \Gamma \).

There are many equivalent formulations of this condition. We mention three.

**Proposition 1.6.** \( \Gamma \) satisfies the (positive) winding condition if and only if any of the following equivalent conditions holds:

1) \( \int_{\Gamma} d^c \varphi \geq 0 \) for all smooth plurisubharmonic functions \( \varphi \) on \( \mathbb{C}^n \).

2) For each polynomial \( P \in \mathbb{C}[z] \), the unique compactly supported solution \( W_P(\Gamma) \) to the equation \( dW_P(\Gamma) = P(\Gamma) \) satisfies \( W_P(\Gamma) \geq 0 \).

3) The linking number \( \text{Link}(\Gamma, Z) \geq 0 \) for all algebraic hypersurfaces \( Z \) contained in \( \mathbb{C}^n - \text{spt}\Gamma \).

**Proposition 1.7.** If \( \Gamma \) is the boundary of a positive holomorphic 1-chain \( V \) in \( \mathbb{C}^n \), then \( \Gamma \) satisfies the positive winding condition.

**Proof:** We have \( \int_{\Gamma} d^c \varphi = \int_V d^c \varphi = \int_V dd^c \varphi \geq 0 \) since \( dd^c \varphi \geq 0 \). \( \Box \)

The following converse of Proposition 1.7 is due to Alexander and Wermer [AW], [W2].

**Main Theorem 1.8.** Let \( \Gamma \) be a scarred 1-cycle in \( \mathbb{C}^n \). If \( \Gamma \) satisfies the (positive) winding condition, then \( \Gamma \) bounds a positive holomorphic 1-chain in \( \mathbb{C}^n \).
This slightly generalizes the theorem in [AW] which applies only to smooth oriented curves. However, the essential point of this paper is to provide a conceptually different proof of the result which has other applications. This proof has two distinct parts which constitute the following two sections.

**Note.** We adopt the following notation throughout the paper. The polynomial hull of a compact subset $K \subset \mathbb{C}^n$ is denoted by $\hat{K}$. The mass of a current $T$ with compact support in $\mathbb{C}^n$ is denoted by $M(T)$.

### 2. A Dual Interpretation

In this section we shall use the Hahn-Banach Theorem to establish a dual interpretation of the positive winding condition. The main result is the following. Recall that if $C$ is a convex cone in a topological vector space $V$, its **polar** is the set $C^0 = \{ v' \in V' : v'(v) \geq 0 \text{ for all } v \in C \}$.

**Theorem 2.1. (The Duality Theorem)** The cone $A$ in the space $\mathcal{E}^1_1(\mathbb{C}^n)$ of smooth 1-forms on $\mathbb{C}^n$, defined by

$$A \equiv \{ d\psi + dc\phi : \psi \in C^\infty(\mathbb{C}^n) \text{ and } \phi \in \mathcal{PSH}(\mathbb{C}^n) \}$$

and the cone $B$ in the dual space $\mathcal{E}^1_1(\mathbb{C}^n)_\mathbb{R}$ of compactly supported one-dimensional currents in $\mathbb{C}^n$, defined by

$$B \equiv \{ S : S = d(T + R) \text{ with } T \geq 0 \text{ and } R \text{ of bidimension } 2,0 + 0,2 \}$$

are each the polar of the other.

Moreover,

(i) The cone $B$ coincides with the cone

$$B' \equiv \{ S : dS = 0 \text{ and } \exists T \geq 0 \text{ with compact support and } dd^cT = -d^cS \},$$
(ii) If $S = d(T + R) \in B$ with $T$ and $R$ as above, then
\[ \text{spt} T \subseteq \hat{\text{spt}} S \] (the polynomial hull of spt$S$).

This result can be restated as follows.

