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ON THE EXISTENCE OF SIGNED SOLUTIONS FOR A QUASILINEAR ELLIPTIC PROBLEM IN $\mathbb{R}^N$

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Dedicated to Luís Adauto Medeiros on his 80th birthday

Abstract

We study the existence of positive and negative weak solutions for the equation

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda f(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$, λ is a positive real parameter and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is bounded from below for a positive constant and "large" at infinity. It is assumed that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous and just superlinear in a neighborhood of the origin.

1 Introduction

In this note we are concerned with the existence of positive and negative solutions for the equation

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where, $1 < p < N$, λ is a positive real parameter and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a function satisfying the following assumptions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf\limits_{x \in \mathbb{R}^N} V(x) = V_0 > 0$;

(V_2) for every $M > 0$

$$|\{x \in \mathbb{R}^N : V(x) \leq M\}| < \infty,$$

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where $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^N$.

We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions which give its behavior only in a neighborhood of the origin:

(f_1) there exists $r \in (p, p^*)$ such that

$$\limsup_{|s|\to 0} \frac{f(s)s}{|s|^r} < +\infty;$$

(f_2) there exists $q \in (p, p^*)$ verifying

$$\liminf_{|s|\to 0} \frac{F(s)}{|s|^q} > 0;$$

(f_3) there exists $\theta \in (p, p^*)$ such that

$$0 < \theta F(s) \leq sf(s) \text{ for } |s| \neq 0 \text{ small};$$

Here $p^* = \frac{pN}{(N-p)}$ is the critical Sobolev exponent and $F(s) = \int_0^s f(t)dt$.

Remark 1.1 *A typical example of a function f satisfying the hypotheses (f_1)–(f_3) is $f(s) = a|s|^{r-2}s + b|s|^{\beta-2}s$ with $p < r \leq q < p^* < \beta$ and a, b are positive constants.*

In recent years, there has been a large amount of work done on problems modeled by equations involving the p-Laplacian operator. Some of these problems arise in diverse areas of applied mathematics and physics, for example in the theory of nonlinear elasticity, glaciology, combustion theory, population biology, nonlinear flow laws, system of Monge-Kantorovich partial differential equations. For additional discussions about problems modeled by p-Laplacian operator, see for example [3], [4], [5], [6], [9] and references therein.

In the case $p = 2$, equation (1.1) arises naturally from the search for standing wave solution for a nonlinear Schrödinger equation, see [8]. Similarly, the search of standing (or traveling) waves in nonlinear equations of Klein-Gordon type leads to the study of (1.1). Still in the case $p = 2$, equation (1.1) appears also in other contexts, for example when one studies reaction-diffusion equations.

Next, we state our main result.

Theorem 1.2 *Assume* $(V_1) - (V_2)$ *and* $(f_1) - (f_3)$. *Then equation (1.1) has one positive solution and one negative solution for all* λ *sufficiently large.*

As is well known, if f was assumed to be superlinear at infinity , that is, in the sense that (f_1) and (f_2) hold as $|u| \to \infty$ and (f_3) holds for $|u|$ large, then equation (1.1) has one positive solution and one negative solution. Here, the assumptions $(f_1) - (f_3)$ refer solely to its behavior in a neighborhood of $u = 0$ and they are sufficient to obtain Theorem 1.2.

The study of problem (1.1) was in part motivated by work of Costa and Wang [2], where they deal with the case $p = 2$ in bounded domain with Dirichlet boundary conditions and without the term of the potential. As in [2], since the natural variational functional associated to (1.1) is not well defined because we do not have information about the function $F(s)$ at infinity, we explored an argument of penalization to obtain a new functional which will be well defined. Using a compact embedding result of Bartsch and Wang [1] and Mountain Pass theorem, we show that this functional has a positive and a negative critical points. After that, by Moser iteration we get an L^∞-estimate for these critical points which depends on parameter λ. Then, for λ sufficiently large, we can conclude that the critical points are solutions of our original problem.

