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REMARKS ON THE ALEXANDER-WERMER THEOREM FOR CURVES

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To Renato Tribuzy on occasion of his 60th birthday

Abstract

We give a new proof of the Alexander-Wermer Theorem that characterizes the oriented curves in $\mathbf{C}^n$ which bound positive holomorphic chains, in terms of the linking numbers of the curve with algebraic cycles in the complement. In fact we establish a slightly stronger version which applies to a wider class of boundary 1-cycles. Arguments here are based on the Hahn-Banach Theorem and some geometric measure theory. Several ingredients in the original proof have been eliminated.

1. The Alexander-Wermer Theorem

We present a different proof of the Alexander-Wermer Theorem [AW], [W$_2$] for curves which uses the Hahn-Banach Theorem and techniques of geometric measure theory. Several ingredients of the original proof are eliminated, such as the reliance on the result in [HL$_1$] that if a curve satisfies the moment condition, then it bounds a holomorphic 1-chain. The arguments given here have been adapted to study the analogous problem in general projective manifolds (cf. [HL$_{3,4,5}$]).

Our arguments will also apply to a more general class of curves which we now introduce.

*Partially supported by the N.S.F.

Definition 1.1. *Let X be a complex manifold and suppose there exists a closed subset $\Sigma(\Gamma)$ of Hausdorff 1-measure zero and an oriented, properly embedded C^1-submanifold of $X - \Sigma(\Gamma)$ with connected components $\Gamma_1, \Gamma_2, \ldots$. If, for given integers $n_1, n_2, \ldots$, the sum $\Gamma = \sum_{k=1}^{\infty} n_k \Gamma_k$ defines a current of locally finite mass in X which is d-closed (i.e., without boundary), and if* sptΓ *has only a finite number of connected components*[1], *then Γ will be called a* **scarred 1-cycle (of class C^1) in** X. *By a unique choice of orientation on each Γ_k we may assume each $n_k > 0$.*

Exemple 1.2. Any real analytic 1-cycle is automatically a scarred 1-cycle (of class C^∞) – see [F, p. 433].

Definition 1.3. *By a* **positive holomorphic 1-chain with boundary** Γ *we mean a sum $V = \sum_{k=1}^{\infty} m_k [V_k]$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible 1-dimensional complex analytic subvariety of $X - \mathrm{spt}\Gamma$ such that V has locally finite mass in X and, as currents in X,*

$$dV = \Gamma$$

Remark 1.4. Standard projection techniques (cf. [Sh], [H]) show that any 1-dimensional complex subvariety W of $X - \mathrm{spt}\Gamma$ automatically has locally finite 2-measure at points of Γ, and furthermore, its current boundary is of the form $dW = \sum \epsilon_k \Gamma_k$ where $\epsilon_k = -1, 0$ or 1 for all k. See [H] and the "added in proof" for the more general case where T is a positive d-closed current on $\mathbf{C}^2 - \mathrm{spt}\Gamma$.

Definition 1.5. *A scarred 1-cycle Γ in $\mathbf{C}^n$ satisfies the* **(positive) winding**

[1] More generally we need only assume that sptΓ is contained in a compact connected set of finite linear measure

condition *if*

$$\frac{1}{2\pi i}\int_\Gamma \frac{dP}{P} \ > \ 0$$

for all polynomials $P \in \mathbf{C}[z]$ with $P \neq 0$ on Γ.

There are many equivalent formulations of this condition. We mention three.

Proposition 1.6. *Γ satisfies the (positive) winding condition if and only if any of the following equivalent conditions holds:*

1) $\int_\Gamma d^c\varphi \ \geq \ 0$ for all smooth plurisubharmonic functions φ on $\mathbf{C}^n$.

2) For each polynomial $P \in \mathbf{C}[z]$, the unique compactly supported solution $W_P(\Gamma)$ to the equation $dW_P(\Gamma) = P_*(\Gamma)$ satisfies $W_P(\Gamma) \geq 0$.