**Theorem 2.1′.** A real 1-dimensional current $S$ with $dS = 0$ and compact support in $\mathbb{C}^n$ satisfies the (positive) winding condition if and only if
\[
S = d(T + R) \quad (2.1)
\]
where $T$ is a positive 1,1 current and $R$ has bidimension 2,0 + 0,2, or equivalently,
\[
\ddc T = -\alpha S \quad (2.2)
\]
for some compactly supported $T \geq 0$. Moreover, for each such $T$,
\[
\text{spt} T \subseteq \hat{\text{spt}} S \quad (2.3)
\]

**Proof:** We will show that $B^0 = A$ and that $B$ is closed. This is enough to conclude that $A$ and $B$ are each the polar of the other because of the bipolar theorem: $(C^0)^0 = \overline{C}$.

\[ \square \]

**Proof that** $A = B^0$. The inclusion $A \subseteq B^0$ is essentially a restatement of Proposition 1.7 – the same proof applies. We need only show $B^0 \subseteq A$. Suppose $\alpha \in B^0$, i.e., $S(\alpha) \geq 0$ for all $S \in B$. Restricting to $S$ of the form $S = dR$ where $R = R^{2,0} + R^{0,2}$ is of bidimension 2,0+2, we see that $S(\alpha) = dR(\alpha)$ must vanish (since $-dR$ is also in $B$). Hence, $\partial \alpha^{1,0} = 0$ and $\overline{\partial} \alpha^{0,1} = 0$. That is, $d\alpha = d^{1,1} \alpha$. In particular, $d^{1,1} \alpha$ is $d$-closed. Therefore, on $\mathbb{C}^n$ the equation $d\alpha = d^{1,1} \alpha = \ddc \varphi$ can be solved for some $\varphi \in C^\infty(\mathbb{C}^n)$. 
Taking $S = dT$ where $T = \delta p \xi \geq 0$ for $p \in \mathbb{C}^n$, yields $(d\alpha)(\delta p \xi) = (d^{1,1}\alpha)(\delta p \xi) \geq 0$. Hence, $dd^c \varphi = d^{1,1} \alpha \geq 0$, i.e., $\varphi \in PSH(\mathbb{C}^n)$. Since $\alpha - dd^c \varphi$ is $d$-closed, there exists $\psi \in C^\infty(\mathbb{C}^n)$ with $\alpha = d\psi + dd^c \varphi$.

To show that $B$ is closed requires several preliminary results.

**Proof of (i).** If $S \in B$, then $dd^c R$ is of bidegree $(n-1, n+1) + (n+1, n-1)$, and hence it must vanish. Therefore, $dd^c T = -dd^c S$, i.e., $S \in B'$. Conversely, if $S \in B'$, then $S - dT$ is $dd^c$-closed and of course also $d$-closed. Note that for $T \geq 0$ and $R$ real and of bidimension $(2, 0) + (0, 2)$, the equations

$$d(T + R) = S$$

and

$$\partial T + \partial R^{n,n-2} = S^{n,n-1}$$

are equivalent. Now the right hand side of the equation $\partial R^{n,n-2} = S^{n,n-1} - \partial T$ is $\partial$-closed. On $\mathbb{C}^n$, this implies that there exists a solution $R$ with compact support.

**Proof of (ii).** Since $T \geq 0$, we know from [DS] that $\text{spt} T \subseteq \overline{\text{spt} dd^c T}$. Of course $\text{spt} dd^c T = \text{spt} d^c S \subseteq \text{spt} S$.

**Lemma 2.2.** If $S = d(T + R) \in B$, then the mass $M(T) = T(dd^c|z|^2) = S(d^c|z|^2)$.

**Proof:** Note that $T(dd^c|z|^2) = (T + R)(dd^c|z|^2) = (d(T + R), d^c|z|^2) = S(d^c|z|^2)$. □
Proposition 2.3. The cone $B$ is closed.

Proof: Suppose $S_j = d(T_j + R_j) \in B$ and $S_j \to S$. Then by Lemma 2.2, $M(T_j) = S_j(d^c z^2) \to S(d^c |z|^2)$, and so the masses $M(T_j)$ are uniformly bounded in $j$. The convergence of $\{S_j\}$ means that all $\text{spt} S_j \subset B(0, R)$ for some $R$. Hence by part (ii) we have $\text{spt} T_j \subset B(0, R)$ for all $j$. By the basic compactness property of positive currents, there is a subsequence with $T_j \to T \geq 0$. Finally, since $dd^c T_j = -d^c S_j$ we have $dd^c T = -d^c S$. Hence, $S \in B' = B$. □

3. The Remainder of the Proof of the Main Theorem

Suppose now that $\Gamma$ is a scarred 1-cycle in $C^n$ which satisfies the positive winding condition. Applying Theorem 2.1 (in its second, “restated” form) with $S = \Gamma$, there exists a compactly supported, positive (1,1)-current $T$ such

$$d(T + R) = \Gamma$$

(3.1)

where $R$ is a current of compact support and bidimension $(2,0)+(0,2)$. We shall show that $R = 0$ and $T$ is a positive holomorphic chain. To proceed we utilize a basic result of Wermel [W1] in a generalized form due to Alexander [A].