The organization of this work is as follows: In Section 2, we give some preliminary results and in Section 3, we prove the main Theorem.

In what follows, $C, C_1, C_2, ...$ will denote positive generic constants, u^+, u^- will be the positive and negative parts of u, respectively and $\|u\|_t$ denotes the usual L^t-norm for a function $u \in L^t(\mathbb{R}^N)$, $1 \leq t \leq \infty$.

2 Preliminary results

As usual, $W^{1,p}(\mathbb{R}^N)$ denotes the Sobolev space of functions in $L^p(\mathbb{R}^N)$ such that their weak derivatives are also in $L^p(\mathbb{R}^N)$ with the norm

$$\|u\|_{1,p}^p = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p),$$

and we consider the subspace $E \subset W^{1,p}(\mathbb{R}^N)$ defined by

$$E = \Big\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p < \infty \Big\},$$

which is a reflexive Banach space endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p + V(x)|u|^p \right)^{1/p}.$$

By assumption (V_1), it follows that the space E is continuously immersed in $W^{1,p}(\mathbb{R}^N)$. Moreover, we have the following result:

Lemma 2.1 *The space E is compactly immersed in $L^s(\mathbb{R}^N)$ for all $s \in [p, p^*)$.*

Proof. See, for example, [1]. □

Since the hypotheses $(f_1)-(f_3)$ give the behavior of f only in a neighborhood of $s=0$, the natural functional associated with the equation (1.1) given by

$$I_\lambda(u)=\frac{1}{p}\|u\|^p-\lambda\int_{\mathbb{R}^N}F(u)$$

is not well defined in E. In order to apply variational methods, let us use here an argument of penalization as in [2] to obtain a new functional. To this end, observe that the conditions (f_1) and (f_2) imply the existence of positive constants C_0, C_1 such that

$$F(u)\le C_0|u|^r \tag{2.2}$$

and

$$F(u)\ge C_1|u|^q \tag{2.3}$$

for $|u|$ small (we may assume that $r\le q$). Now, we consider $\rho(t)$ be an even cut-off function verifying:

$$\rho(t)=\begin{cases}1 & \text{if} \quad |t|\le\delta\\ 0 & \text{if} \quad |t|\ge 2\delta,\end{cases}$$

$t\rho'(t)\le 0$ and $|t\rho'(t)|\le\frac{2}{\delta}$, where δ is chosen such that (2.2), (2.3) and (f_3) hold for $|u|\le 2\delta$. Define

$$G(u)=\rho(u)F(u)+(1-\rho(u))F_\infty(u),$$

where $F_\infty(u)=C_0|u|^r$ and denote $g(u)=G'(u)$. If $|u|\le\delta$, we have $G(u)=F(u)$. It follows from (f_1), that

$$|g(u)|=|F'(u)|=|f(u)|\le C_1|u|^{r-1}.$$

For $|u| \geq 2\delta$ we have $G(u) = F_\infty(u) = C_0|u|^r$, consequently $|g(u)| \leq C_2|u|^{r-1}$. By definition

$$g(u) = \rho'(u)F(u) + \rho(u)F'(u) - \rho'(u)F_\infty(u) + (1-\rho(u))F'_\infty(u).$$

Since $|\rho'(u)u| \leq \frac{2}{\delta}$, by (2.2) we get $|\rho'(u)F(u)| \leq C_0|u|^{r-1}$ for all $\delta \leq |u| \leq 2\delta$. Therefore, there exists $C > 0$ such that

$$|g(u)| \leq C|u|^{r-1} \quad \text{for all} \quad u \in \mathbb{R}. \tag{2.4}$$

2.1 Modified problem

Denoting $g(u) = G'(u)$, let us consider the following modified problem:

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda g(u) \quad \text{in} \quad \mathbb{R}^N. \tag{2.5}$$

By definition of G, the functional associated to (2.5) given by

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda \int_{\mathbb{R}^N} G(u)$$

is well defined in E. It is clear that J_λ is of class C^1 and its critical points are precisely the weak solutions of problem (2.5). In order to apply critical point theory, let us show the following property of function G.