3) The linking number $\mathrm{Link}(\Gamma, Z) \geq 0$ for all algebraic hypersurfaces Z contained in $\mathbf{C}^n - \mathrm{spt}\Gamma$.

Proposition 1.7. *If Γ is the boundary of a positive holomorphic 1-chain V in $\mathbf{C}^n$, then Γ satisfies the positive winding condition.*

Proof: We have $\int_\Gamma d^c\varphi = \int_{dV} d^c\varphi = \int_V dd^c\varphi \geq 0$ since $dd^c\varphi \geq 0$.

$\square$

The following converse of Proposition 1.7 is due to Alexander and Wermer [AW], [W_2].

Main Theorem 1.8. *Let Γ be a scarred 1-cycle in $\mathbf{C}^n$. If Γ satisfies the (positive) winding condition, then Γ bounds a positive holomorphic 1-chain in $\mathbf{C}^n$.*

This slightly generalizes the theorem in [AW] which applies only to smooth oriented curves. However, the essential point of this paper is to provide a conceptually different proof of the result which has other applications. This proof has two distinct parts which constitute the following two sections.

Note. We adopt the following notation throughout the paper. The polynomial hull of a compact subset $K \subset \mathbf{C}^n$ is denoted by $\widehat{K}$. The mass of a current T with compact support in $\mathbf{C}^n$ is denoted by $M(T)$.

2. A Dual Interpretation

In this section we shall use the Hahn-Banach Theorem to establish a dual interpretation of the positive winding condition. The main result is the following. Recall that if C is a convex cone in a topological vector space V, its **polar** is the set $C^0 = \{v' \in V' : v'(v) \geq 0 \text{ for all } v \in C\}$.

Theorem 2.1. (The Duality Theorem) *The cone A in the space $\mathcal{E}^1_{\mathbf{R}}(\mathbf{C}^n)$ of smooth 1-forms on $\mathbf{C}^n$, defined by*

$$A \equiv \{d\psi + d^c\varphi : \psi \in C^\infty(\mathbf{C}^n) \text{ and } \varphi \in \mathcal{PSH}(\mathbf{C}^n)\}$$

and the cone B in the dual space $\mathcal{E}'_1(\mathbf{C}^n)_{\mathbf{R}}$ of compactly supported one-dimensional currents in $\mathbf{C}^n$, defined by

$$B \equiv \{S : S = d(T+R) \text{ with } T \geq 0 \text{ and } R \text{ of bidimension } 2,0+0,2\}$$

are each the polar of the other.

Moreover,

(i) The cone B coincides with the cone

$$B' \equiv \{S : dS = 0 \text{ and } \exists T \geq 0 \text{ with compact support and } dd^cT = -d^cS\}, \text{ and}$$

(ii) If $S = d(T + R) \in B$ with T and R as above, then

$$\mathrm{sptT} \ \subseteq \widehat{\mathrm{sptS}} \quad \text{(the polynomial hull of sptS).}$$

This result can be restated as follows.

Theorem 2.1′. *A real 1-dimensional current S with $dS = 0$ and compact support in $\mathbf{C}^n$ satisfies the (positive) winding condition if and only if*

$$S \ = \ d(T + R) \tag{2.1}$$

where T is a positive 1,1 current and R has bidimension 2,0 + 0,2, or equivalently,

$$dd^cT \ = \ -d^cS \tag{2.2}$$

for some compactly supported $T \geq 0$. Moreover, for each such T,

$$\mathrm{sptT} \ \subseteq \widehat{\mathrm{sptS}} \tag{2.3}$$

Proof: We will show that $B^0 \ = \ A$ and that B is closed. This is enough to conclude that A and B are each the polar of the other because of the bipolar theorem: $(C^0)^0 = \overline{C}$.