Theorem 3.1. Let $\Gamma$ be a scarred 1-cycle of class $C^1$ in $C^n$. Then $\hat{\text{spt}} \Gamma - \text{spt} \Gamma$ is a 1-dimensional complex analytic subvariety of $C^n - \text{spt} \Gamma$.

Proof: Alexander proves in [A] that if $K \subset C^n$ is contained in a compact connected set of finite linear measure, then $\hat{K} - K$ is a 1-dimensional complex analytic subvariety of $C^n - K$. The set $\text{spt} \Gamma$ has finite linear measure and only finitely many connected components. One sees from the definition that it
is possible to make a connected set $K = \text{spt}\Gamma \cup \tau$ of finite linear measure by adding a finite union of piecewise linear arcs $\tau$ contained in the complement of $\text{spt}\Gamma$. Each irreducible component $W$ of the complex analytic curve $\hat{K} - K$ will have locally finite 2-measure at points of $\tau$ and will extend to $\mathbb{C}^n - \text{spt}\Gamma$ as a variety with boundary of the form $\sum_k c_k \tau_k$, where the $c_k$’s are constants and $\tau_k$ are the connected arcs comprising $\tau$ (cf. [HL1], [H]). Suppose this boundary is non-zero. Then $W$ must be contained in the union of the complex lines determined by the real line segments comprising $\partial W \cap \tau$. Since $W$ is irreducible, it is contained in just one such complex line. Constructing $\tau$ so that each connected component of $\tau$ has at least two (complex independent) line segments, we have a contradiction. Thus, for generic choice of $\tau$, the set $\hat{K} - \text{spt}\Gamma$ is a 1-dimensional subvariety of $\mathbb{C}^n - \text{spt}\Gamma$. In particular, this proves that $\hat{K} \subseteq \hat{\text{spt}}\Gamma$. Since $\hat{\text{spt}}\Gamma \subseteq \hat{K}$, we are done.

Let $V_1, V_2, \ldots$ denote the irreducible components of the complex curve given by Theorem 3.1. We are going to prove that $T = \sum_j n_j V_j$ for positive integers $n_j$. For this we first utilize a result from [HL2, p. 182].

**Lemma 3.2.** Suppose $T$ is a positive current of bidimension 1,1 with $dd^c T = 0$ on a complex manifold $X$. If $T$ is supported in a complex analytic curve $W$ in $X$, then $T$ can be written as a sum $T = \sum_j h_j W_j$ where each $W_j$ is an irreducible component of $W$ and $h_j$ is a non-negative harmonic function on $W_j$.

The case needed here is the following.

**Corollary 3.3.** If $T \geq 0$ satisfies $dd^c T = -d^c \Gamma$ on $\mathbb{C}^n$, then on $\mathbb{C}^n - \text{spt}\Gamma$
one has $T = \sum_j h_j V_j$ with $h_j$ harmonic on $V_j$.

We first restrict attention to dimension $n = 2$, where the equation (2.5), namely

$$\partial R^{2,0} = \Gamma^{2,1} - \partial T$$

implies that $R^{2,0}$ is a holomorphic 2-form outside the support of $\Gamma - dT$.

**Lemma 3.4. (n=2)** If $d(T + R) = \Gamma$ with $T \geq 0$ and $R$ of bidimension $(2,0) + (0,2)$, then

$$\text{spt} R \subseteq \text{spt} \Gamma$$

**Proof:** By Theorem 2.1(ii), $R^{2,0}$ is a holomorphic 2-form on $\mathbb{C}^2 - \text{hspt} \Gamma$, and $R^{2,0}$ vanishes outside of a compact subset of $\mathbb{C}^2$. The polynomially convex set $\text{hspt} \Gamma$ cannot have a bounded component in its complement. Therefore, $R^{2,0}$ must vanish on all of $\mathbb{C}^2 - \text{hspt} \Gamma$.