Lemma 2.2 *Under the hypotheses above,* $0 < \alpha G(u) \leq ug(u)$ *for all* $u \neq 0$, *where* $\alpha = min\{r, \theta\}$

Proof. See Lemma 1.1 in [2].

□

The Lemma below shows that the functional J_λ possesses the mountain pass geometry.

Lemma 2.3 *There exist $\varepsilon_0 > 0$ and $C = C(\varepsilon_0, \lambda) > 0$ such that*

$$J_\lambda(u) \geq C > 0 \quad \textit{for} \;\; u \in E, \quad \|u\| = \varepsilon_0$$

and $u_0 \in E$, $\|u_0\| > \varepsilon_0$ verifying $J_\lambda(u_0) \leq 0$.

Proof. Using the fact that $G(u) \leq C|u|^r$ and $r > p$, together with the Sobolev embedding we get

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p - \lambda C\|u\|^r \geq C(\varepsilon_0, \lambda) > 0,$$

for $\|u\| = \varepsilon_0$ sufficiently small. Now, Lemma 2.2 and a straightforward computation show that $G(u) \geq C|u|^\alpha$. Since $\alpha > p$, taking $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have $J_\lambda(t\varphi) \to -\infty$ as $t \to +\infty$. Hence for t sufficiently large, $u_0 = t\varphi$ satisfies $J_\lambda(u_0) \leq 0$.

□

Proposition 2.4 *Under the hypotheses $(V_1) - (V_2)$ and $(f_1) - (f_3)$, problem (2.5) has a nontrivial solution.*

Proof. As consequence from Lemmas 2.1 and 2.2, it is standard to show that the functional J_λ satisfies the Palais-Smale condition. Thus by Lemma 2.3 we can apply the Mountain Pass theorem to get a nontrivial critical point of J_λ.

□

Lemma 2.5 *If $u \in E$ is a critical point of J_λ, then*

$$\|u\|^p \leq \frac{p\alpha}{\alpha - p} J_\lambda(u). \tag{2.6}$$

Proof. This estimate follows directly from Lemma 2.2 and

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda \int_{\mathbb{R}^N} G(u), \quad \|u\|^p = \lambda \int_{\mathbb{R}^N} g(u)u.$$

□

In order to show that solutions of *modified problem* (2.5) are solutions of problem (1.1), we will need the following L^∞ estimate.

Lemma 2.6 *If $u \in E$ is a weak solution of problem (2.5), then $u \in L^\infty(\mathbb{R}^N)$. Moreover, there exists $C = C(r, p, N) > 0$ such that*

$$\|u\|_\infty \leq C(\lambda \|u\|^{r-p})^{1/(p^*-r)} \|u\|. \tag{2.7}$$

Proof. Let u be a weak solution of problem (2.5), that is

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi + \int_{\mathbb{R}^N} V(x)|u|^{p-2} u \varphi = \lambda \int_{\mathbb{R}^N} g(u)\varphi, \tag{2.8}$$

for all $\varphi \in E$. We can assume without loss of generality that u is nonnegative, if necessary changing u by u_+ or u_-. For each $k > 0$ we define

$$u_k = \begin{cases} u & \text{if } u \leq k \\ k & \text{if } u \geq k, \end{cases}$$

$v_k = u_k^{p(\beta-1)} u$ and $w_k = u u_k^{\beta-1}$ with $\beta > 1$ to be determined later. Taking $\varphi = v_k$ in (2.8) and using (2.4) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} u_k^{p(\beta-1)} |\nabla u|^p \quad \leq \quad & -\int_{\mathbb{R}^N} V(x) u_k^{p(\beta-1)} u^p - p(\beta-1) \int_{\mathbb{R}^N} u_k^{p(\beta-1)-1} u \nabla u_k \nabla u \\ & + \lambda C \int_{\mathbb{R}^N} u^r u_k^{p(\beta-1)}. \end{aligned}$$