□

Proof that $A = B^0$. The inclusion $A \subseteq B^0$ is essentially a restatement of Proposition 1.7 – the same proof applies. We need only show $B^0 \subseteq A$. Suppose $\alpha \in B^0$, i.e., $S(\alpha) \geq 0$ for all $S \in B$. Restricting to S of the form $S = dR$ where $R = R^{2,0} + R^{0,2}$ is of bidimension 2,0+0,2, we see that $S(\alpha) = dR(\alpha)$ must vanish (since $-dR$ is also in B). Hence, $\partial\alpha^{1,0} = 0$ and $\overline{\partial}\alpha^{0,1} = 0$. That is, $d\alpha = d^{1,1}\alpha$. In particular, $d^{1,1}\alpha$ is d-closed. Therefore, on $\mathbf{C}^n$ the equation $d\alpha = d^{1,1}\alpha = dd^c\varphi$ can be solved for some $\varphi \in C^\infty(\mathbf{C}^n)$.

Taking $S = dT$ where $T = \delta_p\xi \geq 0$ for $p \in \mathbf{C}^n$, yields $(d\alpha)(\delta_p\xi) = (d^{1,1}\alpha)(\delta_p\xi) \geq 0$. Hence, $dd^c\varphi = d^{1,1}\alpha \geq 0$, i.e., $\varphi \in \mathcal{PSH}(\mathbf{C}^n)$. Since $\alpha - d^c\varphi$ is d-closed, there exists $\psi \in C^\infty(\mathbf{C}^n)$ with $\alpha = d\psi + d^c\varphi$.

□

To show that B is closed requires several preliminary results.

Proof of (i). If $S \in B$, then dd^cR is of bidegree $(n-1, n+1) + (n+1, n-1)$, and hence it must vanish. Therefore, $dd^cT = -d^cS$, i.e., $S \in B'$. Conversely, if $S \in B'$, then $S - dT$ is d^c-closed and of course also d-closed. Note that for $T \geq 0$ and R real and of bidimension $(2,0) + (0,2)$, the equations

$$d(T + R) \;=\; S \tag{2.4}$$

and

$$\partial T + \overline{\partial} R^{n,n-2} \;=\; S^{n,n-1} \tag{2.5}$$

are equivalent. Now the right hand side of the equation $\overline{\partial} R^{n,n-2} = S^{n,n-1} - \partial T$ is $\overline{\partial}$-closed. On $\mathbf{C}^n$, this implies that there exists a solution R with compact support.

□

Proof of (ii). Since $T \geq 0$, we know from [DS] that $\mathrm{sptT} \subseteq \widehat{\mathrm{spt\,dd^c}}\mathrm{T}$. Of course $\mathrm{spt\,dd^cT} = \mathrm{spt\,d^cS} \subseteq \mathrm{sptS}$.

□

Lemma 2.2. *If $S = d(T + R) \in B$, then the mass $M(T) = T(dd^c|z|^2) = S(d^c|z|^2)$.*

Proof: Note that $T(dd^c|z|^2) = (T + R)(dd^c|z|^2) = (d(T + R), d^c|z|^2) = S(d^c|z|^2)$. □

Proposition 2.3. *The cone B is closed.*

Proof: Suppose $S_j = d(T_j + R_j) \in B$ and $S_j \to S$. Then by Lemma 2.2, $M(T_j) = S_j(d^c|z|^2) \to S(d^c|z|^2)$, and so the masses $M(T_j)$ are uniformly bounded in j. The convergence of $\{S_j\}$ means that all $\mathrm{sptS_j} \subset \mathrm{B(0,R)}$ for some R. Hence by part (ii) we have $\mathrm{sptT_j} \subset \mathrm{B(0,R)}$ for all j. By the basic compactness property of positive currents, there is a subsequence with $T_j \to T \geq 0$. Finally, since $dd^cT_j = -d^cS_j$ we have $dd^cT = -d^cS$. Hence, $S \in B' = B$.

□

3. The Remainder of the Proof of the Main Theorem

Suppose now that Γ is a scarred 1-cycle in $\mathbf{C}^n$ which satisfies the positive winding condition. Applying Theorem 2.1 (in its second, "restated" form) with $S = \Gamma$, there exists a compactly supported, positive (1,1)-current T such

$$d(T + R) \;=\; \Gamma \tag{3.1}$$

where R is a current of compact support and bidimension (2,0)+(0,2). We shall show that $R = 0$ and T is a positive holomorphic chain. To proceed we utilize a basic result of Wermer [W_1] in a generalized form due to Alexander [A].