**Lemma 3.5. (n=2)** Each $h_j \equiv c_j$ is constant, and the current $T = \sum_j c_j V_j$ is $d$-closed on $\mathbb{C}^2 - \text{hspt} \Gamma$.

**Proof:** Pick a regular point of one of the components $V_j$, let $\pi$ denote a holomorphic projection (locally near the point) onto $V_j$, and let $i$ denote the inclusion of $V_j$ into $\mathbb{C}^2$. Note that $T$ is locally supported in $V_j$ by Theorem 2.1(ii) while $R$ is locally supported in $V_j$ by Lemma 3.4. Therefore, both of the push-forwards $\pi_* T$ and $\pi_* R$ are well defined. Now $\pi_* R$, being of bidimension $(2,0) + (0,2)$ in $V_j$ must vanish. However, $T = h_j V_j$ satisfies $\pi_* T = h_j$. Since $d(T + R) = 0$, the push-forward $\pi_* d(T + R) = d\pi_* (T + R) = dh_j$ must also vanish, i.e., each $h_j = c_j$ is constant. This proves:
Corollary 3.6. \((n=2)\) The current \(T = \sum_j c_j V_j\) on \(\mathbb{C}^2 - \text{spt}\Gamma\) has locally finite mass across \(\text{spt}\Gamma\) and its extension \(T^0\) by zero across \(\text{spt}\Gamma\) satisfies

\[
dT^0 = \sum_j r_j \Gamma_j \quad \text{on} \quad \mathbb{C}^2
\]

for real constants \(r_j\).

**Proof:** See Remark 1.4. \(\Box\)

Another corollary of Lemma 3.5 is the following.

**Corollary 3.7.** \(\text{spt}R \subseteq \text{spt}\Gamma\)

**Proof:** By (2.5) the current \(R^{2,0}\) is a holomorphic 2-form on \(\mathbb{C}^2 - \text{spt}\Gamma\) since \(dT = 0\) there. Since \(R^{2,0}\) vanishes at infinity, this proves the result. \(\Box\)

**Completion of the case \(n=2\).** Now

\[
T + R = T^0 + \chi T + R \quad (3.2)
\]

where \(\chi\) is the characteristic function of \(\text{spt}\Gamma\) and \(\chi T + R\) has support in \(\text{spt}\Gamma\).

We also have

\[
d(\chi T + R) = \sum_j (n_j - r_j) \Gamma_j \quad \text{on} \quad \mathbb{C}^2 \quad (3.3)
\]

Let \(\rho\) denote a local projection onto a regular point of \(\Gamma_j\). Then \(\rho_*(\chi T + R)\) is a well defined current on \(\Gamma_j\), but of dimension 2. Hence it must vanish. Since \(\rho_*\) commutes with \(d\), this proves that \((n_j - r_j) \Gamma_j\) must vanish. Hence, \(r_j = n_j\) for all \(j\), and so \(d(\chi T + R) = 0\) and \(dT^0 = d(T + R)\) by equations (3.2) and (3.3). This proves that \(dT^0 = \Gamma\) by (3.1).
Proof for the case \( n \geq 3 \). The general case follow easily from the case where \( n = 2 \). Consider a generic linear projection \( \pi : \mathbb{C}^n \to \mathbb{C}^2 \) so that each mapping \( V_j \to \pi V_j \) is one-to-one. Then the current \( T = \sum_j h_j V_j \) in \( \mathbb{C}^n - \text{spt} \Gamma \) projects to the current \( \pi_T = \sum_j \tilde{h}_j \pi(V_j) \) in \( \mathbb{C}^2 - \pi(\text{spt} \Gamma) \) where \( \tilde{h}_j \circ \pi = h_j \). Since each \( \tilde{h}_j = c_j \) is constant, so is each \( h_j \). Now the current \( T^0 = \sum_j c_j V_j \) satisfies \( dT^0 = \sum_j r_j \Gamma_j \) and again by projecting we conclude that \( r_j = n_j \).

□

References


Rice University
Departments of Mathematics
P. O. Box 1892
Houston, TX 77251
E-mail: harvey@rice.edu

State University of New York At Stony Brook
Department of Mathematics
NY, USA
E-mail: blaine@math.sunysb.edu