Observing that the first and the second terms in the right side of the inequality above are not positive, we have

$$\int_{\mathbb{R}^N} u_k^{p(\beta-1)} |\nabla u|^p \leq \lambda C \int_{\mathbb{R}^N} u^r u_k^{p(\beta-1)} = \lambda C \int_{\mathbb{R}^N} u^{r-p} w_k^p. \tag{2.9}$$

By Gagliardo-Nirenberg-Sobolev inequality and (2.9), we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N} w_k^{p^*}\right)^{p/p^*} &\leq C_1\int_{\mathbb{R}^N}|\nabla w_k|^p\\ &\leq C_2\int_{\mathbb{R}^N} u_k^{p(\beta-1)}|\nabla u|^p + C_3(\beta-1)^p\int_{\mathbb{R}^N} u^p u_k^{p(\beta-2)}|\nabla u_k|^p\\ &\leq C_4\beta^p\int_{\mathbb{R}^N} u_k^{p(\beta-1)}|\nabla u|^p\\ &\leq \lambda C_5\beta^p\int_{\mathbb{R}^N} u^{r-p}w_k^p,\end{aligned}$$

where we have used that $1 \leq \beta^p$, $(\beta-1)^p \leq \beta^p$ and the definition of u_k. Now, using the Hölder inequality we have

$$\left(\int_{\mathbb{R}^N} w_k^{p^*}\right)^{p/p^*} \leq \lambda\beta^p C_5\left(\int_{\mathbb{R}^N} u^{p^*}\right)^{(r-p)/p^*}\left(\int_{\mathbb{R}^N} w_k^{pp^*/(p^*-r+p)}\right)^{(p^*-r+p)/p^*}.$$

Since that $|w_k| \leq |u|^\beta$ and E is continuously immersed in $L^{p^*}(\mathbb{R}^N)$ we get

$$\left(\int_{\mathbb{R}^N} |uu_k^{\beta-1}|^{p^*}\right)^{p/p^*} \leq \lambda\beta^p C_6\|u\|^{r-p}\left(\int_{\mathbb{R}^N} u^{\beta pp^*/(p^*-r+p)}\right)^{(p^*-r+p)/p^*}.$$

Choosing $\beta = 1+\frac{p^*-r}{p}$ we have $\beta pp^*/(p^*-r+p) = p^*$. Thus,

$$\left(\int_{\mathbb{R}^N} |uu_k^{\beta-1}|^{p^*}\right)^{p/p^*} \leq \lambda\beta^p C_6\|u\|^{r-p}\|u\|_{\beta\alpha^*}^{p\beta},$$

where $\alpha^* = pp^*/(p^*-r+p)$. By Fatou's Lemma, we obtain

$$\|u\|_{\beta p^*} \leq (\lambda\beta^p C_6\|u\|^{r-p})^{1/p\beta}\|u\|_{\beta\alpha^*}. \tag{2.10}$$

For each $m = 0, 1, 2, ...$ let us define $\beta_{m+1}\alpha^* = p^*\beta_m$ where $\beta_0 = \beta$. Applying the procedure before for β_1, by (2.10) we have

$$\begin{aligned}\|u\|_{\beta_1 p^*} &\leq (\lambda\beta_1^p C_6\|u\|^{r-p})^{1/p\beta_1}\|u\|_{\beta_1\alpha^*}\\ &\leq (\lambda\beta_1^p C_6\|u\|^{r-p})^{1/p\beta_1}(\lambda\beta^p C_6\|u\|^{r-p})^{1/p\beta}\|u\|_{\beta\alpha^*}\\ &\leq (\lambda C_6\|u\|^{r-p})^{1/p\beta+1/p\beta_1}(\beta)^{1/\beta}(\beta_1)^{1/\beta_1}\|u\|_{p*}.\end{aligned}$$