Theorem 3.1. *Let Γ be a scarred 1-cycle of class C^1 in $\mathbf{C}^n$. Then $\widehat{\mathrm{spt}\Gamma} - \mathrm{spt}\Gamma$ is a 1-dimensional complex analytic subvariety of $\mathbf{C}^n - \mathrm{spt}\Gamma$.*

Proof: Alexander proves in [A] that if $K \subset \mathbf{C}^n$ is contained in a compact connected set of finite linear measure, then $\widehat{K} - K$ is a 1-dimensional complex analytic subvariety of $\mathbf{C}^n - K$. The set $\mathrm{spt}\Gamma$ has finite linear measure and only finitely many connected components. One sees from the definition that it is possible to make a connected set $K = \mathrm{spt}\Gamma \cup \tau$ of finite linear measure by adding a finite union of piecewise linear arcs τ contained in the complement of $\mathrm{spt}\Gamma$. Each irreducible component W of the complex analytic curve $\widehat{K} - K$ will have locally finite 2-measure at points of τ and will extend to $\mathbf{C}^n - \mathrm{spt}\Gamma$ as a variety with boundary of the form $\sum_k c_k \tau_k$, where the c_k's are constants and τ_k are the connected arcs comprising τ (cf. [HL$_1$], [H]). Suppose this boundary is non-zero. Then W must be contained in the union of the complex lines determined by the real line segments comprising $\partial W \cap \tau$. Since W is irreducible, it is contained in just one such complex line. Constructing τ so that each connected component of τ has at least two (complex independent) line segments, we have a contradiction. Thus, for generic choice of τ, the set $\widehat{K} - \mathrm{spt}\Gamma$ is a 1-dimensional subvariety of $\mathbf{C}^n - \mathrm{spt}\Gamma$. In particular, this proves that $\widehat{K} \subseteq \widehat{\mathrm{spt}\Gamma}$. Since $\widehat{\mathrm{spt}\Gamma} \subseteq \widehat{K}$, we are done.

$\square$

Let $V_1, V_2, \ldots$ denote the irreducible components of the complex curve given by Theorem 3.1. We are going to prove that $T = \sum_j n_j V_j$ for positive integers n_j. For this we first utilize a result from [HL$_2$, p. 182].

Lemma 3.2. *Suppose T is a positive current of bidimension 1,1 with $dd^c T = 0$ on a complex manifold X. If T is supported in a complex analytic curve W in X, then T can be written as a sum $T = \sum_j h_j W_j$ where each W_j is an irreducible component of W and h_j is a non-negative harmonic function on W_j.*

The case needed here is the following.

Corollary 3.3. *If $T \geq 0$ satisfies $dd^c T = -d^c\Gamma$ on $\mathbf{C}^n$, then on $\mathbf{C}^n - \mathrm{spt}\Gamma$*

one has $T = \sum_j h_j V_j$ *with* h_j *harmonic on* V_j*.*

We first restrict attention to dimension $n = 2$, where the equation (2.5), namely

$$\overline{\partial} R^{2,0} \; = \; \Gamma^{2,1} - \partial T$$

implies that $R^{2,0}$ is a holomorphic 2-form outside the support of $\Gamma - dT$.

Lemma 3.4. **(n=2)** *If* $d(T+R) = \Gamma$ *with* $T \geq 0$ *and* R *of bidimension* $(2,0)+(0,2)$*, then*

$$\mathrm{sptR} \; \subseteq \; \widehat{\mathrm{spt}\Gamma}$$

Proof: By Theorem 2.1(ii), $R^{2,0}$ is a holomorphic 2-form on $\mathbf{C}^2 - \widehat{\mathrm{spt}\Gamma}$, and $R^{2,0}$ vanishes outside of a compact subset of $\mathbf{C}^2$. The polynomially convex set $\widehat{\mathrm{spt}\Gamma}$ cannot have a bounded component in its complement. Therefore, $R^{2,0}$ must vanish on all of $\mathbf{C}^2 - \widehat{\mathrm{spt}\Gamma}$.