Observing that $\beta_m = \chi^m\beta$ where $\chi = \frac{p^*}{\alpha^*}$, by iteration we obtain

$$\|u\|_{\beta_m p^*} \leq (\lambda C_6\|u\|^{r-p})^{\frac{1}{p\beta}\sum_{i=0}^m \chi^{-i}}\beta^{\frac{1}{\beta}\sum_{i=0}^m \chi^{-i}}\chi^{\frac{1}{\beta}\sum_{i=0}^m i\chi^{-i}}\|u\|_{p*}.$$

Since $\chi > 1$ and $\lim_{m\to\infty} \frac{1}{p\beta}\sum_{i=0}^{m} \chi^{-i} = \frac{1}{p^*-r}$, we can take the limit as $m \to \infty$ to get

$$\|u\|_\infty \leq C_7(\lambda\|u\|^{r-p})^{1/(p^*-r)}\|u\|.$$

Thus the proof is complete.

□

3 Proof of the main theorem

In order to obtain a positive and a negative solution of problem (1.1), let us consider the following auxiliary functionals

$$J_{1,\lambda}(u) = \frac{1}{p}\|u\|^p - \lambda\int_{\mathbb{R}^N} G_1(u)$$

and

$$J_{2,\lambda}(u) = \frac{1}{p}\|u\|^p - \lambda\int_{\mathbb{R}^N} G_2(u)$$

respectively, where

$$G_1(u) = \begin{cases} G(u) & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} \qquad \text{and} \qquad G_2(u) = \begin{cases} G(u) & \text{if } u \leq 0 \\ 0 & \text{if } u > 0. \end{cases}$$

Since G_1 and G_2 have the same properties that G, by Proposition 2.4 it follows that $J_{i,\lambda}$ has a critical point u_i at the level

$$c_{i,\lambda} = \inf_{h\in\Gamma_i}\max_{t\in[0,1]} J_{i,\lambda}(h(t)),$$

where $\Gamma_i \doteq \{h \in C(E,\mathbb{R}) : J_{i,\lambda}(h(1)) \leq 0\}$ for $i = 1, 2$. Besides, the solution u_1 is positive and u_2 is negative.

Lemma 3.1 *Let u_i be a critical point of $J_{i,\lambda}$ at the level $c_{i,\lambda}$ for $i = 1, 2$. Then there exists $C > 0$ such that*

$$\|u_{i,\lambda}\| \leq C\lambda^{-\frac{1}{q-p}}. \tag{3.11}$$

Proof. Let us show (3.11) only for $u_{1,\lambda}$ because the same argument works with $u_{2,\lambda}$. Notice that

$$c_{1,\lambda} \leq d_{1,\lambda}, \tag{3.12}$$

where

$$d_{1,\lambda} = \inf_{u>0} \max_{t\geq 0} J_{1,\lambda}(tu).$$

Since $J_{1,\lambda}(u) = J_\lambda(u)$ for $u \geq 0$, from Lemma 2.5 we obtain

$$\|u_{1,\lambda}\|^p \leq \frac{p\alpha}{\alpha - p} d_{1,\lambda}. \tag{3.13}$$

We claim that

$$d_{1,\lambda} \leq C\lambda^{-\frac{p}{q-p}},$$

for some $C > 0$. To this, we consider the functional

$$\Phi_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda \int_{\mathbb{R}^N} C_1|u|^q,$$

where $C_1 > 0$ is such that $G(u) \geq C_1|u|^q$ for $|u| \leq 2\delta$. Define

$$e_\lambda = \inf_{u\neq 0} \max_{t\geq 0} \Phi_\lambda(tu).$$

If $u \in E, u \neq 0$, we have that

$$\max_{t\geq 0} \Phi_\lambda(tu) = \left(\frac{1}{p} - \frac{1}{q}\right)(q\lambda C_0)^{-p/(q-p)} \frac{\|u\|^{pq/(q-p)}}{\|u\|_q^{pq/(q-p)}}.$$

Since the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is continuous, it follows that

$$\inf_{u\neq 0} \frac{\|u\|^{pq/(q-p)}}{\|u\|_q^{pq/(q-p)}} = \eta > 0.$$

Hence

$$e_\lambda = \inf_{u\neq 0} \max_{t\geq 0} \Phi_\lambda(tu) = \left(\frac{1}{p} - \frac{1}{q}\right)(q\lambda C_1)^{-p/(q-p)}\eta. \tag{3.14}$$

We can also see that Φ_λ has a ground state u_λ^* at the level e_λ. Thus,

$$\left(\frac{1}{p}-\frac{1}{q}\right)\|u_\lambda^*\|^p = \Phi_\lambda(u_\lambda^*) = e_\lambda = \left(\frac{1}{p}-\frac{1}{q}\right)(q\lambda C_0)^{-p/(q-p)}\eta,$$

which implies that

$$\|u_\lambda^*\| \leq C_1\lambda^{-1/(q-p)}. \tag{3.15}$$

with C_1 independent of λ. Consequently $(\lambda\|u_\lambda^*\|^{q-p}) \leq C_1$. By the same argument used in the proof of Lemma 2.6 and (3.15) we have

$$\|u_\lambda^*\|_\infty \leq C(\lambda\|u_\lambda^*\|^{q-p})^{1/(p^*-q)}\|u_\lambda^*\| \leq C_2\|u_\lambda^*\| \leq C_3\lambda^{-\frac{1}{q-p}} \leq 2\delta,$$

for $\lambda > 0$ sufficiently large. From this L^∞-estimative, we can conclude that

$$d_{1,\lambda} \leq \inf_{0<u\leq 2\delta}\max_{t\geq 0} J_{1,\lambda}(tu) \leq \inf_{0<u\leq 2\delta}\max_{t\geq 0}\Phi_\lambda(tu) = \inf_{u\neq 0}\max_{t\geq 0}\Phi_\lambda(tu) = e_\lambda. \tag{3.16}$$

It follows from (3.14) and (3.16) that

$$d_{1,\lambda} \leq C\lambda^{-\frac{p}{q-p}}.$$

This together with (3.13) implies the estimate (3.11).

□

Proof of the Theorem 1.2: Substituting (3.11) in (2.7) we obtain

$$\|u_{i,\lambda}\|_\infty \leq C\lambda^{(q-p^*)/((q-p)(p^*-r))}, \quad \text{for} \quad i = 1, 2.$$

Since $q < p^*$, there exists $\lambda_0 > 0$ such that

$$\|u_{i,\lambda}\|_\infty \leq 2\delta,$$

for all $\lambda > \lambda_0$. Consequently, $u_{1,\lambda}$ is a positive solution and $u_{2,\lambda}$ is a negative solution of our original problem (1.1).

□

Remark 3.2 *In [2], the authors have obtained the existence of sign-changing solutions by using the abstract critical point theory in partially ordered Hilbert space which involves the density of the Banach space $C(\Omega)$ of continuous functions in the Hilbert space $H_0^1(\Omega)$, more precisely, they have used the fact that the cone of the positive functions in $C(\Omega)$ has nonempty interior. However, this framework imposes stronger hypothesis, that is, boundedness of the domain and smoothness of the nonlinearity. In [7], using properties of invariants set of descending flow as in [10] and the references therein, the authors have obtained signed-changing solution for the case $p = 2$ where they have used strongly the fact that the spectrum from the operator $-\Delta + V$ is well characterized and this is not the case when $p \neq 2$.*

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