$\square$

Lemma 3.5. **(n=2)** *Each* $h_j \equiv c_j$ *is constant, and the current* $T = \sum_j c_j V_j$ *is* d*-closed on* $\mathbf{C}^2 - \mathrm{spt}\Gamma$.

Proof: Pick a regular point of one of the components V_j, let π denote a holomorphic projection (locally near the point) onto V_j, and let i denote the inclusion of V_j into $\mathbf{C}^2$. Note that T is locally supported in V_j by Theorem 2.1(ii) while R is locally supported in V_j by Lemma 3.4. Therefore, both of the push-forwards $\pi_* T$ and $\pi_* R$ are well defined. Now $\pi_* R$, being of bidimension $(2,0)+(0,2)$ in V_j must vanish. However, $T = h_j V_j$ satisfies $\pi_* T = h_j$. Since $d(T+R) = 0$, the push-forward $\pi_* d(T+R) = d\pi_*(T+R) = dh_j$ must also vanish, i.e., each $h_j = c_j$ is constant. This proves:

Corollary 3.6. **(n=2)** *The current* $T = \sum_j c_j V_j$ *on* $\mathbf{C}^2 - \mathrm{spt}\Gamma$ *has locally finite mass across* $\mathrm{spt}\Gamma$ *and its extension* T^0 *by zero across* $\mathrm{spt}\Gamma$ *satisfies*

$$dT^0 \;=\; \sum_j r_j \Gamma_j \qquad \text{on} \;\; \mathbf{C}^2$$

for real constants r_j.

Proof: See Remark 1.4.

□

Another corollary of Lemma 3.5 is the following.

Corollary 3.7. $\mathrm{sptR} \;\subseteq\; \mathrm{spt}\Gamma$

Proof: By (2.5) the current $R^{2,0}$ is a holomorphic 2-form on $\mathbf{C}^2 - \mathrm{spt}\Gamma$ since $dT = 0$ there. Since $R^{2,0}$ vanishes at infinity, this proves the result.

□

Completion of the case n=2. Now

$$T + R \;=\; T^0 + \chi T + R \tag{3.2}$$

where χ is the characteristic function of $\mathrm{spt}\Gamma$ and $\chi T + R$ has support in $\mathrm{spt}\Gamma$. We also have

$$d(\chi T + R) \;=\; \sum_j (n_j - r_j)\Gamma_j \qquad \text{on} \;\; \mathbf{C}^2 \tag{3.3}$$

Let ρ denote a local projection onto a regular point of Γ_j. Then $\rho_*(\chi T + R)$ is a well defined current on Γ_j, but of dimension 2. Hence it must vanish. Since ρ_* commutes with d, this proves that $(n_j - r_j)\Gamma_j$ must vanish. Hence, $r_j = n_j$ for all j, and so $d(\chi T + R) = 0$ and $dT^0 = d(T + R)$ by equations (3.2) and (3.3). This proves that $dT^0 = \Gamma$ by (3.1) .

Proof for the case $n \geq 3$. The general case follow easily from the case where $n = 2$. Consider a generic linear projection $\pi : \mathbf{C}^n \to \mathbf{C}^2$ so that each mapping $V_j \to \pi V_j$ is one-to-one. Then the current $T = \sum_j h_j V_j$ in $\mathbf{C}^n - \mathrm{spt}\Gamma$ projects to the current $\pi_* T = \sum_j \widetilde{h}_j \pi(V_j)$ in $\mathbf{C}^2 - \pi(\mathrm{spt}\Gamma)$ where $\widetilde{h}_j \circ \pi = h_j$. Since each $\widetilde{h}_j = c_j$ is constant, so is each h_j. Now the current $T^0 = \sum_j c_j V_j$ satisfies $dT^0 = \sum_j r_j \Gamma_j$ and again by projecting we conclude that $r_j = n_j$.

□

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